THE STRUCTURE OF INDIRECT TAXATION
AND ECONOMIC EFFICIENCY *

A.B. ATKINSON and J.E. STIGLITZ

University of Essex, Colchester, England and Yale University, New Haven, U.S.A.

First version received October 1971, revised version received November 1971

The recent literature on indirect taxation has been characterised by two disjoint strands. On the one hand, there are the advocates of the replacement of differentiated indirect taxes by a uniform tax on all commodities (such as a value added tax). Their case is based in part on administrative simplicity, but rests largely on the belief that a uniform tax is more conducive to economic efficiency. On the other hand, there is the literature on “optimal commodity taxation” arguing that different commodities ought to be taxed at different rates, since this reduces the dead weight loss. This line of argument, which was first put forward by Ramsey (1927) and later extended by Samuelson (1951), has been the subject of a number of recent papers. ¹ Although both advocates and critics of differentiated indirect taxes have been primarily concerned with economic efficiency, the debate has never really been joined: each side has discussed the issue as though the other did not exist. The results of Ramsey have been ignored or dismissed as being of little practical significance (cf. Prest, 1967 and Musgrave, 1959). On the other hand, the recent studies in optimal taxation have made little at-

* The authors are grateful to J.A.Mirrlees for his very helpful comments on an earlier version of this paper. It has also benefitted a great deal from comments made at seminars at the universities of Essex, Kent, Southampton, York, University College, London and Nuffield College, Oxford. Stiglitz’s research was supported under grants from the Ford Foundation and the National Science Foundation.

¹ The revival of interest in this area owes much to the paper by Diamond and Mirrlees (1971); see also Stiglitz and Dasgupta (1971), Dixit (1970) and Lerner (1970).
tempt to relate their findings to the conventional views — to show to what extent they are simply alternative forms of conventional maxims for the design of the tax system and to what extent they are in fact contradictory.

The purpose of this paper is to present a new formulation of the optimal tax problem which gives more insight into the structure of the solution and provides more easily interpreted results. Using this formulation, we try to clarify the relationship between the results on optimal taxation and the conventional wisdom, making clear where and under what conditions they are in agreement. Moreover, using this new approach it is possible to calculate the optimal tax structure corresponding to empirically estimated demand function, and some numerical results are presented.

1. The conventional wisdom

A number of criteria for evaluating alternative tax structures have been proposed: (a) efficiency; (b) equity; (c) administrative simplicity; (d) flexibility (usefulness for stabilization policies). This paper focuses primarily on the first of these considerations, since it is the efficiency aspects that have received most attention. The analysis does, however, have important implications for the conflict between efficiency and equity and these are discussed briefly in the final section. (In a sequel to this paper, the distributional arguments and the relationship between direct and indirect taxation are examined in more detail.) The last two considerations — administrative simplicity and flexibility — are not discussed, but it should be emphasised that this does not imply that in our judgment these other effects are not of importance. Against any gains from differentiation on the first two accounts must be set some judgment of the political and economic benefits to be had from the simpler administrative structure associated with uniform taxes.

\footnote{In particular, once the principle of differentiation is accepted, the tax system may be subjected to the pressures of special interest groups; each group would argue that special considerations dictate that the tax on its commodity (its factor use) be lowered. The tax structure eventually emerging might well be based as much on relative strengths of these pressure groups as on relative dead weight losses.}
The conventional analysis of the efficiency arguments presented in most textbooks is based on a partial equilibrium model of a single market (see fig. 1). As a result of the tax at rate $t$, the supply curve shifts up from $SS$ to $S'S'$. The tax revenue is $AP'CB$. The excess loss of consumer surplus is $PP'F$ and of producer surplus is $PCF$. The total deadweight loss for a given revenue ($R$) may be approximated for small taxes by (see Bishop, 1968, p. 211)

$$\frac{R^2}{2q \times \left( \frac{1}{\varepsilon_d} + \frac{1}{\varepsilon_s} \right)},$$

where $\varepsilon_d$ and $\varepsilon_s$ denote the elasticities of demand and supply, and $q \times x$ denotes expenditure on the commodity. From this are derived the following maxims: to minimise distortion we should tax those goods which
(i) have a low price elasticity of demand, (ii) have a low price elasticity of supply, (iii) form an important part of people's budgets. 3

This geometric analysis gives somewhat similar results to those reached by Ramsey in one of the special cases he considered. The relationship between them has, however, been obscured by the confusion in much of the literature of two different questions:
(a) If taxes can only be imposed on one commodity (or a subset of commodities), which should be chosen? This is in effect the question considered by Hicks.
(b) If there is more than one taxable commodity, what should be the relative tax rates on different commodities? This is the question considered by Ramsey.

