Cowles Foundation Paper 353

Zeitschrift für Nationalökonomie, Suppl. 1 (1971), 11—22

A Model of a Continuing State with Scarce Capital*

By

Tjalling C. Koopmans, New Haven, Conn., USA

Abstract

The paper discusses a disaggregative model of a stationary state with a von Neumann type technology. The model allows for scarce primary inputs, and for consumption good outputs that enter into a utility function. The stationary state results from maximization of the sum of all future utilities discounted by a given annual discount factor α, either in a sufficiently distant future regardless of the initial capital stock, or at all times if the initial stock is just right. The dependence of the self-preserving capital stock on α is discussed.

1. Introduction and Summary

Recent decades have seen extensive development of linear and, more generally, convex optimization models for application to economic problems. Initially formulated in terms of quantities, these models have also led to important insights concerning valuations of goods and services, associated with optimal use of resources. The variables in question have been variously called "objectively determined valuations" (Kantorovich), "variables of the dual problem" (Dantzig), "efficiency prices" (Koopmans), "shadow prices", "implicit prices", etc.

By and large the models have fallen into two categories. In the first category, consisting of static, single-period, models, all variables either carry the same dating, or are timeless. In the second category, of dynamic models, the variables are dated, and an optimal path over time of the system studied is to be determined.

There is an intermediate category of models in which variables are dated, and a wide range of variation over time is indeed technologically

---

* Abstract of a paper presented to the Symposium on National Economy Modeling, organized by the Siberian Branch of the Academy of Sciences of the U. S. S. R., held in Novosibirsk, June 22—27, 1970, and subsequently to the Conference on the von Neumann Model, organized by the Institute for Advanced Studies, Vienna, Austria, July 6—7, 1970. The research described in this paper was carried out under grants from the National Science Foundation and from the Ford Foundation.

feasible. However, the typical model in this category concentrates on paths with constancy or proportional variation of the quantity and/or price variables. In von Neumann's original model, this was achieved by explicit constraint. Since that model did not recognize consumption as an end of economic activity, it did not allow social (consumers' or planners') preferences to have an effect on the path that solves the problem posed in the model.

The structure of the present model again permits constant as well as variable paths, but the problem posed is to choose the initial capital stock in such a way that constancy over time of the quantity variables is an outcome of an optimization that reflects assumed social preferences with regard to both the composition and the timing of consumption. As a result, the valuations implicit in such an optimal continuing state reflect both the scarcity of labor and other resources, and that of capital (produced means of production).

Analogy with the results of similar studies of one-good, one-resource (labor), models would suggest that the method of this report can be extended to continuing states of proportional growth, either with constant technology, or with a uniform rate of technological advance in all processes.

Precedents for one-sector models are described in Section 2. A corresponding disaggregative model is described in Section 3, its optimal continuing state defined and analyzed in Section 4. In Section 5 equivalent characterizations of the optimal continuing state are stated, that may help in interpreting the problem posed in the model and in computing its solution. Section 6 discusses the possibility of more than one solution. Section 7 lists conjectures about possible applications.

I am indebted to David Gale for pointing out that there is considerable substantive overlap between the present discussion and an investigation by Sutherland [1967, 1970]. The principal difference is that Sutherland's study is formulated as a search for continuing states that maximize an objective function for an infinite horizon. The present approach starts from one- or two-period optimization problems with a stationarity condition expressed by the constraints, and utilizes a von Neumann technology, both of these features being aimed at possible computational application. Statement (iii) of Section 4 indicates the mathematical link between the two approaches.

2. Aggregative (One-sector) Analogue

The point of departure is the following continuous-time one-sector optimization problem.

Data: Numbers ϱ, ζ with $0 < \varrho < 1$, $0 < \zeta$.
Functions $u(y)$, $0 < y$, and $g(z)$, $0 \leqq z$, with
$u'(y) > 0$, $u''(y) < 0$, $\lim_{y \to 0} u'(y) = \infty$,
$g(0) = 0$, $\varrho < g'(0)$, $g''(z) < 0$.

