

PORTFOLIO ALLOCATION WITH MANY RISKY ASSETS

1. Portfolio separation theorems

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Earlier discussions of portfolio allocation made three crucial assumptions:

(1) There is only one commodity, or if there are many commodities, relative prices among the commodities are constant.

(2) There is only period.¹

(3) There are only two assets, one risky and one safe.²

These assumptions seem to provide a natural starting-point for a theory of portfolio allocation. But if this is to be the case, it must be possible for us to remove these assumptions without destroying the basic qualitative propositions that have been derived. That this may not be the case is suggested by the following considerations:

(1) When there are only two assets, one of them risky, one of them safe, if an individual wishes to increase the variance of this portfolio, then he must increase his purchases of the risky asset; if there are two risky assets, he can change the proportions in which he holds the two risky assets. Will it be possible then to extend the Arrow propositions [1] about the wealth elasticity of the demand for money to a world in which there is more than one risky asset?

(2) When there is more than one commodity, relative prices between commodities can change; one asset may be more highly correlated with the price of one commodity, the other with the price of another commodity. What is meant, in this context, by a safe asset?

(3) Exactly analogous questions arise in a multi-period context where the

¹That is, the assets are purchased at the beginning of the period and mature at the end of the period.

²The Markowitz-Tobin mean variance analysis did allow there to be a large number of risky assets; but a single 'mutual fund' of these risky assets could be formed, so that it was *as if* there were only a single risky asset.

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rate of interest in the future is unknown. A long-term bond is 'safe' in terms of consumption in the long term, a short-term bond is safe in terms of consumption in the short-term.

Moreover, several important economic phenomena are left unaccounted for by the single period, single consumption good, two asset model:

- (1) 'Hedging', e.g. in commodity markets, foreign exchange markets, etc.
- (2) 'Liquidity' the 'ease' with which some assets are converted into consumption — has played a central role in the discussions of the demand for money, but almost no part at all in formal modelling of portfolio analysis.
- (3) 'Term structure of interest rates' — the relation between short- and long-term interest rates — again is a major issue in monetary theory in which considerations of uncertainty clearly are important, yet without a multi-period model, little can be said.

In the next four lectures, I shall attempt to sketch a basis for extending portfolio theory to remove the three assumptions noted above. We shall see that although the simpler results of the two asset-one commodity-one period model do not extend in a straightforward manner, we are able to derive some interesting propositions in this broader context.

We begin our discussion with the problems introduced by allowing there to be more than one risky asset. Two questions naturally pose themselves:

- (1) The first is a question of aggregation: when can we form a single 'mutual fund', which can act as a single risky asset? Note that the assumption that it is possible to aggregate risky assets is implicit throughout most of macro-economics. Keynes [2] and Hicks [3] for instance, seem to have lumped together equities and long-term bonds, and treated them in terms of a single 'market'.
- (2) When it is not possible to form a mutual fund, what can we say about the demand for different assets, about the structure of portfolios, etc.?

These two questions have been answered in two papers [4,5] by Professor Cass and myself. The mathematical argument is somewhat long, and rather than simply repeat what we have already written there, let me attempt to develop some intuitive feel for the kind of result that can be obtained.

1. Aggregation

The first question we must ask ourselves is: for what purposes do we wish to form an aggregate? A second, related question is: what do we wish to aggregate together, e.g. all risky assets, or all bonds? To form meaningful aggregates, in general relative prices among the constituents of the aggregate must remain constant. But if our object is the analysis of certain macro-economic

relations, the kinds of macro-economic policy changes often contemplated may result in changes in, for instance, the price of long-term bonds relative to equities, or of different equities relative to one another.

There are fortunately some simple economic problems in which we can avoid these difficulties, and it is best to begin with them: suppose we are interested in how individuals' allocation of their portfolio among different assets depends on their wealth. The set of assets available to each individual is assumed to be the same. We can ask then: under what circumstances will the ratios in which different individuals hold different risky assets be independent of the level of wealth? For in these circumstances, we can form a single 'mutual fund' of risky assets which will be purchased by all individuals, regardless of their wealth. Moreover, in that case, the relative demands for different assets will be independent of the distribution of income, and we can predict changes in holdings of individual assets simply from a knowledge of the changes in the aggregate risky portfolio.