In the former case, we wish to tax the commodity for which the dead weight loss is lowest for a given revenue, and here maxims (i)–(iii) apply. In the Ramsey case, we wish to minimize the total dead weight loss over all taxable commodities, so that for each commodity the marginal dead weight loss associated with raising a marginal dollar of tax revenue must be the same. In the case of a perfectly elastic supply this requires (for small taxes)

$$\frac{t_i}{q_i} = \text{constant for all commodities } i = 1, \ldots, n,$$

or that the (ad valorem) tax rates be inversely proportional to the elasticity of demand in each industry. (Note that in this case the importance of the good in consumers' budgets is not relevant.) It is on the Ramsey question that we focus in this paper.

This partial equilibrium analysis is clearly unsatisfactory in view of

---

3 The following passage from Hicks (1968) perhaps comes closest to giving a fair representation of the conventional wisdom: "for a given revenue the loss of surplus will be larger, the larger is the elasticity of demand or supply; if either is completely inelastic the loss of surplus falls to zero, and there is no tendency to substitute any other good for the taxed commodity, the outlay tax becomes equivalent to a lump sum taken from the taxpayer ... in all ordinary circumstances, however, there will be some loss of surplus. This loss will also vary (this time inversely) with the amount spent on the article, i.e. its importance in consumption. For to raise a given revenue from an "unimportant" commodity, very high rates of tax may be required; with any normal elasticity of demand or supply the loss of surplus will be severe" (p. 149).
the restrictive assumptions on which it is based. In particular, it requires (a) the absence of income effects, and (b) the independence of demand functions. There has therefore been considerable scepticism about its applicability. Prest (1967), for example, dismisses the Ramsey results with the comment that “such restrictive assumptions have to be made in order to derive a solution, that they appear to have little practical significance” (although he offers nothing in its place). In contrast to the restrictive partial equilibrium analysis, the results of Ramsey, Samuelson, Boiteux (1951) (and more recently Diamond and Mirrlees) are in many respects more general. In particular, they have led to the important finding that the optimal tax structure requires that (for infinitesimal taxes) the compensated demand for each good be reduced by the same proportion. 4 However, while this provides considerable insight into the form of the solution, it does not yield any simple qualitative propositions about the optimal tax structure. It does not, for example, suggest which goods should be taxed more heavily — or indeed whether a differentiated tax structure is in fact optimal. Moreover, it does not readily permit the calculation of optimal rates of taxation on the basis of empirically estimated demand functions.

The aim of this paper is to derive results midway in generality between those obtained from partial equilibrium analysis and the Ramsey—Samuelson results. The results we derive allow straightforward statements to be made in the case where all consumers are identical about the effect of efficiency considerations on the structure of indirect taxation and facilitate the estimation of the optimal tax structure. The basic model is described in the next section; the central results are set out in section 3; and the implications are discussed in the remainder of the paper.

2. The model

2.1. Assumptions about production

In this paper we focus on the role of demand factors in determining the optimal structure of indirect taxation, and therefore make the sim-

4 From what it would have been had producer prices been charged. This result is still dependent on certain restrictive assumptions: e.g. constant returns to scale in the private sector.
pllest possible assumptions about production. Most importantly, we assume constant returns to scale, which precludes any discussion of the role of supply elasticities. For ease of exposition, it is also assumed that producer prices are fixed for all commodities and labour (the only factor supplied by households), although the results in no way depend on this assumption. Writing \( q_i \) for the consumer price of good \( i \), \( p_i \) for the producer price, we have \( q_i = p_i + t_i \). We assume without loss of generality that one good (leisure) is not taxed and that the wage is unity.

2.2. Assumptions about consumption

A consumer is assumed to maximise a function \( U(x, L) \) subject to the budget constraint

\[
\sum_{i=1}^{n} q_i x_i = L ,
\]

(2.1)

where \( L \) is the amount of labour supplied and \( x_i \) is the amount of the consumption good purchased. Writing \( \alpha \) for the marginal utility of income, this gives the first-order conditions

\[
U_i = \alpha q_i \quad i = 1, \ldots, n
\]

(2.2a)

\[
-U_L = \alpha .
\]

(2.2b)

2.3. Social welfare function

The general assumption made is that the government maximises a social welfare function which is individualistic and impersonal: \( W[U^1, U^2, U^3, \ldots, U^m] \) where \( U^k \) is the utility of the \( k \)th man. However, in order to focus on the efficiency aspects, we assume here that all consumers are identical, which means that we can consider the welfare of a ‘representative’ individual:

\[
U(x, L) .
\]

---

5 For a discussion of the role of supply considerations, see Stiglitz and Dasgupta (1971)
6 We shall assume that these producer prices correctly reflect social costs, i.e. there are no externalities or “imperfections of competition”.
2.4. Government budget constraint

It is assumed that the purpose of commodity taxation is to raise a certain revenue $R$ (which will purchase a fixed quantity of any of the goods at producer prices):

$$ R = \sum_{i=1}^{n} t_i x_i = \sum_{i=1}^{n} (q_i - p_i) x_i = L - \sum_{i=1}^{n} p_i x_i. \quad (2.3) $$

