Cowles Foundation Paper 366b

C. B. McGuire and Roy Radner (eds.), Decision and Organization

CHAPTER 4

REPRESENTATION OF PREFERENCE ORDERINGS OVER TIME[1]

TJALLING C. KOOPMANS

1. Preference over Time

In Section 1 of Chapter 3 we have argued the desirability of formalizing the idea of consumer's preference in terms of a *preference ordering* on a *prospect space*, before discussing the possibility of representing such an ordering by a *utility function*. The considerations there adduced have still greater force with regard to problems of evaluative comparison of growth paths for an indefinite future. If one interprets this as an infinite future, neither the concept of a utility function depending on infinitely many variables, nor that of a preference ordering on a space of infinitely many dimensions, has an obvious intuitive meaning. To start from the more basic one – the preference ordering – is therefore even more desirable in that case, in that it may help avoid implicit assumptions one is not aware of.

In the present chapter, therefore, the propositions of Chapter 3 are applied to the representation of preference orderings over time. Because of the close connections between the two chapters, the notations are almost identical, and a single list of references to the literature appears at the end of Chapter 4.

Before getting into details, a word is in order on the question whose preference is being studied. This question concerns the interpretation and relevance of the analysis, as distinct from the logical connections between

[1] This chapter reports on research under a grant from the National Science Foundation. It revises and extends Sections 5–9 of KOOPMANS (1966). I am indebted to Kenneth Arrow and Gerard Debreu for extremely valuable comments.

the properties of the ordering and the mathematical form of its representation. In regard to preference over time, the simplest interpretation of the orderings that have been studied most thus far is a normative one. One looks at various possible preference orderings that may be adopted, by whatever decision process, for the planning of an economy with a constant population size. New problems arise if population is expected to grow indefinitely or to keep changing in other ways.

Another possible interpretation is that one wishes to study descriptively the preference ordering of an individual with regard to his life-time consumption program, assuming that such an ordering is implicit in his decisions. For this interpretation the finite life span and the bequest motive need to be considered as well. For applications of such a preference ordering, see YAARI (1964).

Finally – the ultimate goal of a theory of preference over time for an economy with private wealth – one may wish to examine whether, or under what conditions, an aggregate preference ordering over time can be imputed, on an "as if" basis, to a society of individual decision-makers each guided by his own preference ordering over time.

In all these interpretations, normative or descriptive, the most intriguing problems arise from the fact that the future has a beginning but no discernible end. In contrast to this central problem, the question whether to use a discrete or a continuous time concept seems in the present state of knowledge primarily a matter of research tactics rather than of substance. So far the indications are that axiomatic analysis is somewhat simpler if one chooses discrete time. On the other hand, the maximization of a utility function of a given form under given technological constraints is often simpler with continuous time. We shall therefore here choose discrete time on the basis of expedience without further excuse or explanation.

2. Postulates Concerning a Preference Ordering over Time

We shall adopt a set of five postulates about a preference ordering $\succsim$ on a space ${}_1\mathscr{X}$ of *programs*, that is, of infinite sequences, denoted

$$ {}_1x \equiv (x_1, x_2, x_3, \ldots), \tag{2.1} $$

of vectors

$$ x_t \equiv (x_{t1}, x_{t2}, \ldots, x_{tn}) \tag{2.2} $$

associated with successive time periods $t = 1, 2, 3, \ldots$. The *program space*

${}_1\mathscr{X}$ is the space of all such sequences, in which each vector x_t is a point of the same (single period) *choice space* $\mathscr{X}$. Thus the components x_{ti} of x_t refer to a list of commodities, characteristics, or activities (as the case may be), which is the same for all t.

The postulates are modeled after those used in two earlier studies by KOOPMANS (1960) and by KOOPMANS, DIAMOND and WILLIAMSON (1964). The main difference is that the former studies presupposed the existence of a continuous representation. In the present study, the postulates refer to a continuous ordering, and the proximate aim of the study is to derive the existence of a continuous representation. Further differences will be noted in connection with the third and fifth postulates.

The problem of logical independence of the postulates is not investigated. The formulation and sequence of postulates is chosen primarily from the point of view of naturalness of interpretation. One case of recognized dependence between postulates is noted in Footnote 4.

It will be useful occasionally to employ short notations for finite or infinite segments of the program sequence, as follows,

$$(2.3) \qquad {}_1x \equiv (x_1, {}_2x) \equiv (x_1, \ldots, x_{t-1}, {}_tx) \equiv ({}_1x_{t-1}, {}_tx).$$

In an infinite-dimensional space such as ${}_1\mathscr{X}$, the choice of the distance function is crucial for the meaning of the continuity concept implied in it. We shall adopt the function[2]

$$(2.4) \qquad D({}_1x, {}_1y) \equiv \sup_t d(x_t, y_t)$$

where $d(x_t, y_t)$ is the distance between the t-th period installments x_t, y_t of the programs ${}_1x$, ${}_1y$, according to the definition

$$(2.5) \qquad d(x_t, y_t) \equiv \max_i |x_{ti} - y_{ti}|.$$

POSTULATE 1 (Continuity). *The program space* ${}_1\mathscr{X}$ *is the space of all programs* ${}_1x$ *such that, for all* t, x_t *is in a choice space* $\mathscr{X}$, *which is a connected subset of n-dimensional Euclidean space. On the program space there exists a complete preference ordering* $\succsim$, *which is continuous with regard to the distance function* (2.4).

[2] The symbol $\sup_t d_t$ denotes the largest of the numbers d_t, $t = 1, 2, 3, \ldots$, if there is a largest, or the smallest number not exceeded by any d_t if there is no largest. Such a number exists whenever $\mathscr{X}$ is bounded, that is, when the range of $d(x, y)$ for all x, y in $\mathscr{X}$ is bounded. If $\mathscr{X}$ is unbounded we admit the possibility that $D({}_1x, {}_1y) = \infty$.

POSTULATE 2 (Sensitivity). *There exists a program* ${}_1x$ *in* ${}_1\mathscr{X}$ *and a vector* y_1 *in* $\mathscr{X}$ *such that*

$${}_1x = (x_1, x_2, x_3, \ldots) \succ (y_1, x_2, x_3, \ldots).$$

The first purpose of P2 is to exclude the trivial case where all programs in ${}_1\mathscr{X}$ are equivalent. However, P2 does more than that. It also excludes orderings in which the standing of any program ${}_1x$ relative to other programs is independent of any vector x_t pertaining to any specific period t, but does depend on the asymptotic behavior of x_t as t tends to infinity.[3]