Problem: Maximize $\int_0^\infty e^{-\varrho t} u(y_t)\, d t$ *subject to*

$y_t \geqq 0,\ z_t \geqq 0,\ z_0 = \zeta,$
$y_t = g(z_t) - \dot{z}_t,$ for all $t \geqq 0.$
Denote the solution by $(\hat{y}_t, \hat{z}_t),\ 0 \leqq t < \infty.$

Interpretations: In all interpretations, (a), (b), (c),

y_t = consumption flow per worker, at time t,
$u(y_t)$ = utility* flow per worker, at time t,
z_t = capital stock per worker, at time t,
$f(z)$ (see below) = output flow per worker when $z_t = z$,
where $f(0) = 0,\ f'(z) > 0,\ f''(z) < 0.$

(a) For constant labor force and technology (F. Ramsey [1928]),

ϱ = continuous-time discount rate } applied to future
$e^{-\varrho}$ = continuous-time discount factor } utility flows

$g(z) = f(z).$

(Ramsey emphasizes the limiting case $\varrho = 0$, which requires additional specifications.)

(b) For constant technology and with labor force growing at exogenous exponential rate $\lambda > 0$ (Cass [1965], Koopmans [1965]),

ϱ = discount rate applied to per-worker utility flow,

$g(z) = f(z) - \lambda z$ where λz is the investment needed to keep the per-worker capital stock at a constant level.

(c) Further variants introduce technological progress, either labor-augmenting (also called Harrod-neutral) (Mirrlees [1967]) or product-augmenting (Inagaki [1966]).

Characteristics of solution: For all $\zeta > 0$, and for given ϱ, the optimal capital path $\hat{z}_t$ asymptotically approaches the same level $\hat{z}(\varrho)$, which is selfpreserving in that $\zeta = \hat{z}(\varrho)$ implies $\hat{z}_t = \hat{z}(\varrho)$ for all t. Furthermore

---

* Use of the term "utility" in the present context represents a way of speaking, a result of terminological history in which the meaning changes while the term persists. In a one-sector model where y is a number, the important operational meaning of $u(y)$ is that the ratio $u'(y_1)/u'(y_2)$ represents the relative weight given to an extra unit of consumption for a generation consuming at a low level y_1, say, compared with an extra unit for a generation with a higher level of consumption y_2, regardless of the time at which either generation lives. For further comment, see Koopmans (1967 b). In a model with several consumption goods, the function $u(y)$ in addition serves as an indicator of social preferences with regard to alternative compositions of consumption.

$\hat{z}(\varrho)$ increases as ϱ decreases, and* $\hat{z}(0) \equiv \hat{z}$ is the "golden rule level" of the per worker capital stock, which if maintained over time yields the largest indefinitely sustainable consumption per worker (see Fig. 1).

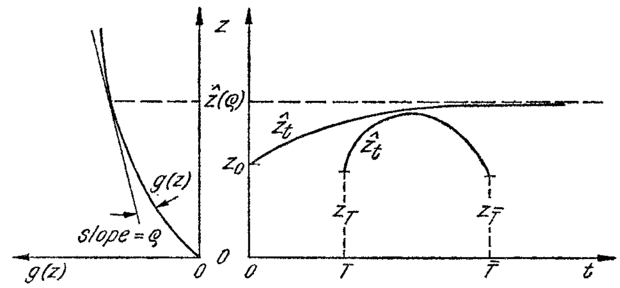

Fig. 1

The variant in which one maximizes $\int_T^{\bar{T}} e^{-\varrho t} u(y_t)\, dt,\ 0 \leqq T < \bar{T} < \infty,$ subject to given z_T, $z_{\bar{T}}$ will, for sufficiently large $\bar{T} - T$, give an optimal capital path $\hat{z}_t$ that bulges out toward, and in a central portion of $[T, \bar{T}]$ is close to, the self-preserving level $\hat{z}(\varrho)$ (catenary property, Cass [1966], Samuelson [1965]). For further references and details see Koopmans [1967 a].

3. Disaggregative Model Based on a Constant Von-Neumann Technology

We consider various discrete-time optimization problems which are variants or composites of the following single-period problem:

Data:

List of k productive processes, each defined by a unit activity.

List of ℓ capital goods.

List of m resources (primary inputs).

List of n consumption goods.

Column vectors** $z_1 \geq 0$, $z_2 \geqq 0$ of order ℓ, representing given initial and prescribed terminal capital stock for the single period considered.

Column vector $w > 0$ of order m, representing resource flows available during the period.

A, B, C, D, nonnegative matrices of order (ℓ, k), (ℓ, k), (m, k), (n, k), respectively, representing capital stock input, capital stock output, resource input flow, and consumption good output flow coefficients for

* $\equiv$ denotes equality by definition.