3. Derivation of optimal tax formula

The problem faced by the government is to choose $t_i$ ($i = 1, \ldots, n$) to maximise $U(x, L)$ subject to (2.3) and the conditions for individual utility maximisation (2.1) and (2.2). Following the approach of Ramsey we can regard the $t_i$ and $\alpha$ as functions of $x_i$ and $L$ from equations (2.2) and frame the problem in terms of choosing $(x_i, L)$ to maximise the Lagrangian.

$$ U(x, L) + \mu \left[ \sum_{i=1}^{n} q_i x_i - L \right] - \lambda \left[ R + \sum_{i=1}^{n} p_i x_i - L \right]. $$

This formulation differs from that of Diamond and Mirrlees (1971), who worked with the indirect utility function and the tax rates as control variables.

If we define $-L$ as good 0, we may write the Lagrangian in vector notation

$$ U + \frac{\mu}{\alpha} U' x - \lambda (R + px), $$

where $U'$ denotes the vector $U_i$ ($i = 0, \ldots, n$) and the $q_i$ have been eliminated using conditions (2.2). The first order conditions are

$$ \left( 1 + \frac{\mu}{\alpha} \right) U' + \frac{\mu}{\alpha} U'' x = \lambda p, \quad (3.2) $$

The budget constraint has to be introduced separately as it does not appear in equations (2.2).
where \( U'' \) denotes the matrix \( U_{ij} \) (\( i, j = 0, ..., n \)). Let us define

\[
H^k = \sum_{i=0}^{n} \frac{(-U_{ik})x_i}{U_k},
\]

i.e. \( H^k \) is the sum of the elasticities of the marginal utility of \( x_k \) with respect to each of the commodities. Then the first-order condition can be written as

\[
q_k [\alpha + \mu(1 - H^k)] = \lambda p_k \quad k = 0, ..., n . \tag{3.3}
\]

But from the normalisation \( t_0 = 0 \),

\[
\mu = \frac{\lambda - \alpha}{1 - H^0} , \tag{3.4}
\]

so that the optimal tax rates \( t_k^* \) as a percentage of consumer prices are characterised by \(^8\)

\[
\frac{t_k^*}{p_k + t_k^*} = \lambda - \alpha \frac{H^k - H^0}{\lambda - H^0} . \tag{3.5}
\]

While this equation does not in general provide an explicit formula for the optimal tax rate (since the \( H^k \) depend on the tax rates), it does allow us to draw a number of conclusions about the optimal structure of taxation. The implications of equation (3.5) will be the subject of the remainder of the paper. \(^9\)

\(^8\) It can be seen that the assumption of fixed producer prices does not affect this result: if the government revenue constraint were replaced by a production constraint \( F(x) = 0 \), the analysis would go through as before with \( F_l \) replacing \( p_l \). Since \( F_l \) is homogeneous of degree zero, equation (3.2) is unaffected.

\(^9\) Equation (3.5) can also be obtained from the results of Samuelson, Diamond and Mirrlees by inverting their formulae (see the appendix).
4. Implications of basic optimal tax formula — uniform taxation?

One of the main questions of policy importance is whether or not a uniform structure of taxation would be desirable. \(^{10}\) From equation (3.5) we can determine the conditions under which efficiency considerations would require a uniform tax. It is in fact immediately clear that a sufficient condition for \(t^*_k = t^*_j\) all \(k, j \geq 1\) is that the indifference map be homothetic; i.e. all the indifference curves are identical in shape; they are simply radial "blow-ups" of any given indifference curve. To see this, observe that if the indifference map is homothetic, there exists a representation of the indifference map which is homogeneous of degree one in all arguments together. Thus \(U_j\) is homogeneous of degree zero, i.e. \(\Sigma_i U_{ik} x_i = 0 = H^k\) all \(k\).

Homotheticity of the entire indifference map is not, however, necessary for the optimal tax to be uniform. Let us consider first the case where the marginal utility of leisure is independent of the consumption of every commodity. Then \(t^*_k = t^*_j\) all \(k, j \geq 1\) implies either \(-H^0 = \infty\) or that \(H^k = H^j\). The first condition means that

\[
- \frac{U_{00} L}{U_0} = \infty,
\]

or that the elasticity of marginal utility of labour is infinite, which implies a completely inelastic supply of labour. The implications of the second condition can be seen as follows: differentiate the first order conditions and budget constraint (2.1) and (2.2), to obtain

\[
\begin{bmatrix}
U_{11} & U_{12} & \ldots & -q_1 \\
U_{21} & U_{22} & \ldots & -q_2 \\
\vdots & \vdots & \ddots & \vdots \\
-q_1 & -q_2 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
dx_1 \\
dx_2 \\
\vdots \\
d\alpha
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\vdots \\
-dE
\end{bmatrix},
\]