Finally, interest in this possibility arises from the fact that, conceptually at least, it allows us to think of the problem of portfolio allocation as a two-stage process: first decide on what relative proportions to invest in the different risky assets, and then decide on what proportion of our total portfolio to invest in money (a safe asset) and what proportion in the 'mutual fund'. Because of this property, theorems showing that relative proportions in which different individuals purchase different risky assets are the same are referred to as portfolio 'separation' theorems.

Formally, our problem may be stated as follows. An individual faces a set of assets, whose return per dollar invested is given by $\rho_i(\theta)$ in state θ , $i = 1, \dots, n$. θ may either be a continuous or discrete variable.³ The individual wishes to maximize his expected utility $EU(W(\theta))$. His terminal wealth in state θ , $W(\theta)$, depends on his portfolio allocation. If W_0 is his initial wealth, and a_i is the proportion of ones portfolio allocated to the i th asset,

$$\sum_i a_i = 1, \quad (1)$$

then

$$W(\theta) = W_0 \left(\sum_i a_i \rho_i(\theta) \right). \quad (2)$$

If there is a safe asset, we shall let it be the first, i.e.

$$\rho_1(\theta) \equiv \rho_m \quad (3)$$

all θ .

³When θ is a discrete variable, we write $\rho_{i\theta}$ as the return to security in state θ .

Thus, if U is concave, the necessary and sufficient condition for optimal portfolio allocation is

$$EU'(\rho_i(\theta) - \rho_1(\theta)) = 0, \quad i = 2, \dots, n. \quad (4)$$

Eq. (4) may be solved for a_i as a function of W_0 . Our problem is, under what conditions is

$$\frac{a_i}{n} \quad i = 2, \dots, n, \quad (5)$$

$$\sum_{j=2} a_j$$

independent of W_0 .

One obvious set of conditions is that the demand functions for the different assets be linear in W_0 :

$$Z_i = \beta_i + \gamma_i W_0 \quad (6)$$

with

$$\beta_i \gamma_j = \gamma_i \beta_j,$$

where Z_i is the demand (in dollars) for the i th asset i :

$$Z_i = a_i W_0.$$

Special cases of (6) are, of course, $\beta_i = \beta_j = 0$: a_i (for all i) is independent of W_0 ; and $\gamma_i = \gamma_j = 0$, demand for each risky asset is independent of the level of wealth.

It is fairly easy to guess a class of utility functions which yields linear demand functions:

$$U' = (A + BW)^{-c} \quad (7)$$

and the limiting case of (7):

$$U' = ce^{-Aw} \quad (7')$$

To see that (7) does in fact yield linear demand curves, consider first the case of $A = 0$. Then the first order conditions for utility maximization can be written

$$E \left\{ B [\rho_m W_0 + \sum_{j=2}^n a_j^* (\rho_j - \rho_m) W]^{-c} (\rho_i - \rho_1) \right\} = 0, \quad i = 2, \dots, n \quad (8)$$

or

$$E \left\{ \rho_m + \sum_{j=2}^n a_j^* (\rho_j - \rho_m)^{-c} (\rho_i - \rho_m) \right\} = 0. \quad (8')$$

The solution to (8) does not depend on W_0 . Hence $\beta_i = 0$, $\gamma_i = a_i^*$, $i = 2, \dots, n$, $a_1 = 1 - \sum_{j=2}^n a_j^*$.

The general case can easily be converted into the form (8). Pretend that all our wealth in excess of W_0^* were allocated in constant proportions to the different assets, so

$$Z_1 = W_0^* + (1 - \sum_{j=2}^n a_j) (W_0 - W_0^*)$$

$$z_i = a_i (W_0 - W_0^*) \quad i = 2, \dots, n.$$

Then the first order conditions may be written

$$E \left\{ [A + B \rho_m W_0^* + (\rho_m + \sum_{j=2}^n (a_j (\rho_j - \rho_m)) (W_0 - W_0^*))]^{-c} (\rho_i - \rho_m) \right\} = 0.$$

Now if we choose

$$W_0^* = - \frac{A}{B \rho_m}$$

(9) reduces to (8'), and the result is immediate.⁴

One special case of (7) has been the focus of much attention: if $c = 1$, we obtain the quadratic utility function.