Next we introduce two independence postulates, P3′ and P3″, both of which will be maintained throughout Sections 2–6. In Section 7 we comment briefly on the case where P3″ is omitted. In these postulates we employ an arbitrary but fixed reference program,

$$(2.6) \qquad {}_1z = (z_1, {}_2z) = (z_1, z_2, {}_3z),$$

to define five orderings, induced by $\succsim$ on factor spaces of ${}_1\mathscr{X}$, and denoted $\succsim_1^z$, $\succsim_2^z$, ${}_1\succsim_2^z$, $\succsim_3^z$, $\succsim_2^z$, as follows:

$$(2.7) \qquad \begin{cases} x_1 \succsim_1^z y_1 & \text{means } (x_1, {}_2z) \succsim (y_1, {}_2z) \\ {}_2x \; {}_2\succsim^z \; {}_2y & \text{means } (z_1, {}_2x) \succsim (z_1, {}_2y) \\ (x_1, x_2) \; {}_1\succsim_2^z (y_1, y_2) & \text{means } (x_1, x_2, {}_3z) \succsim (y_1, y_2, {}_3z) \\ {}_3x \; {}_3\succsim^z \; {}_3y & \text{means } (z_1, z_2, {}_3x) \succsim (z_1, z_2, {}_3y) \\ x_2 \succsim_2^z y_2 & \text{means } (z_1, x_2, {}_3z) \succsim (z_1, y_2, {}_3z) \end{cases}$$

POSTULATE 3′ (Limited Independence). *The two orderings* $\succsim_1^z$, $\succsim_2^z$ *are independent of the reference program* ${}_1z$.

POSTULATE 3″ (Extended Independence). *The ordering* ${}_1\succsim_2^z$ *is independent of* ${}_1z$.

For convenient reference, we also introduce

POSTULATE 3 (Complete Independence). *Both* P3′ *and* P3″ *hold.*[4]

---

[3] A simple example of such an ordering $\succsim$ satisfying all postulates except P2 is that in which $\mathscr{X}$ is one-dimensional and $\succsim$ is represented by $\lim_{T\to\infty} \sup_{t \geq T} x_t$. This ordering looks only at the highest consumption level that is, ultimately, and again and again thereafter, at least temporarily reached or arbitrarily closely approached. (Note the contrast between succinct mathematical notation and involved equivalent verbal statement$_1$.)

[4] By GORMAN (1968b) (see Footnote 10 of Chapter 3), the independence of ${}_1\succsim_2^z$ and ${}_2\succsim^z$ implies that of $\succsim_1^z$.

Whenever one or both of P3′, P3″ are assumed in what follows, the corresponding orderings will be denoted $\succsim_1$, $\succsim_2$, ${}_1\succsim_2$. Note that ${}_1\succsim_2$ would have been denoted $\succsim_{1,2}$ in Chapter 3.

In the earlier studies referred to above, the implications of P3′ were pursued at length, those of P3 only briefly mentioned. In this study, the emphasis is reversed.

Neither P3′ nor P3″ can be regarded as realistic. Taken together, they will be found to preclude all complementarity between the consumption of different periods. P3′ by itself will be seen to permit a limited complementarity among the utility levels to be associated with consumption in successive periods, but still no complementarity between individual commodities or activities in different periods. P3 or P3′ should therefore be looked upon as first approximations, made to facilitate exploration of the implications of the fourth postulate, the real objective of this study:

POSTULATE 4 (Stationarity). *There exists a first period vector x_1^* in $\mathscr{X}$ with the property that the programs*

$$ {}_1x = (x_1^*, {}_2x) = (x_1^*, x_2, x_3, \ldots) $$

$$ {}_1y = (x_1^*, {}_2y) = (x_1^*, y_2, y_3, \ldots) $$

are such that ${}_1x \succsim_1 {}_1y$ if and only if the programs[5]

$$ {}_1v = (v_1, v_2, v_3, \ldots) = (x_2, x_3, x_4, \ldots) = {}_2x, $$

$$ {}_1w = (w_1, w_2, w_3, \ldots) = (y_2, y_3, y_4, \ldots) = {}_2y, $$

defined by $v_t \equiv x_{t+1}$, $w_t \equiv y_{t+1}$, $t = 1, 2, \ldots$, are such that ${}_1v \succsim_1 w$.

Before interpreting this postulate in less formal language, we note that, if one particular $x_1 = x_1^*$ in $\mathscr{X}$ has this property, then by P3′ every x_1 in $\mathscr{X}$ has this property. Using this, P4 says that if two programs ${}_1x$, ${}_1y$ have a common first-period vector $x_1 = y_1$, then the programs ${}_1v$, ${}_1w$ obtained by deleting x_1 from ${}_1x$ and from ${}_1y$, respectively, and advancing the timing of all subsequent vectors by one period, are ordered in the same way as ${}_1x$, ${}_1y$.

It is worth emphasizing that in this statement nothing is said or implied about the ordering of "then future" programs ${}_2x$, ${}_2y$ that may be applied

[5] In the notations ${}_2x$, ${}_2y$ as used here, there is no longer a necessary connection between the presubscript of ${}_2x$ and the timing of the first installment x_2 of that program. That is, x_2 simply means the vector that happened to represent second period consumption in the program ${}_1x$. In the program ${}_2x = {}_1v$, that same consumption occurs in the first period. With this point established, the notations ${}_1v$, ${}_1w$ will no longer be needed in what follows.

after the first period has elapsed. That is, no question of consistency or inconsistency of orderings adopted at different points in time is raised.[6] Only the ordering $\succsim$ applying "now" is under discussion. Applied repeatedly, P4 implies that the present ordering of two programs $(x_1, \ldots, x_{t-1}, {}_t x) \equiv ({}_1 x_{t-1}, {}_t x)$ and $({}_1 x_{t-1}, {}_t y)$ that start to differ in a designated way only from some point t in time onward is independent both of what that point in time is, and of what the common values ${}_1 x_{t-1}$ up to that point are.

The fifth and last postulate asserts, roughly, that the end result of an infinite sequence of improvements starting from some given program is itself an improvement over that program. If only a finite number of future periods is affected by all but a finite number of the improvements, such an assertion is already implied in P1, P3′, P4. For simplicity we will refer only to a sequence of improvements made to successive vectors in the program, taken one at a time. A similar postulate has been used by DIAMOND (1965). An alternative postulate in terms of improvements affecting several periods at a time is briefly considered in Subsection 6* below.

POSTULATE 5 (Monotonicity). *If* ${}_1 x$, ${}_1 y$ *are programs such that, for all* $t = 1, 2, \ldots, (x_1, x_2, \ldots, x_{t-1}, y_t, y_{t+1}, y_{t+2}, \ldots) \precsim (x_1, x_2, \ldots, x_{t-1}, x_t, y_{t+1}, y_{t+2}, \ldots)$ *then* ${}_1 y \precsim {}_1 x$.

It can be shown that, given all other postulates, P5 is implied in the following stronger postulate, used in a previous study (KOOPMANS (1960)).

POSTULATE 5′ (Extreme Programs). *There exist in* ${}_1\mathscr{X}$ *a best and a worst program.*

There is some interest in avoiding that stronger statement wherever possible, with a view to problems of optimal growth under continuing technical change.