\(^{10}\) From this point we set \(p_j = 1\) all \(j\) (without loss of generality), so that uniform taxation implies \(t^*_k = t^*_j\) all \(k, j \geq 1\).
where $E = \sum_{i=1}^{n} q_i x_i$ = total expenditure; defining $H_{ki} = -\frac{U_{ki} x_i}{U_k}$ we obtain

(by appropriate normalisation)

$$
\begin{bmatrix}
H_{11} & H_{12} & \cdots & 1 \\
H_{21} & H_{22} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
q_1 x_1 & q_2 x_2 & \cdots & 0
\end{bmatrix} \begin{bmatrix}
d \ln x_1 \\
d \ln x_2 \\
\vdots \\
d \ln \alpha
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
dE
\end{bmatrix}
$$

(4.1)

Denote by $D$ the determinant of the matrix of coefficients of the left hand side of (4.1). Then,

$$
D \left[ \frac{d \ln x_1}{dE} - \frac{d \ln x_2}{dE} \right] = 0 \begin{bmatrix}
H_{12} & H_{13} & \cdots & 1 \\
H_{22} & H_{23} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
q_1 x_1 & 0 & \cdots & 0
\end{bmatrix} - H_{21} 0 \begin{bmatrix}
0 \\
0 \\
\vdots \\
q_1 x_1
\end{bmatrix} \begin{bmatrix}
1 \\
1 \\
\vdots \\
0
\end{bmatrix}
$$

$$
= (-1)^n \left\{ \begin{bmatrix}
H_{12} & H_{13} & \cdots & 1 \\
H_{22} & H_{23} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
q_1 x_1 & 1 & \cdots & 0
\end{bmatrix} + \begin{bmatrix}
H_{11} & H_{13} & \cdots & 1 \\
H_{22} & H_{23} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
q_1 x_1 & 1 & \cdots & 0
\end{bmatrix} \right\}
$$

$$
= (-1)^n \left\{ \begin{bmatrix}
H_{12} + H_{11} & H_{13} & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \right\}
$$

$$
= (-1)^n \left\{ \begin{bmatrix}
H_{12} + H_{11} & H_{13} & \cdots & \Sigma_i H_{1i} & 1 \\
0 & 0 & \cdots & \Sigma_i H_{2i} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \Sigma_i H_{2i} & 1
\end{bmatrix} \right\} = 0,
$$

(since $H^k = \Sigma_i H_{ki}$), i.e. we require equal, and hence unitary, expenditure elasticities of all commodities. If $U_{i0} \neq 0$ for some $i$, then $t_i$ may equal $t_j$ even without unitary expenditure elasticities. The effects of the in-
teractions among the commodities directly may just be offset by the interactions between the commodities and leisure. (This is illustrated in a three commodity example given in section 8.)

In this section we have examined the conditions under which the optimal tax structure will be uniform. It is clear that there is no presumption that these conditions are likely to obtain and in the following sections we examine how the optimal tax rates depend on the characteristics of different commodities.

5. Implications of optimal tax formula — two polar cases

We now show how the results of Ramsey and others can be obtained as polar cases of formula (3.5). Assume first that there is constant marginal disutility of labour. Then \( H^0 = 0 \) and the optimal tax is given by

\[
\frac{t_k^*}{1 + t_k^*} = \frac{\lambda - \alpha}{\lambda} H_k^k .
\]  

(5.1)

If in addition we assume that \( U_{ij} = 0 \) \( (i \neq j) \), we can see that \( H_k^k \) is inversely proportional to the elasticity of demand. \(^{11}\) We have, therefore, obtained the Ramsey result that the optimal taxes should be inversely proportional to the price elasticity of demand.

On the other hand, if we suppose that \( (-H^0) \) tends to infinity, which corresponds to the case of a completely inelastic supply of labour, then

\[
\frac{t_k^*/(1 + t_k^*)}{t_j^*/(1 + t_j^*)} = \frac{H_k^j / (-H^0)}{H_j^j / (-H^0)} + 1 \rightarrow 1 ,
\]

i.e. uniform rate of tax on all goods. Since a uniform rate of tax on all goods is equivalent to a tax on labour alone, this corresponds to the conventional prescription that where there is a factor which is completely inelastically supplied, this should bear all the tax.

\(^{11}\) From differentiating the first-order conditions, we obtain \( U_d \frac{\partial^2 \pi}{\partial q \partial \pi} = \alpha \) (since \( \alpha \) is constant), so \( H_k^k = (1/\alpha) \).
Our formula (3.5) can be seen therefore as the kind of “weighted average” that Lerner (1970) has suggested might exist. There are two “extreme” optimal tax systems: the uniform tax and taxes proportional to \( H^k \). Where between these two extremes the optimal tax system lies will depend on \((-H^0)\).