Similarly, to see that (7') yields linear demand curves, we let $a_i W_0$, $i = 2, \dots, n$ be constant. Then the first-order conditions can be written

⁴ The analogy between this family of demand functions and the Stone-Geary system should be apparent.

$$E(\rho_i - \rho_m) \left(\exp(-A([W_0 - \sum_{i=2}^n a_i W_0] \rho_m + \sum_{i=2}^n a_i W_0 \rho_i)) \right) = 0 \quad (11)$$

or

$$E(\rho_i - \rho_m) (\exp - A \left(\sum_2^n a_i W_0 (\rho_i - \rho_m) \right)) = 0,$$

which we solve for $a_i W_0$, $i = 2, \dots, n$.

This function is known as the 'constant absolute risk aversion' utility function. It has the property, as we have just seen, that the wealth elasticity of demand for risky assets is zero.

That the utility functions (7) allow for the formation of a single mutual fund of risky assets has now been established. Two questions remain:

(a) Are these the only functions which yield linear demand functions?

(b) Are linear demand functions the only ones which allow the formation of mutual funds?

The answer to the first question is yes. There are, I suppose, a number of ways of going about showing this. The most straightforward, the most analogous to how these problems are solved in ordinary demand theory, would be the following: since demand functions are derived from utility functions by maximizing subject to a budget constraint, i.e. by taking derivatives, we could attempt to 'integrate back' to get our utility function. An example of this approach will be given in a later lecture.

We shall take here another simpler approach. We shall consider a special class of securities — the so-called Arrow-Debreu securities — which we shall make more extensive use of later; we shall show that if, for this special class, it is possible to form a mutual fund regardless of the values of the probabilities of different states or the returns to different securities, then the utility function must be of the form (7).

The Arrow-Debreu securities are securities which pay in one state only. If we have a set of n securities and n states of nature, we can find a set of Arrow-Debreu securities which yield exactly the same opportunity set. In fact, the intersection of the budget plane for $W_0 = 1$ with each of the axis yields the set of Arrow-Debreu securities (figure 1). Mathematically, if ρ_i are the returns per dollar invested in the i th Arrow-Debreu security,

$$[\rho_{ij}] \begin{bmatrix} 1/\hat{\rho}_1 \\ \vdots \\ 1/\hat{\rho}_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (12)$$

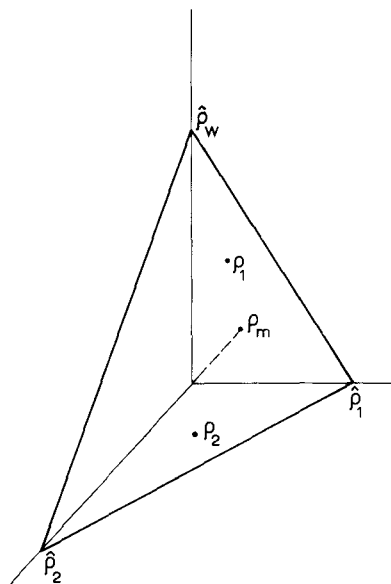


Fig. 1.

In this case, we can write the conditions for expected utility maximisation as

$$U'(a_i \hat{p}_i, W_0) \pi_i \hat{p}_i = U'(Z_i \hat{p}_i) \pi_i \hat{p}_i = \lambda, \quad (13)$$

where λ is the marginal expected utility of an increment in initial wealth.

Taking the total differential of (13), we obtain

$$U'' \hat{p}_i dZ_i \pi_i \hat{p}_i = d\lambda. \quad (14)$$

Dividing (14) by (13), we obtain

$$\frac{U'' \hat{p}_i}{U'} dZ_i = \frac{d\lambda}{\lambda}. \quad (15)$$

We define

$$D_i = U'(Z_i \hat{p}_i) / U''(Z_i \hat{p}_i) \hat{p}_i. \quad (16)$$

Then (15) can be rewritten

$$\frac{dZ_i}{dW_o} = D_i \frac{d \ln \lambda}{dW_o}. \quad (17)$$

Summing up over all i , we have

$$\sum_i \frac{dZ_i}{dW_o} = 1 = \frac{d \ln \lambda}{dW_o} \sum_i D_i. \quad (18)$$