On the basis of the postulates set out, we seek to construct a representation of $\succsim$ on the entire program space ${}_1\mathscr{X}$, or on as large a subspace of it as we can. Our strategy will be first to find such representations on suitably chosen subspaces of ${}_1\mathscr{X}$.

3. Representation of $\succsim$ on Any Subspace of Ultimately Identical Programs

Since the space ${}_1\mathscr{X}$ is infinite-dimensional, Proposition 1 cannot be directly applied to the ordering $\succsim$ given on it.[7] For this reason, we shall

[6] For a discussion of that question, see STROTZ (1957).

[7] While it is true that Proposition 1 can be extended to infinite-dimensional spaces having the topological property of "separability" (see DEBREU (1954, 1964)), the distance function (2.4) does not endow ${}_1\mathscr{X}$ with that property.

in the present section study $\succsim$ on the subspace ${}_1\mathscr{X}_T^z$ of all programs of the form

$$ {}_1x = ({}_1x_T\,,\, {}_{T+1}z)\,, \tag{3.1}$$

where ${}_1z$ is again an arbitrary but fixed reference program. Since programs in this subspace differ only in the segments ${}_1\mathscr{X}_T$, the ordering $\succsim$ on ${}_1\mathscr{X}$ restricted to the subspace ${}_1\mathscr{X}_T^z$ induces an ordering of sequences ${}_1x_T$ of length T on the space ${}_1\mathscr{X}_T$. We shall denote this ordering by ${}_1\succsim_T$. In Subsection 3* we shall prove

RESULT D. *For all T, the ordering ${}_1\succsim_T$ is independent of ${}_1z$, and is represented by a function of the form*

$$ U_T({}_1x_T) = u(x_1)+\alpha u(x_2)+ \cdots +\alpha^{T-1}u(x_T), \quad 0<\alpha<1\,. \tag{3.2}$$

Here $u(x)$ is a continuous function defined on $\mathscr{X}$, and both α and $u(x)$ are independent of T.

The proof proceeds through a succession of statements which we label (Da), (Db), …, recording in each case the postulates and/or previous results used in the proof. The notations for induced orderings extend those of (2.7).

(Da; P3′, P4) The induced ordering ${}_t\succsim^z$ of sequences ${}_tx$, defined by restricting $\succsim$ to the set of programs $({}_1z_{t-1}\,,\, {}_tx)$ is independent of ${}_1z$ and of t.

(Db; P3′, P4) The induced ordering $\succsim_t^z$ of vectors x_t is independent of ${}_1z$ and of t.

(Dc; P3, P4) The induced ordering ${}_{t-1}\succsim_t^z$ of vectors $(x_{t-1}\,,\, x_t)$ is independent of ${}_1z$ and of t.

(Dd; C, Db, Dc) The induced ordering ${}_1\succsim_T^z$ of sequences ${}_1x_T$ is independent of ${}_1z$, and is represented by a continuous function of the form

$$ U_T({}_1x_T) = u_1(x_1)+u_2(x_2)+ \cdots +u_T(x_T)\,, \tag{3.3}$$

unique up to a linear transformation similar to (5.5) in Chapter 3.

(De; Dd, P4) One can choose the $u_i(x_i)$ in (3.3) in such a way that (3.2) holds with $\alpha>0$, where α is unique, and where $u(x)$ is unique up to a linear transformation

$$ u^*(x) = \beta+\gamma u(x), \qquad \gamma>0\,. \tag{3.4}$$

(Df; De, P5) $\alpha<1$.

3. Proof of Result D.* Clearly the continuity of $\succsim$ entails the continuity of all restricted orderings induced by it.

(*Da*). P3′ allows us to write

$$\gtrsim_1^z = \gtrsim_1, \qquad {}_2\gtrsim^z = {}_2\gtrsim. \tag{3.5}$$

Using the symbol $\Leftrightarrow$ to denote logical equivalence, these statements are made explicit by

$$\text{for all } {}_2x^*, x_1, y_1, \qquad (x_1, {}_2z) \gtrsim (y_1, {}_2z) \Leftrightarrow (x_1, {}_2x^*) \gtrsim (y_1, {}_2x^*), \tag{3.6}$$

$$\text{for all } x_1^*, {}_2x, {}_2y, \qquad (z_1, {}_2x) \gtrsim (z_1, {}_2y) \Leftrightarrow (x_1^*, {}_2x) \gtrsim (x_1^*, {}_2y). \tag{3.7}$$

In particular, choosing for x_1^* in (3.7) the x_1^* occurring in P4, we have from P4

$$\text{for all } {}_2x, {}_2y, \qquad (z_1, {}_2x) \gtrsim (z_1, {}_2y) \Leftrightarrow {}_2x \gtrsim {}_2y, \tag{3.8}$$

an implication which can be applied once more to give

$$(z_1, z_2, {}_3x) \gtrsim (z_1, z_2, {}_3y) \Leftrightarrow (z_2, {}_3x) \gtrsim (z_2, {}_3y) \Leftrightarrow {}_3x \gtrsim {}_3y, \text{ etc.}$$

These results are summarized in

$$ {}_t\gtrsim^z = {}_t\gtrsim = \cdots = {}_2\gtrsim = \gtrsim \qquad t = 2, 3, \ldots, \tag{3.9}$$

keeping in mind the notational practice explained in Footnote 5.

(*Db*). From (3.8) and (3.6), we have, for all ${}_1x^*$,

$$(z_1, x_2, {}_3z) \gtrsim (z_1, y_2, {}_3z) \Leftrightarrow (x_2, {}_3z) \gtrsim (y_2, {}_3z) \Leftrightarrow$$
$$\Leftrightarrow (x_2, {}_3x^*) \gtrsim (y_2, {}_3x^*) \Leftrightarrow (x_1^*, x_2, {}_3x^*) \gtrsim (x_1^*, y_2, {}_3x^*).$$

This reasoning and its repetition yield

$$\gtrsim_t^z = \gtrsim_t = \cdots = \gtrsim_2 = \gtrsim_1, \quad t = 1, 2, 3, \ldots. \tag{3.10}$$

(*Dc*). We now bring in P3″, written as ${}_1\gtrsim_2^z = {}_1\gtrsim_2$. Together with (3.8) this implies that, for all ${}_1x^*$,

$$(z_1, x_2, x_3, {}_4z) \gtrsim (z_1, y_2, y_3, {}_4z) \Leftrightarrow (x_2, x_3, {}_4z) \gtrsim (y_2, y_3, {}_4z) \Leftrightarrow$$
$$\Leftrightarrow (x_2, x_3, {}_4x^*) \gtrsim (y_2, y_3, {}_4x^*) \Leftrightarrow (x_1^*, x_2, x_3, {}_4x^*) \gtrsim (x_1^*, y_2, y_3, {}_4x^*).$$

Since this can again be repeated, we have

$$ {}_{t-1}\gtrsim_t^z = {}_{t-1}\gtrsim_t = \cdots = {}_2\gtrsim_3 = {}_1\gtrsim_2, \quad t = 2, 3, \ldots. \tag{3.11}$$

(*Dd*). We consider ${}_1\gtrsim_T^z$, and note that $\gtrsim_t^z$, $t = 1, \ldots, T$ and ${}_{t-1}\gtrsim_t^z$, $t = 2, \ldots, T$, are all independent of ${}_1z$. By P2, $\gtrsim_1$ permits $x_1 \succ_1 y_1$, and by (3.10) a similar statement holds for $\gtrsim_t$, $t = 2, 3, \ldots$. The premises of Result C of Section 6 in Chapter 3 are therefore satisfied, and the representation

(3.3) follows. Hence ${}_1\succsim^z_T$ is independent of z, and we write ${}_1\succsim_T$ from here on.