6. Implications of optimal tax formulae – direct additivity and strong separability

The formulation of the optimal tax in equation (3.5) suggests a special case which allows easily interpreted results to be obtained: direct additivity of the utility function. This implies that there exists some monotonic transformation of the utility function such that \( U_{ij} = 0 \) for \( i \neq j \): i.e. \( H^k \) may be written

\[
H^k = \left( \frac{-U_{kk}x_k}{U_k} \right),
\]

(which is invariant with respect to monotonic transformations of the utility function). \(^{12}\)

By differentiating equations (2.2), we can see that this is inversely proportional to the income elasticity of demand for good \( k \) (see Houthakker, 1960):

\[
\left( \frac{-U_{kk}x_k}{U_k} \right) \frac{1}{x_k} \frac{\partial x_k}{\partial m} = -\frac{1}{\alpha} \frac{\partial \alpha}{\partial m}.
\]

Moreover, if we assume that \( U_{ij} < 0 \) for all \( i = 0, ..., n \), it follows that \( \lambda > \alpha \) if a positive revenue is to be raised. \(^{13}\) We have therefore the use-

\(^{12}\) In the general case, replace \( U \) by \( V(U) \), so that \( V_i = V' U_i \) and \( V_{ij} = V' U_{ij} + V' U_i U_j \), then

\[
\Sigma_i \frac{(-V_{ik})x_i}{V_k} = \Sigma_i \frac{(-U_{ik})x_i}{U_k} - \frac{V'}{V} \Sigma_i U_i x_i,
\]

but the second term disappears (using the budget constraint), establishing that \( H^k \) is invariant.

\(^{13}\) Since \( H^k > 0 \) and \( H^0 < 0 \), so that the tax rates are all positive of \( \lambda > \alpha \), negative if \( \lambda < \alpha \). For discussion of these restrictions, see Green (1961).
ful result that when the utility function is directly additive, the optimal tax rate depends inversely on the income elasticity of demand. This clearly has important implications for the conflict between equity and efficiency which are discussed further below.

Examples of the solution for directly additive functions are:

**Example 1.** The direct addilog function

\[ U(x_i, L) = (\bar{L} - L)^{\beta_0} + \frac{1}{1 - \beta_i} \sum_{i=1}^{n} x_i^{1-\beta_i}, \]

\[ \beta_i > 0, \quad i = 1, ..., n. \]

In the case \( \beta_i = \beta, i = 1, ..., n \) this has unitary expenditure elasticities, and we can deduce that the optimal tax system is proportional. In the more general case where the \( \beta_i \) are different, the tax rate will increase with \( \beta_i \).

**Example 2.** Stone–Geary utility function

\[ V(L) + \sum_{i=1}^{n} \beta_i \log(x_i - c_i). \quad \sum \beta_i = 1 \]

This function was considered by Diamond and Mirrlees (1971), but they were unable to say more than that the optimal tax would not be uniform. Using the approach adopted here, we can see that

\[ H^k = \frac{x_k}{(x_k - c_k)} = \frac{\text{total expenditure on good } k}{\text{"luxury" expenditure on good } k}. \]

This suggests that the optimal tax will be high on those goods which are basically necessities and low on luxury goods.

Direct additivity is a restrictive assumption. It is, however, considerably less restrictive than the assumptions required for partial equilibrium analysis to be valid (for \( H^0 \neq 0 \), direct additivity does not imply zero cross-price effects). Moreover, there are some grounds for believing that
direct additivity is reasonably consistent with empirical evidence on demand — at least for broad commodity groups.

In view of the fact that additivity is undoubtedly more appealing at the level of broad commodity groups than of individual commodities, it is perhaps best to interpret our results in this way. Suppose that the utility function is strongly separable:

\[ U(x) = F\left[U^1(x^1) + U^2(x^2) + \ldots\right] , \]

where \( x^i \) denotes a subset of commodities \( x_{i1}, x_{i2}, \ldots \). Then the optimal tax rates taking the commodity groups as a whole (regarding them as a composite commodity) are given by (3.5). It seems likely for administrative or other reasons that commodities would in fact have to be grouped for tax purposes, although the groups would not necessarily coincide with the subgroups \( x^i \). Where this does not apply, we can regard the determination of the optimal tax structure as a two-stage process: what should be the relative taxes within a group, and what should be the average tax rates between groups? We may note that where there are only two goods in a subgroup \( x^i \), the relative tax rates depend simply on the relative expenditure elasticities: from the first order conditions

\[
\begin{bmatrix}
H_{11} & H_{12} & 1
\end{bmatrix}
\begin{bmatrix}
\frac{d \ln x_{11}}{dE_1} \\
\frac{d \ln x_{21}}{dE_1}
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix}
\]

where \( E_1 \) is the total expenditure on the group of commodities. Thus

\[
D \left[ \frac{d \ln x_{11}}{dE_1} - \frac{d \ln x_{21}}{dE_2} \right] = H_{12} - H_{22} + H_{11} - H_{21}
\]

\[= H^1 - H^2 ,\]

(since \( H^1 = H_{11} + H_{12} \), etc.).
7. Applications of additivity and separability: savings and risk-taking

In this section we discuss briefly two cases where the direct additivity results seem particularly applicable.