Solving for $d \ln \lambda / dW_o$:

$$\frac{d \ln \lambda}{dW_o} = \frac{D_1}{\sum D_i}.$$

and substituting back into (17), we obtain

$$\frac{dZ_i}{dW_o} = \frac{D_i}{\sum D_i}.$$

If the demand functions are linear (i.e. of the form (6)), then

$$\gamma_i = D_i / \sum D_j \quad (20)$$

must be independent of W_o :

$$\frac{1}{\sum D_j} \frac{dD_i}{dW_o} - \frac{D_i}{(\sum D_j)^2} = \sum_j \frac{dD_j}{dW_o} = 0$$

or

$$\gamma_i = \frac{dD_i}{dW_o} / \sum \frac{dD_j}{dW_o}.$$

From (20) we have

$$\frac{d \ln D_i}{dW_o} = \frac{d \ln D_j}{dW_o} \quad \text{all } i, j.$$

But if we let W_i = income in state i :

$$W_i = \hat{\rho}_i \alpha_i W_o$$

$$\frac{dD_i}{dW_o} = \frac{dD_i}{dW_i} \hat{\rho}_i \gamma_i = \frac{dD_i}{dW_i} \frac{\hat{\rho}_i D_i}{\sum_j D_j}.$$

Hence

$$\frac{dD_i}{dW_i} \hat{\rho}_i = \frac{1}{\rho_i} \frac{dU'/U''}{dW} \hat{\rho}_i = \frac{dU'/U''}{dW}$$

must be the same for all i . But by choosing π, ρ appropriately, we can make W_i take on any values desired.

Hence

$$\frac{dU'/U''}{dW}$$

must be a constant. Integrating, we obtain

$$U'/U'' = A' + B'W$$

or

$$\frac{U''}{U'} = \frac{1}{A' + B'W} . \quad (21)$$

Integrating we obtain (7). In the special case of $B = 0$ we have

$$\ln U' = (1/A')W + C'$$

$$U' = C'e^{(1/A')W}$$

i.e. we obtain (7a).

Let us review what we have established so far:

(1) If there is a safe asset, then if the utility function is of the form (7), then the demand functions are of the form (6) (linear), and the ratios in which the different risky assets are purchased remains unchanged as wealth changes. The ratio depends only on the parameter c .

(2) If, regardless of the probabilities of different states occurring or of the pattern of returns of the different securities, the demand functions are linear, then the utility function must be of the form (7).

We still have not provided a complete answer to our original question: when can we separate out the decision of what risky assets to purchase from the question of what proportion of our wealth to invest in safe, what proportion in risky assets? Linearity was only presented as an obvious sufficient condition for this. Somewhat surprisingly, it turns out that linearity is not

only a sufficient condition, but it is also necessary.⁵ The class of utility functions (7) are the only ones which allow aggregation of the risky assets, unless we wish to impose some conditions on the structure of returns of the assets.

We could have begun our analysis of the conditions for aggregation from the ‘other side’: instead of asking for what utility functions is aggregation possible, regardless of the pattern of returns of the assets, we could have asked: for what kinds of patterns of returns of the assets is aggregation possible, regardless of the utility function?

If the distribution of returns of the different assets were such that we could summarize all possible portfolios in terms of two parameters – mean and variance – then we could aggregate all the risky assets together. For the optimal portfolio clearly must minimize the variance for any given mean. If σ is the variance-covariance matrix per dollar of return, and μ_i is the mean return of the i th asset, then the variance may be written (using the obvious vector notation)

$$W_o^2 a \cdot \sigma \cdot a'$$

and the mean is just

$$W_o(\rho_m + \sum_{i=2}^n a_i \cdot (\mu_i - \rho_m)). \quad (23)$$

Thus we form the Lagrangian

$$W_o^2 \sum_i \sum_j a_i \sigma_{ij} a_j - q [W_o(\sum_i a_i (\mu_i - \rho_m) + \rho_M) - M].$$

Optimality requires

$$2W_o^2 \sum_i a_i \sigma_{ij} - W_o q (\mu_j - \rho_m) = 0,$$

or

$$a = \frac{q(\mu - \rho_m)\sigma^{-1}}{2W_o} \quad (24)$$

⁵For a proof of this, the reader is referred to Cass-Stiglitz [4].

where $(\mu - \rho_m)$ is the vector whose i th element is $\mu_i - \rho_m$. Hence a_i/a_j , the ratio in which two risky assets are purchased, depends only on $(\mu - \rho_m)\tau^{-1}$ and not at all on q , W_0 or M .