(*De*). By (3.8) and (3.3), ${}_2\succsim_T$ is represented on ${}_2\mathscr{X}_T$ by either of the functions

$$u_2(x_2)+u_3(x_3)+\cdots+u_T(x_T),$$

$$u_1(x_2)+u_2(x_3)+\cdots+u_{T-1}(x_T).$$

It follows, along the lines of the uniqueness proof of Proposition 2, that, for all x in $\mathscr{X}$ and all $T\geqq 3$,

$$u_{t+1}(x)=\beta_t+\alpha u_t(x), \quad t=1,\ldots,T-1, \quad \alpha>0.$$

Since we are free to choose each $u_t(x)$, $t=2,\ldots,T-1$, so as to have $\beta_t=0$ for all t, (3.2) results, with $u(x)=u_1(x)\equiv u^{(T)}(x)$, say, which might still depend on T. However, by comparing the representation (3.2) of ${}_1\succsim_T$ in terms of $u^{(T)}(x)$ with that in terms of $u^{(T+1)}(x)$ obtained from (3.2) with $T+1$ sustituted for T, one finds, again using the uniqueness argument, that the same $u^{(T)}(x)=u(x)$ can be used for all $T\geqq 3$, and hence, by holding ${}_3x$ constant, also for $T=1, 2$.

(*Df*). The proof of *Df* will be given in Section 4.

4. Representation of $\succsim$ on the Space of Ultimately Constant Programs

In this section we choose a favorable ground on which to face the infinite horizon by first restricting ourselves to the space ${}_{\text{con}}\mathscr{X}$ of *constant programs*

(4.1) $${}_{\text{con}}x\equiv(x, x, x, \ldots),$$

that is, of programs ${}_1x$ for which $x_t=x$ for all t.

The points of ${}_{\text{con}}\mathscr{X}$ are in a one-to-one correspondence

(4.2) $${}_{\text{con}}x\leftrightarrow x$$

to those of $\mathscr{X}$. Because for all x, x' in $\mathscr{X}$,

(4.3) $$D({}_{\text{con}}x, {}_{\text{con}}x')=d(x, x'),$$

this correspondence preserves the distance function, and therewith the continuity concept. Moreover, if x, y are vectors of $\mathscr{X}$ such that $y\precsim_1 x$, then, by *Db* and P5, if ${}_{\text{con}}x_T$ denotes the sequence $(x, x, \ldots, x)$ of T identical vectors x,

(4.4) $${}_{\text{con}}y\precsim(x, {}_{\text{con}}y)\precsim\cdots\precsim({}_{\text{con}}x_T, {}_{\text{con}}y)\precsim\cdots\precsim{}_{\text{con}}x.$$

The continuous ordering $\succsim_1$ on $\mathscr{X}$ is therefore transformed by the correspondence (4.2) into the ordering $\succsim$ restricted to ${}_{\text{con}}\mathscr{X}$. In particular,

RESULT E. *Any continuous representation $u(x)$ of $\succsim_1$ on $\mathscr{X}$ is at the same time a continuous representation of $\succsim$ restricted to ${}_{\text{con}}\mathscr{X}$.*

Note that only limited independence (P3′) was used in the proof of Result E.

Next we consider the space $\mathscr{X}_{\text{con}}$ of *ultimately constant programs*, that is, of programs such that, for some $T \geqq 0$,

$$(4.5) \qquad {}_1x = ({}_1x_T, {}_{\text{con}}x) = (x_1, \ldots, x_T, x, x, \ldots)$$

(for $T=0$ the term ${}_1x_T$ is absent). One readily verifies that the reasoning that led to Result *De* also applies in any subspace $\mathscr{X}^{(T)}_{\text{con}}$ of $\mathscr{X}_{\text{con}}$ consisting of programs (4.5) *with a fixed T*. The only difference consists of an added term in (3.2). One now finds for all $T \geqq 2$ a continuous representation of $\succsim$, restricted to $\mathscr{X}^{(T)}_{\text{con}}$, by the function

$$(4.6) \qquad u(x_1) + \alpha u(x_2) + \ldots \alpha^{T-1} u(x_T) + f_T(u(x)), \quad 0 < \alpha,$$

where $f_T(u)$ is continuous and increasing. From this representation we can derive two representations of $\succsim$ restricted to $\mathscr{X}^{(T-1)}_{\text{con}}$, one (4.7a) by setting $x_1 = x_1^*$ and applying P4, the other (4.7b) by setting $x_T = x$, as follows,

$$(4.7a) \qquad U^{(a)}({}_1x) \equiv \alpha u(x_1) + \cdots + \alpha^{T-1} u(x_{T-1}) + f_T(u(x))$$

$$(4.7b) \qquad U^{(b)}({}_1x) \equiv u(x_1) + \cdots + \alpha^{T-2} u(x_{T-1}) + \alpha^{T-1} u(x) + f_T(u(x)).$$

By Result C these representations are, for all $T \geqq 3$, unique up to a linear transformation. Comparison of the first terms shows that

$$U^{(a)}({}_1x) = \alpha U^{(b)}({}_1x) + \beta,$$

which implies that

$$f_T(u) = \alpha^T u + \alpha f_T(u) + \beta.$$

Since $f_T(u)$ is increasing, we must have $\alpha < 1$, that is, *Df* above, thus completing the proof of Result D. Solving for $f_T(u)$ and dropping the constant term, we have

RESULT F. *On the space $\mathscr{X}_{\text{con}}$ of ultimately constant programs $\succsim$ is represented by the continuous function (with $0 < \alpha < 1$)*

$$(4.8) \quad U({}_1x) = U({}_1x_T, {}_{\text{con}}x) \equiv u(x_1) + \alpha u(x_2) + \cdots + \alpha^{T-1} u(x_T) + \frac{\alpha^T}{1-\alpha} u(x),$$

unique up to a linear transformation.

Note that in this function T itself depends on the given ultimately constant program ${}_1x$. For definiteness one can specify that $T+1$ is the earliest time from which onward ${}_1x$ is constant. However, the same value of $U({}_1x)$ is obtained if one allows $T+1$ to be any time, earliest or not, beyond which ${}_1x$ is constant. It is for that reason that the function (4.8) represents $\succsim$ on the space $\mathscr{X}_{\text{con}}$ for *all* ultimately constant programs, regardless of the values of their "minimal" T.