7.1. Taxation of savings

Suppose that lifetime utility for the representative individual is given by the sum of instantaneous utility from consumption \( x_i \) in period \( i \) (regarded as a composite good) and the disutility of effort from working in period 0:

\[
V(L_0) + \sum_{i=1}^{n} U(x_i).
\]

We can deduce that when the elasticity of marginal utility is constant, the optimal tax is a uniform tax on consumption in all periods. If the elasticity of marginal utility rises as consumption rises, and if the optimal plan involves a rising level of consumption (as it will if there is a positive interest rate and no time discounting), the optimal tax rate will be higher on consumption at later dates — a uniform consumption tax would need to be supplemented by a tax on interest. (This case is one in which consumption today is a luxury good and savings a necessity.) If, on the other hand, the elasticity of marginal utility falls with rising consumption, a uniform consumption tax should be supplemented by a subsidy on interest. If further we add a term in bequests, then the optimal tax will be higher on bequests than on consumption only if bequests are a necessity (i.e. the proportion of wealth allocated to bequests falls as wealth increases).

7.2. Risk-taking

Suppose that a person earns \( L \) in period 0, and saves this for consumption in the next period. He allocates an amount \( z_1 \) to a safe asset (yielding \( r \) with certainty) and \( z_2 \) to a risky asset yielding an uncertain pattern of returns \( \tilde{R} \). His income in state \( \theta \) is

\[
Y(\theta) = (1 + \tilde{R})z_2 + (1 + r)z_1.
\]
His expected utility is

\[ V(L) + EU(Y) \]

Although the utility function is additive in \( Y \), it is not additive in \( z_1 \) and \( z_2 \). It is, however, separable between \( z_1 \) and \( z_2 \) on the one hand and \( L \) on the other, and we can apply the result given in the previous section. Thus we obtain the interesting result that optimal taxation requires that the risky industry be taxed at a higher or lower rate than the safe as the expenditure elasticity of demand for the risky industry is less than or greater than unity. This result may be reinterpreted in terms of properties of the utility function: the risky industry should be taxed at a higher or lower rate than the safe as there is increasing or decreasing relative risk aversion, if and only if there is constant relative risk aversion ought we to tax both industries at the same rate. \(^{14}\)

8. Complementarity with leisure and the optimal tax structure

While direct additivity may seem reasonable for broad commodity groups, it may be less appealing to assume that the marginal utility of leisure is independent of the consumption of different goods. Moreover, it has commonly been suggested that the degree of complementarity with leisure is one determinant of the optimal tax structure. Prest, for example, says that:

"indirect taxes on commodities which do not soak up a large fraction of the expenditure from marginal earnings (i.e. commodities not highly competitive with leisure ...) earn higher marks than those which do. Ideally, one would like to tax those goods which are in joint demand with leisure, i.e. where the elasticity of demand for leisure is negative with respect to their price" (1967, p. 376).

\(^{14}\) For a more extensive discussion of the taxation of safe and risky industries, see Stiglitz (1970). Unfortunately, these appealing results carry over to cases with more than one risky asset only in those situations where the portfolio separation theorem obtains (see Cass and Stiglitz, 1971).
In order to explore the dependence of the optimal tax rate on the degree of complementarity with leisure we consider a model with 3 goods – 2 consumption goods and the (untaxed) factor labour. This model has earlier been discussed by Corlett and Hague (1953) and Harberger (1964); however, they carried out their analysis in terms of the properties of the demand functions rather than of the utility function. Following the same procedure as in section 4, we can write (where \( I \) denotes unearned income and good zero is taken as leisure (\(- L\))

\[
D \left[ \frac{d \log x_1}{dI} - \frac{d \log x_2}{dI} \right] = \begin{vmatrix}
H_{00} & H_{02} & 1 \\
H_{10} & H_{12} & 1 \\
H_{20} & H_{22} & 1
\end{vmatrix} + \begin{vmatrix}
H_{00} & H_{01} & 1 \\
H_{10} & H_{11} & 1 \\
H_{20} & H_{21} & 1
\end{vmatrix}
\]

\[
= \begin{vmatrix}
H_{00} & H^0 & 1 \\
H_{10} & H^1 & 1 \\
H_{20} & H^2 & 1
\end{vmatrix},
\]

where \( D > 0 \) from second-order conditions. From this it follows that

\[
(H^1 - H^0) = (H^2 - H^0) \left[ \frac{H_{10} - H_{00}}{H_{20} - H_{00}} \right] - \frac{D}{[H_{20} - H_{00}]} \left[ \frac{d \log x_1}{dI} - \frac{d \log x_2}{dI} \right]. \tag{8.1}
\]

From (3.5) we can see that the relative optimal tax rates depend on \( H^k - H^0 \), so that (8.1) allows us to deduce the conditions under which the tax rates will be higher on one good than on another.