Hence a_i/a_j , the ratio in which two risky assets are purchased, depends only on $(\mu - \rho_m)\tau^{-1}$ and not at all on q , W_0 or M .

The question then arises: when can we summarize all possible portfolios in terms of the mean and variance of the portfolio? Clearly, if all portfolios differ from one another only in mean and variance, it must be true that if $F(x)$ is the distribution function for any security, the distribution function for any other security — or any linear combination of securities — is just $F(A + Bx)$. But the only class of securities for which this is true are the stable Pareto-Levy distributions, the only one of which that has a finite variance is the normal distribution. (Another member with infinite variance is the Cauchy distribution.)

We have thus found the conditions for aggregation for the two polar cases: when can we form aggregates regardless of the utility function, imposing only restrictions on the returns, and when can we form aggregates regardless of the pattern of returns, imposing only restrictions on the utility function.

The question naturally arises: can we impose weak restrictions on, say, the utility function, and find that then a wide class of patterns of returns will ‘do’, or can we impose weak restrictions on, say, the pattern of returns, and find that a wide class of utility functions will work? There are, of course, all kinds of restrictions that we might consider, but the following two sets of results suggest that it is not likely that interesting results are to be had in this direction.

First, let us consider a case where we cannot summarize the set of possible distributions in terms of two parameters. Is it not possible, however, that individuals look only at mean and variance, so that we can form an aggregate? To see whether this is a restrictive assumption or not, we must ask: what is the class of utility functions for which individuals only look at means and variances? The answer — not surprisingly — turns out that the utility function must be quadratic: a special case of our utility functions (7). In other words, if the utility function is not quadratic, it is always possible to find two distributions with the same mean and same variance, but between which the individual is not indifferent.

A somewhat more interesting case arises when the variances of the assets in the portfolio are small (and accordingly means do not differ by much) and the individual is not allowed to sell short (or allowed to sell short only in “small” amounts).⁶ Then we can approximate

See footnote next page.

$$EU \approx U(\hat{W}) + U'(\hat{W})E(W - \hat{W}) + \frac{U''(\hat{W})}{2} E(W - \hat{W})^2,$$

where \hat{W} is between the maximum possible value of EW and the value of EW of the minimum variance portfolio. A sufficient condition for such a Taylor series approximation to be valid is that U''' exists. Thus, the solution to his expected utility maximization problem can be given by (24), where

$$q \approx \frac{1}{V}$$

where $V = -U''/U'$ is the measure of absolute risk aversion.

Note that if U''' exists, V will be a continuous (indeed, a differentiable) function of W , so that the portfolio allocation will not depend ‘essentially’ on the choice of \hat{W} in the relevant region.

This ‘small variance’ justification of mean-variance analysis is probably of greater interest than either of the other two—normal distribution or quadratic utility functions—but still, it is difficult to see its validity for the analysis of stockmarket portfolios, where means of securities are hardly close to one another and variances are hardly small.

Another approach is to restrict the set of securities somewhat; for instance, if there are as many securities as states of nature, the class of utility functions under which we can form a single aggregate of risky assets is again the utility functions (7) and only those utility functions.

The above discussion has centered on the simplest of economic questions: if there is a safe asset and if the opportunity set of all individuals is the same and if the prices of the different assets are unchanged, when can we form an ‘aggregate’ of the risky assets such that when wealth changes, the ratios in which the different risky assets are held remains unchanged.

When we can form a single aggregate of risky assets, it is clear that the proposition referred to earlier for the single risky asset, that the elasticity of the demand for money is greater than, equal to or less than unity as relative

⁶The reason for this restriction should be clear: even if the variance of the return per dollar invested is small, by selling short (buying securities on margin) one can obtain a high leverage and a very risky portfolio. Actually, we require not only that the variance be small, but that all higher moments around the mean go to zero faster than the variance does. It is possible to construct sequences of distributions for which this property is not satisfied.

risk aversion is increasing, constant or decreasing, is valid even if there are many risky assets.

But is it not possible for that proposition to be valid, even if an aggregate of risky assets cannot be formed? Clearly, the standard proof will not work. But cannot a more general proof be found? Or is the validity of the Arrow proposition limited to situations either where the utility function is of the form (7), or where the distribution of returns of the different assets are described by a multivariate normal distribution? It is to this question that we turn in the next lecture.

References

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