5. Representation of $\succsim$ on the Space of Programs Bounded in Utility

It is now possible to indicate a large subspace of the program space on which the ordering $\succsim$ is represented by

$$(5.1) \qquad U({}_1x) \equiv \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t), \quad 0<\alpha<1.$$

We shall call a program ${}_1x$ *bounded in utility* if there exist vectors $\underline{x}$, $\bar{x}$ in $\mathscr{X}$ with $\underline{x} \prec_1 \bar{x}$ such that

$$(5.2) \qquad \underline{x} \precsim_1 x_t \precsim_1 \bar{x} \text{ for all } t = 1, 2, \ldots.$$

We can then show

Proposition 3. *On the space* ${}_1\mathscr{X}^*$ *of all programs bounded in utility, the ordering* $\succsim$ *is represented by the continuous function* (5.1).

It is to be noted that for ultimately constant programs the function (5.1) is identical with that in (4.8). Hence Proposition 3 includes Result F.

5. Proof of Proposition 3.* We first note that if ${}_1x$ is bounded in utility, then,

$$u(\underline{x}) \leqq u(x_t) \leqq u(\bar{x}) \text{ for all } t,$$

and, since $0<\alpha<1$, the series in (5.1) is absolutely convergent, hence its sum exists and is continuous with respect to ${}_1x$.

Now let ${}_1x$ and ${}_1y$ be two programs bounded in utility, and define bounds applicable to both ${}_1x$ and ${}_1y$ by

$$\underline{z} \equiv \begin{cases} \underline{x} \text{ if } \underline{x} \precsim_1 \underline{y} \\ \underline{y} \text{ if } \underline{y} \prec_1 \underline{x} \end{cases}, \qquad \bar{z} \equiv \begin{cases} \bar{x} \text{ if } \bar{x} \succsim_1 \bar{y} \\ \bar{y} \text{ if } \bar{y} \succ_1 \bar{x} \end{cases},$$

$$\underline{u} \equiv u(\underline{z}), \qquad \bar{u} \equiv u(\bar{z}), \qquad \text{so } \underline{u} \leqq \bar{u}.$$

Assume first that $U({}_1x) > U({}_1y)$, and write

$$U({}_1x) - U({}_1y) \equiv 3\Delta > 0 .$$

For comparison purposes we consider two programs

$${}_1x^{(T)} = ({}_1x_T, {}_{\text{con}}\underline{z}), \qquad {}_1y^{(T)} = ({}_1y_T, {}_{\text{con}}\bar{z}),$$

where T is chosen large enough to have

$$\left(\sum_{t=T+1}^{\infty} \alpha^{t-1}\right)(\bar{u} - \underline{u}) = \alpha^T \cdot \frac{\bar{u} - \underline{u}}{1-\alpha} \leqq \Delta .$$

Because of $\underline{u} \leqq u(x_t) \leqq \bar{u}$ and similar inequalities for y_t we then have

$$U({}_1x) - U({}_1x^{(T)}) = \sum_{t=T+1}^{\infty} \alpha^{t-1}(u(x_t) - \underline{u}) \leqq \Delta, \qquad U({}_1y^{(T)}) - U({}_1y) \leqq \Delta ,$$

and therefore

$$U({}_1x^{(T)}) - U({}_1y^{(T)}) \geqq \Delta > 0 .$$

Since ${}_1x^{(T)}$, ${}_1y^{(T)}$ are ultimately constant, this implies ${}_1x^{(T)} \succ {}_1y^{(T)}$ by Result F. But then, using P5, ${}_1x \succsim {}_1x^{(T)} \succ {}_1y^{(T)} \succsim {}_1y$, which yields

$$U({}_1x) > U({}_1y) \text{ implies } {}_1x \succ {}_1y , \tag{5.3}$$

confirming the representation (5.1) in this case.

Assume next that, for two programs ${}_1x$, ${}_1y$ bounded in utility,

$$U({}_1x) = U({}_1y) \text{ but } {}_1x \prec {}_1y . \tag{5.4}$$

Then there exists t_0 such that

$$x_{t_0} \prec_1 y_{t_0}, \text{ so } u(x_{t_0}) < u(y_{t_0}), \tag{5.5}$$

because "$x_t \succsim_1 y_t$ for all t" would contradict "${}_1x \prec {}_1y$" by P5.

Using the connectedness of $\mathscr{X}$, we draw an arc $\mathscr{A}$ in $\mathscr{X}$ connecting x_{t_0} and y_{t_0} (see Fig. 1). Then, by the continuity of $u(x)$, we can find a point x on $\mathscr{A}$ such that

$$u(x) = u(x_{t_0}) \tag{5.6}$$

while, for each $\delta > 0$, there exists x' on $\mathscr{A}$ such that

$$d(x, x') \leqq \delta , \tag{5.7a}$$

$$u(x') > u(x) . \tag{5.7b}$$

Using P_1 we can choose δ such that, if (5.7a) holds,

$$_1x \prec {_1x'} \equiv ({_1x_{t_0-1}}, x', {_{t_0+1}x}) \prec {_1y}.$$

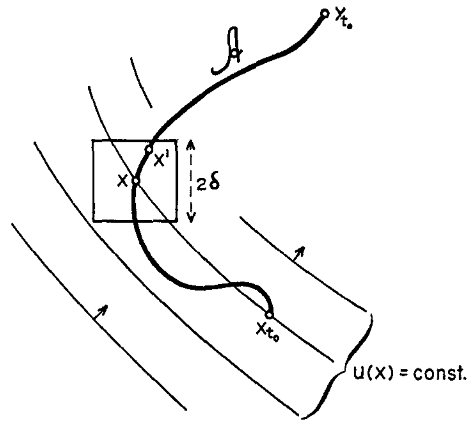

Fig. 1

But then, by (5.7b) and (5.1),

$$U({_1x'}) > U({_1x}) = U({_1y}) \text{ but } {_1x'} \prec {_1y},$$

a contradiction of (5.3). Hence (5.4) is false, and

$$U({_1x}) = U({_1y}) \text{ implies } {_1x} \sim {_1y},$$

confirming (5.1) in this case as well. Since the third case, $U({_1x}) < U({_1y})$, is symmetric to the first, the proof is now complete.

6. Concluding Remarks on the Representation of $\succsim$

The representations we have found show unexpectedly strong implications of the postulates used. It turns out that offsetting program changes in future periods can be determined on the basis of just two mathematical data,

(i) the function $u(x)$ which allows the comparison of "utility differences" within the same period, and

(ii) a constant discount factor α which extends that comparison to utility differences in different periods.

The representation may be called cardinal in the sense that only increasing linear transformations, applied simultaneously to $u(x)$ and to $U({}_1x)$, will preserve these simple properties.