Equation (8.1) tells us that when we relax the assumption of direct additivity the optimal tax rate depends not only on the difference in the income elasticities but also on whether \( H_{10} \approx H_{20} \). This can be interpreted as follows: \( H_{ik} \) is the elasticity of the marginal utility of good \( i \) with respect to an increase in leisure. If this is high, the good can be said to be complementary with leisure, and according to (8.1) the tax rate
on this good should cet. par. be high. If the marginal utility of tennis racquets increases proportionately more with a rise in leisure than the marginal utility of food, then the former should be taxed more heavily.\textsuperscript{15}

The notion of complementarity introduced in the previous paragraph follows that of Edgeworth and Pareto and differs from the more usual Hicksian definition, which is framed in terms of the compensated elasticity of demand.\textsuperscript{16} In the present 3 good model, we can see that (defining \( \eta_{ij} = q_j S_{ij} / x_i \) where \( S_{ij} \) is the Slutsky term)

\[
\frac{1}{D} [\eta_{20} - \eta_{10}] = \begin{vmatrix}
H_{10} & H_{11} & 1 & H_{10} & H_{12} & 1 \\
H_{20} & H_{21} & 1 & H_{20} & H_{22} & 1 \\
(-x_0 U_0) & -x_1 U_1 & 0 & (-x_0 U_0) & -x_2 U_2 & 0 \\
H_{01} & H^1 & 1 & \sim \nH_{02} & H^2 & 1 \\
(-x_0 U_0) & 0 & 0 \\
\end{vmatrix}
\]

This gives the result reached by Corlett and Hague, Harberger and others, that the good with the highest cross elasticity with labour will be taxed less heavily; i.e. we should tax more heavily goods which are complementary with leisure. It is important, however, to emphasise that it has nothing to do with leisure per se. The general principle is that if we have one untaxed good, we should tax more heavily that good most complementary with it, since it is a way of indirectly “taxing” the untaxed good. It just happens that we are here assuming that leisure is untaxed.

\textsuperscript{15} Thus, for example, it is a sufficient condition for the optimal tax to be uniform for the income elasticities to be identical and for \( H_{10} = H_{20} \). However, this is not necessary and we may have \( \tau^1_1 = \tau^2_1 \) when the income elasticity of 1 is higher than that of 2, but \( H_{10} \) is greater than \( H_{20} \). As we noted earlier, homotheticity is not required for taxes to be uniform.

\textsuperscript{16} Although it should be noted that in the form used here \( (H_{01} \geq H_{02}) \), it is invariant with respect to monotonic transformations of \( U \).
9. Numerical illustration of optimal tax calculations

One of the advantages of the approach adopted here is that it readily allows us to calculate the optimal structure of indirect taxes from empirically estimated demand functions. In particular, the assumption of direct additivity is made in much of the empirical work on consumer demand and a number of studies employ the special functional forms examined in section 6. There is, however, the difficulty in using such estimates that we require for our analysis a simultaneous estimation of the labour supply function together with the commodity demand functions, and this is not in general available. For this reason, estimates have to be based on assumed values for the elasticity of the supply of labour and we concentrate here on the case \( H^0 = 0 \) (where the supply of labour is completely elastic) since we have seen that in this case the divergences from uniform tax rates are greatest (in the other polar case, \( H^0 \to -\infty \), a uniform tax is optimal).

The optimal structure of taxation depends on the terms \( H^k \) and in general these will be functions of the tax rates — we have to allow for the fact that changing the tax rates will affect the income elasticities of demand. In the special case of the direct addilog utility function, however, the terms \( H^k \) are constant and we can calculate the optimal tax rates directly. In table 1 we present the optimal tax rates derived from the estimates given of direct addilog demand functions by Houthakker (1960). In each case the tax rates are normalised so that the tax rate for durables is 10%. The differences between the tax rates on different commodities are quite substantial: the range between the rate for the most heavily taxed commodity and that for the lowest is 3:1 in Sweden, 6:1 in Canada and 4:1 for the OEEC. As one would expect from the earlier results, it is the 'necessities' — food and rent — that are taxed heavily, whereas durable goods with a high income elasticity are taxed at a low rate. At least as far as food is concerned, the pattern is very different from that actually in force in most countries.