Since $\alpha < 1$ the present postulates do not permit expression of the ethical principle of treating all future generations' utilities on a par with present utilities. A way has been found to include that limiting case in models of optimal growth by retreating to the notion of a partial ordering. VON WEIZSÄCKER (1965) has proposed to call a program ${}_1x$ better than a program ${}_1y$ if there exists a $T \geqq 1$ such that

$$\sum_{t=1}^{T'} u(x_t) > \sum_{t=1}^{T'} u(y_t) \text{ for all } T' \geqq T.$$

This criterion has been called the *overtaking criterion* by GALE (1967). Under appropriate conditions, it has permitted determination of an optimal path which turns out to be comparable with, and better than, every other feasible path (KOOPMANS (1965, 1967a)).

Returning to the case of a complete ordering, with a discount factor $\alpha < 1$, it is conceivable that the representation (5.1) can be extended on the basis of the present postulates to larger sets of programs not all bounded in utility. In Subsection 6* we allude to a reasoning from a strengthened monotonicity postulate that permits an extension to all programs for which the sum (5.1) exists.

It will be clear that, if $u(x)$ is unbounded on $\mathscr{X}$, then there exist programs for which the sum (5.1) diverges. In such cases the representation (3.2) restricted to a class of ultimately identical programs, all "divergent in utility," may still be valuable. It would permit formulating a partial optimality criterion in which a path is found to stand comparison with all other feasible paths differing from it in a finite number of future periods only. Other considerations would then have to be brought to bear on the choice of the class of ultimately identical programs.

6*. *An Alternative Monotonicity Postulate.* One might wish to strengthen P5 to

POSTULATE 5'' (Strong Monotonicity). *If* ${}_1x$, ${}_1y^{(i)}$, $i = 1, 2, \ldots$, *are programs such that*

$$\left.\begin{array}{l} {}_1y^{(i)} \precsim {}_1y^{(i+1)}, \\ {}_1y^{(i)}_{t_i} = {}_1x_{t_i}, \quad 0 \leqq t_i < t_{i+1}, \end{array}\right\} \text{ for all } i = 1, 2, \ldots,$$

then, ${}_1y \precsim {}_1x$.

This postulate considers successive improvements each extending over an arbitrary number of periods, but where the set of periods affected by successive improvements becomes more and more remote in time. It allows one, for any program ${}_1x$ for which the sum (5.1) exists, to construct an equivalent constant program ${}_{\mathrm{con}}x$ such that $U({}_1x) = U({}_{\mathrm{con}}x)$, thus extending the representation (5.1) to all programs for which that sum exists. Conversely, for any program ${}_1x$ equivalent to a constant program, the sum (5.1) does exist.

7. Limited Independence, Time Perspective and Impatience

If instead of complete independence (P3) we postulate only limited independence (P3′), Proposition 2 is not available, and we must fall back on Result A. A study along these lines was made in two consecutive papers by KOOPMANS (1960) and by KOOPMANS, DIAMOND and WILLIAMSON (1964). The postulates of that study were the analogues of the present postulates of continuity (P1), sensitivity (P2), limited independence (P3′), stationarity (P4), and the existence of extreme programs (P5′), applied to a given utility function $U({}_1x)$ rather than to an ordering.[8]

A theorem by DIAMOND (1965, p. 173) now allows us to obtain all the results of the previous study from the present postulates P1′ (see Footnote 8), P2, P3′, P4, P5′ as applied to an ordering $\succsim$ on ${}_1\mathscr{X}$. The resulting representation $U({}_1x)$ of $\succsim$ is found to satisfy a *recursive relation*

$$U({}_1x) = V(u(x_1), U({}_2x)), \tag{7.1}$$

where $V(u, U)$ is a continuous function defined on the product of two nondegenerate intervals, which is increasing in each of its variables. This *aggregator function* indicates how the single-period utility $u(x_1)$ of the first installment x_1 of ${}_1x$ and the utility $U({}_2x)$ of the sequel ${}_2x$ (were that sequel to start immediately) are combined to form the utility of the entire program ${}_1x$. In particular, if P3 holds, $V(u, U) = u + \alpha U$.

The representation (7.1) is *ordinal* in the sense that any pair of continuous increasing functions Φ, φ with the appropriate domains will define an

[8] Apart from this difference, P1 was strengthened to make P1′, say, by adding two statements: (a) that the continuity on ${}_1\mathscr{X}$ of $U({}_1x)$ is uniform on each equivalence set, (b) that $\mathscr{X}$ is bounded and convex. The latter was used in the proof that the range $\mathscr{U}$ of $U({}_1x)$ is an interval. Alternatively, that result could have been obtained by adding to P5′ that among the extreme programs there are a best and worst constant program, or by deriving that statement in turn from a variant of P5 restricted to $\sim$.

alternative representation

(7.2) $$U^*({}_1x) \equiv \Phi(U({}_1x)) = \Phi(V(u(x_1), U({}_2x)) = V^*(u^*(x_1), U^*({}_2x)),$$

say, where

(7.3) $$u^*(x) \equiv \varphi(u(x)), \quad V^*(u^*, U^*) \equiv \Phi(V(\varphi^{-1}(u^*), \Phi^{-1}(U^*))).$$

This being so, the question arises what takes the place of the discount factor α, the existence of which was derived in Section 3 from P3. In particular, what corresponds to the inequality $\alpha < 1$ crucial to convergence of the representation (5.1)?

It is readily seen from (7.2) and (7.3) (KOOPMANS (1960, Section 14*)) that, in the case of a differentiable function $V(u, U)$, the discount factor associated with a constant program ${}_{\text{con}}x = (x, x, x, \ldots)$,

(7.4) $$\alpha(x) \equiv \left(\frac{\partial V(u, U)}{\partial U}\right)_{u=u(x),\, U=U({}_{\text{con}}x)}$$

satisfies $0 \leqq \alpha(x) \leqq 1$, and is invariant under differentiable increasing scale changes for u and U. Moreover, as distinct from the representation (5.1), $\alpha(x)$ in (7.4) can vary with x. The limited independence postulate P3′ therefore allows scope for the idea already expressed by Irving FISHER (1930, Chapter IV, Sections 3 and 6) with regard to individual preferences: that the discount factor may depend on the level of present and prospective income.

As an illustration, let $\mathscr{X}$ be the closed unit interval $\mathscr{I} = (0, 1)$, let $u(x) = x$, and consider the aggregator function

(7.5) $$V(x, U) = U + (x - U)(a - bx + cU),$$

where we require that

(7.6) $$b,\ c,\ a - 2b,\ a - b - c,\ 1 - a - 2c > 0.$$

Then, if we assign to U the same range $\mathscr{I}$, $V(x, U)$ is increasing in both variables, and

(7.7) $$V(0, 0) = 0, \qquad V(1, 1) = 1.$$

Finally, since $U({}_{\text{con}}x) = x$ is the only root U of $U = V(x, U)$ in the range $\mathscr{I}$,

(7.8) $$\alpha(x) = 1 - a + (b - c)x.$$

Hence the direction of change of $\alpha(x)$ with increasing income x is given by the sign of $(b - c)$. Following FISHER (1930, Chapter IV, Section 6), many but not all economists I have consulted regard an increasing $\alpha(x)$ as the normal case. This implies that the ratio of the marginal utility of future consumption to that of present consumption increases as the level of the constant consumption flow ${}_{\text{con}}x$ is raised. Examples where the sign of $d\alpha(x)/dx$ depends on x can also be constructed.

While $\alpha(x)$ is defined only for constant programs, there is a generalization[9] of the convergence condition $\alpha<1$ in (5.1) to the present case that applies in the entire range of $V(u, U)$. It is found that there exists a transformation function Φ (here one takes $\varphi(u)=u$) such that the function $V^*(u, U^*)$ in (7.3) satisfies (dropping asterisks)

$$V(u, U')-V(u, U) \leqq U'-U \text{ whenever } U'>U\,. \tag{7.9}$$

This inequality has been called the *(weak) time perspective* property of the utility scale resulting from the transformation Φ. It says that the utility difference between two programs, measured in a suitable scale, does not increase (and generally diminishes) if both programs are postponed by one or more periods, while the same consumption or the same sequence of consumptions is inserted in the gaps so created. This inequality between utility differences is satisfied by a class of scales linked by transformations that include nonlinear as well as all linear transformations. For this reason, a representation $U({}_1x)$ satisfying (7.1) where $V(u, U)$ has the property (7.9) has been called *quasi-cardinal*.

There are indications that the weak inequality sign ($\leqq$) in (7.9) can be strengthened to strict inequality ($<$), referred to as *strong time perspective*, without strengthening the postulates. If so, it follows that the function $U({}_1x)$ can be reconstructed from a pair of functions $u(x)$, $V(u, U)$ implied in it. The example (7.5), (7.6) has the strong time perspective property as it stands, without requiring a prior scale change.

Precisely because it compares utility differences between pairs of programs, the time perspective inequality, strong or weak, does not by itself predict the choice within any one pair of programs. However, by elementary steps of reasoning, (7.9) implies a second family of ordinal inequalities, of which the simplest representative is

$$\begin{cases} \text{if} \quad u=u(x)<u'=u(x'), \quad U({}_{\text{con}}x) \leqq U \leqq U({}_{\text{con}}x')\,, \\ \text{then } V(u', V(u, U)) \underset{(=)}{>} V(u, V(u', U))\,. \end{cases} \tag{7.10}$$

[9] This generalization has been derived from statement (*a*) in Footnote 8. The proof uses a variant of the theory of Haar measure.

This inequality, weak or strong depending on whether the inequality (7.9) is weak or strong, has been called an *impatience* inequality. It indicates that if the single-period utility of a vector x' exceeds that of a vector x, then any program $(x', x, {}_3x)$, in which ${}_3x$ is selected from a wide class of "continuations," is preferred (or equivalent) to the corresponding program $(x, x', {}_3x)$ in which the better item is moved from first to second place. The class of continuations ${}_3x$ permitted in (7.10) consists of all those which, if started immediately, would be ranked between ${}_{\text{con}}x$ and ${}_{\text{con}}x'$. This condition should be read in conjunction with Result E of Section 4, which holds also under the present assumptions.

The impatience inequality holds for a wider range of U-values than that indicated in (7.10), and can be generalized to the interchange of two disjoint segments ${}_tx_{t'}$, ${}_sx_{s'}$ of a program, that need not be of equal length or contiguous in time.

8. Nonstationary Orderings and Eventual Impatience

DIAMOND (1965) has studied the implications of postulates similar to those of this chapter, with the main difference that no explicit stationarity postulate corresponding to our P4 is present. However, a certain comparability over time is introduced by assuming, in one interpretation, that there is only a single consumption good ($\mathscr{X}$ is the closed unit interval $\mathscr{C}$), more of which is always better. In another interpretation leading to the same mathematical analysis, there is a given single-period utility function $u(x)$ mapping $\mathscr{X}$ onto $\mathscr{C}$, which is the same for all t. For simplicity, we shall adopt the notation of an ordering $\succsim$ of all programs ${}_1x$ on the denumerable product space $\mathscr{C} \times \mathscr{C} \times \ldots = {}_1\mathscr{C}$, say, that corresponds to the first interpretation. The nonstationarity then applies to the way in which the sequences ${}_1x$ of scalars x_t enter into $\succsim$.

Diamond's postulates then can be shown[10] to be equivalent to specializations, to the case $\mathscr{X} = \mathscr{C}$, of our P1, P3, supplemented by a postulate P6 implying similar specializations of P2, P5, P5'.

POSTULATE 6 (General Monotonicity.) *If* $x_t \geqq y_t$ *for all* t, $x_t > y_t$ *for some* t, *then* ${}_1x \succ {}_1y$.

From these assumptions he derives the following property of *eventual impatience*: For any given program ${}_1x$ and any number $\varepsilon > 0$, there exists

[10] Using the results of GORMAN (1968b) referred to in Footnote 10 of Chapter 3.

a T such that

(8.1) $\quad {}_1x \succ (x_t, {}_2x_{t-1}, x_1, {}_{t+1}x)$ for all $t \geqq T$ with $x_1 \geqq x_t + \varepsilon$.

In words, the interchange with x_1 of any x_t which occurs sufficiently far into the future, and which falls short of x_1 by at least ε, diminishes the utility of the program ${}_1x$. This subtle result, which at first sight appears to miss its aim by a hair's breadth, is both vindicated and complemented by another theorem, attributed to Yaari, which hits the mark exactly (see DIAMOND (1965, p. 176)). It states that P6 and the present specialization of P1 taken together are incompatible with the statement

$$\text{for all } t \text{ and all } {}_1x \text{ in } {}_1\mathscr{X},\ {}_1x \sim (x_t, {}_2x_{t-1}, x_1, {}_{t+1}x),$$

that expresses "equal treatment of all generations."

Similar but somewhat stronger conclusions are obtained by Diamond by changing the distance function underlying P1 to

$$D^*({}_1x, {}_1y) = \sum_{t=1}^{\infty} (\tfrac{1}{2})^t d(x_t, y_t).$$

The results are stronger presumably because this modification explicitly reduces the weight attached, in the definition of continuity, to given consumption differences in a more distant future.

9. Concluding Remarks

The main results of the studies reported in this chapter appear to be two-fold.

In the first place the studies show a sequence of instances of increasing generality, in which a complete and continuous preference ordering of consumption programs for an infinite future necessarily gives a decreasing, or eventually decreasing, weight to consumption in a more distant future. Somewhat fancifully, one may say that the real numbers appear to be a sufficiently rich set of labels to accommodate in a continuous manner all infinite sequences of consumption vectors *only* if one gradually or eventually decreases the weight given to the more distant vectors in the preference ordering to be represented.