In table 2 we present results based on the linear expenditure system, using the estimates of Stone (1954) for the United Kingdom for 1920–1938. In this case the \( H^k \) are functions of the tax rates, since the demand functions are given by

\[
q_k x_k = q_k c_k + \beta_k / \alpha.
\]
However, in the case where $H^0 = 0$, $\alpha$ is independent of the tax rate and equation (3.5) reduces to the quadratic

$$
\left( \frac{q_k}{p_k} \right)^2 \frac{\alpha \beta_k c_k}{\beta_k} (\lambda - \alpha) - \frac{q_k \alpha}{p_k} + \lambda = 0.
$$

This determines the optimal ad valorem tax rates $t^e_k (= 1 - (p_k/q_k))$, and sample calculations are given in table 2 for a range of values of

<table>
<thead>
<tr>
<th>Commodity groups</th>
<th>Ad valorem tax rates (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda/\alpha = 1.025$</td>
</tr>
<tr>
<td>Meat, fish, dairy products and fats</td>
<td>11.1</td>
</tr>
<tr>
<td>Fruits and vegetables</td>
<td>8.2</td>
</tr>
<tr>
<td>Drink and tobacco</td>
<td>10.1</td>
</tr>
<tr>
<td>Household running expenses</td>
<td>5.3</td>
</tr>
<tr>
<td>Durable goods</td>
<td>5.6</td>
</tr>
<tr>
<td>Other goods and services</td>
<td>6.2</td>
</tr>
</tbody>
</table>

Notes:
(a) Based on estimates given by Stone (1954), table 1 of $\hat{c_k}$, $\hat{\beta_k}$ and $1/\alpha$ (= total expenditure minus 'committed' expenditure).
(b) Relationship between producer and consumer prices based on that for 1938 as obtained from National Income and Expenditure (1947). Groups (1) and (2) were combined for this purpose.
(c) Group (4) includes rent, fuel and light, non-durable household goods and domestic service. Group (5) includes clothing, household durables, vehicles, transport and communication services.
\(\lambda/\alpha\) (reflecting different levels of revenue). (The sources of the estimates of \(c_k, \beta_k, p_k\) and \(1/\alpha\) are described in the notes to table 2.) Again the range of tax rates is wide and food (although not in this case housing) is taxed much more heavily than durables.

It should be emphasised that these calculations are presented only to illustrate the application of the theoretical approach developed in the earlier sections. The use of alternative specifications of the demand equations, or of alternative estimates of the same forms, may well give rather different results.

10. Concluding comments

The principal conclusion we have reached is that if direct additivity is a reasonable assumption for broad commodity groups, then the optimal structure of taxation from an efficiency viewpoint is one that taxes more heavily goods which have a low income elasticity of demand. This result generalises the conventional wisdom based on partial equilibrium analysis, which can be obtained as a special case where the supply of labour becomes completely elastic. Moreover, in terms of the debate about the introduction of a uniform system of indirect taxes referred to at the beginning of the paper, we have seen that there is no general presumption in favour of uniform taxation on grounds of allocative efficiency.

The analysis suggests two important areas for further research, which we plan to examine in a sequel to the paper. Firstly, although we have shown that uniform taxation cannot be justified by appeal to considerations of allocative efficiency, it may still be true that the welfare loss involved in using uniform rather than optimal taxes may be small. Secondly, the conclusion that goods with a low income elasticity should be taxed heavily brings out very sharply the conflict between equity and efficiency considerations. The recognition of equity objectives would be expected to lead to important modifications of the conclusions. 17

17 One important contribution of Diamond and Mirrlees (1971) is to extend the Ramsey—Samuelson analysis to the many-consumer case; however, like the analysis for the single consumer, it does not readily allow conclusions to be drawn about which goods should be taxed more heavily.
Appendix

Equation (3.5) can be written

\[ t^*_k = \alpha q_k C_1 H_k + C_2 U_k, \]

where \( C_1 = \frac{\lambda - \alpha}{\alpha \lambda} \frac{1}{1 - H^0}, \)
\( C_2 = \frac{-H^0}{\lambda - \alpha} \frac{1}{1 - H^0} \)

so

\[ t^*_k = -C_1 \sum_i U_{ik} x_i + C_2 U_k \]

and

\[ 0 = -C_1 \sum_i U_i x_i. \]

These equations can be written

\[(t^*, 0) = V(-C_1 x, C_2)',\]

where ' denotes the transposition and

\[ V = \begin{bmatrix} U_{ij} & U_i \\ U_j & 0 \end{bmatrix}. \]

Hence \((-C_1 x, C_2) = V^{-1}(t^*, 0).\)

But

\[ V^{-1} = C_3 \begin{bmatrix} S_{ij} & \frac{\partial x_i}{\partial I} \\ \frac{\partial x_j}{\partial I} & 0 \end{bmatrix}, \]

where \( S_{ij} \) denote the Slutsky terms. We thus obtain by inverting (3.5) the familiar result

\[ \sum_i S_{ik} t_i = -C_1/C_3 x_k \]

obtained by Samuelson (1951): the compensated demand for each good should be reduced by the same proportion (for infinitesimal taxes).
References

Harberger, A.C., 1964, Taxation, resource allocation, and welfare, in: The role of direct and indirect taxes in the federal revenue system (Princeton University Press).
Houthakker, H.S., 1960, Additive preferences, Econometrica 28, 244–257.
1947, National income and expenditure of the United Kingdom 1938 to 1946, H.M.S.O. Cmdn. 7099.