Secondly, the studies containing the stationarity postulate P4 have produced interesting special forms for the utility function $U({}_1x)$ in terms of simpler functions $u(x)$ and possibly $V(u, U)$, that facilitate the use of $U({}_1x)$ in models of optimal economic growth, and may perhaps suggest further parametrization or other specialization for econometric studies of individual consumption plans over time.

The use of substantive terms such as "consumption," "preference," "time" in what is essentially a formal mathematical analysis may hinder the perception of other possible applications in which one or more of these terms are inappropriate. The stationarity postulate, however, strongly suggests temporal or other consecutiveness in the vectors x_t, $t = 1,2, \ldots$, as a condition for meaningful application. In DIAMOND's (1965) study, where stationarity in the aggregation of single-period utilities is dropped, consecutiveness is immaterial in spite of appearances to the contrary in the formulation of some of the postulates. What is interpreted as eventual impatience if t stands for time is therefore also open to the wider interpretation that in *any* permutation of the vectors in the infinite sequence x_t, $t = 1,2, \ldots$, the weight given to vectors further up in the sequence must eventually decrease.

References

ARROW, K. J. (1952), "The Determination of Many-Commodity Preferences Scales by Two-Commodity Comparisons," *Metroeconomica, 4*, 105–115.

ARROW, K. J. (1963), *Social Choice and Individual Values*, (2nd ed.), Wiley, New York.

AUMANN, R. (1964a), "Subjective Programming," Chapter 12 in SHELLY and BRYAN (eds.), *Human Judgments and Optimality*, Wiley, New York, pp. 217–242.

BLASCHKE, W. and G. BOL (1938), *Geometrie der Gewebe*, Springer, Berlin.

DEBREU, G. (1954), "Representation of a Preference Ordering by a Numerical Function," Chapter 11 in THRALL, COOMBS, and DAVIS (eds.), *Decision Processes*, Wiley, New York, pp. 159–165.

DEBREU, G. (1959), *Theory of Value*, Wiley, New York.

DEBREU, G. (1960), "Topological Methods in Cardinal Utility Theory," Chapter 2 in K. J. ARROW, S. KARLIN, and P. SUPPES (eds.), *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, pp. 16-26.

DEBREU, G. (1964), "Continuity Properties of Paretian Utility," *International Economic Review, 5*, 285–293.

DIAMOND, P. A. (1965), "The Evaluation of Infinite Utility Streams," *Econometrica, 33*, 170–177.

EILENBERG, S. (1941), "Ordered Topological Spaces," *American Journal of Mathematics, 63*, 39–45.

FISHER, I. (1930, original edition), *The Theory of Interest*, reprinted by Augustus Kelley, New York, 1961.

GALE, D. (1967), "On Optimal Development in a Multi-Sector Economy," *Review of Economic Studies, 34*, 1–18.

GALE, D. (1968), "A Mathematical Theory of Optimal Economic Development," *Bulletin of the American Mathematical Society, 74*, 207–223.

GOLDMAN, S. M. and H. UZAWA (1964), "A Note on Separability in Demand Analysis," *Econometrica, 32*, 387–398.

GORMAN, W. M. (1959a), "Separable Utility and Aggregation," *Econometrica, 27*, 469–481.

GORMAN, W. M. (1959b), "The Empirical Implications of a Utility Tree: A Further Comment," *Econometrica, 27*, 489.

GORMAN, W. M. (1965), "Conditions for Additive Preferences," (unpublished),

GORMAN, W. M. (1968a), "Conditions for Additive Separability," *Econometrica, 36*, 605–609.

GORMAN, W. M. (1968b), "The Structure of Utility Functions," *Review of Economic Studies, 35*, 367-390.

KOOPMANS, T. C. (1960), "Stationary Ordinal Utility and Impatience," *Econometrica, 28*, 287–309.

KOOPMANS, T. C. (1964), "On Flexibility of Future Preference," Chapter 13 in SHELLY and BRYAN (eds.), *Human Judgments and Optimality*, Wiley, New York, pp. 243–254.

KOOPMANS, T. C., P. A. DIAMOND, and R. E. WILLIAMSON (1964b), "Stationary Utility and Time Perspective," *Econometrica, 32*, 82–100.

KOOPMANS, T. C. (1965), "On the Concept of Optimal Economic Growth," in *The Econometric Approach to Development Planning*, North Holland, Amsterdam, and Rand McNally, Chicago, (a reissue of *Pontificiae Academiae Scientiarvm Scripta Varia*, Vol. XXVIII, 1965), pp. 225–300.

KOOPMANS, T. C. (1966), "Structure of Preference over Time," Cowles Foundation Discussion Paper No. 206.

KOOPMANS, T. C. (1967a), "Objectives, Constraints and Outcomes in Optimal Growth Models," *Econometrica, 35*, 1–15.

KOOPMANS, T. C. (1967b), "Intertemporal Distribution and 'Optimal' Aggregate Economic Growth," Chapter 5 in FELLNER *et al.*, *Ten Economic Studies in the Tradition of Irving Fisher*, Wiley, New York, pp. 95–126.

KOOPMANS, T. C., P. A. DIAMOND, and R. E. WILLIAMSON (1964), "Stationary Utility and Time Perspective, "*Econometrica, 32*, 82-100.

LANCASTER, K. J. (1966a), "A New Approach to Consumer Theory," *Journal of Political Economy, 74*, 132–157.

LANCASTER, K. J. (1966b), "Change and Innovation in the Technology of Consumption," *American Economic Review, 56*, 14–23.

LEONTIEF, W. (1947a), "Introduction to a Theory of the Internal Structure of Functional Relationships," *Econometrica, 15*, 361–373.

LEONTIEF, W. (1947b), "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," *Bulletin of the American Mathematical Society, 53*, 343–350.

RADER, T. (1963), "The Existence of a Utility Function to Represent Preferences," *Review of Economic Studies, 30*, 229–232.

SAMUELSON, P. A. (1947), "Some Special Aspects of the Theory of Consumer's Behavior," Chapter 7 in *Foundations of Economic Analysis*, Harvard University Press, Cambridge, pp. 172–202.

STROTZ, R. H. (1956), "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies, 23*, 165–180.

STROTZ, R. H. (1957), "The Empirical Implications of a Utility Tree," *Econometrica*, *25*, 269–280.

STROTZ, R. H. (1959), "The Utility Tree–A Correction and Further Appraisal," *Econometrica*, *27*, 482–488.

VON WEIZSÄCKER, C. C. (1965), "Existence of Optimal Programmes of Accumulation for an Infinite Time Horizon," *Review of Economic Studies*, *32*, 85–104.

WOLD, H. (1943), "A Synthesis of Pure Demand Analysis, Part II," *Skandinavisk* Aktuaritidskrift, pp. 220–263.

YAARI, M. E. (1964), "On the Consumer's Lifetime Allocation Process," *International Economic Review*, *5*, 304–317.