** $a \geq b$ denotes "$a \geqq b$ but not $a = b$".

the unit activities. In each matrix, at least one positive coefficient occurs in each row. In each of the input matrices A, C, and in the composite output matrix $\begin{bmatrix} B \\ D \end{bmatrix}$, a positive coefficient occurs in each column.

A strictly concave function $u(y)$, where $y = (y_{\cdot 1}, y_{\cdot 2}, \ldots, y_{\cdot n}) \geqq 0$ is a consumption flow (column) vector of order n. We require $\frac{\partial u}{\partial y_{\cdot i}} > 0$, $i = 1, \ldots, u$, for all $y \geqq 0$.

Problem $\overline{P}_1 = P_1(z_1, z_2, w; u)$: *Maximize* $u(y)$ subject to

$$\begin{aligned} x &\geqq 0 \\ -Ax &\geqq -z_1 \\ Bx &\geqq z_2 \\ -Cx &\geqq -w \\ Dx &\geqq y \end{aligned}$$

Interpretation: The components $x_{\cdot i}$, $i = 1, \ldots, k$, of the column vector x are activity levels for the corresponding processes. A positive element in the j-th column of D, $d_{ij} > 0$ for some i, indicates that the j-th process produces at least one consumption good. Such a process may well have positive coefficients b_{ij} as well, indicating that capital goods a_{ij} required by the process are not completely used up in production, but leave a vector of (perhaps somewhat more worn) capital goods b_{ij}. If $d_{ij} = 0$ for all i, the j-th process is worth having in the technology only if capital is created, improved, stored or maintained by it.

The maximization aims at that consumption vector $\hat{y}$, compatible with the given technology, the given resource availability w and the specified transformation of the capital stock from z_1 to z_2, that maximizes the objective function (social preference indicator) $u(y)$.

Tabular representation: $\overline{P}_1$ and similar problems will be represented in summary form, along the lines of the following table, as used by Tucker [1957].

	x	$\geqq$
q_1	$-A$	$-z_1$
q_2	B	z_2
s	$-C$	$-w$
p	D	y

The first column records the row vectors of efficiency prices (dual variables) associated with an optimum $(\hat{x}, \hat{y})$ of $\overline{P}_1$. To write down the conditions defining these prices note that, since $u'(y) > 0$ for all $y \geqq 0$, we have

$$D\hat{x} = \hat{y}. \tag{1}$$

Therefore, $\overline{P}_1$ is equivalent to
Problem $P_1 = P_1\,(z_1, z_2, w; v)$: *Maximize* $v\,(x) \equiv u\,(D\,x)$ *subject to* $x \geqq 0$ *and the primal constraints of*

	x	$\geqq$
q_1	$-A$	$-z_1$
q_2	B	z_2
s	$-C$	$-w$
$\leqq$	$-v'(x)$	

The constraints defining the prices then are, first, the dual constraints

$$-q_1 A + q_2 B - s\,C \leqq -v'(x) \equiv -p\,D \tag{2}$$

summarized in the last line of the Tucker tableau for P_1. Here $v'(x)$ is the row vector with components $\frac{\partial v}{\partial x_{.i}}$, $i = 1, \ldots, k$, and $p \equiv u'(y)$ is the row vector with components $\frac{\partial u}{\partial y_{.j}}$, $j = 1, \ldots, m$, with $y = D\,x$. Then, any set of nonnegative values of the primal (x) and dual (q_1, q_2, s, p) variables that satisfies the primal and dual constraints and, secondly, the complementary slackness condition

$$-q_1 z_1 + q_2 z_2 - s\,w + p\,y = 0, \tag{3}$$

where again $p = u'(y)$, $y = D\,x$, represents an optimum of P_1. See Tucker [1957], which is based on Kuhn and Tucker [1950].

4. Definition of an Optimal Continuing State

We seek to define a vector analogue of the self-preserving capital stock $\hat{z}\,(\varrho)$ of Section 1 using only models with finite, and actually quite short, horizons.

$P_2 = P_2\,(z, w; v, \alpha)$: *Maximize* $v\,(x_1) + \alpha\,v\,(x_2)$ *subject to* x_1, $x_2 \geqq 0$ *and the primal constraints of*

	x_1	x_2	$\geqq$
q_1	$-A$		$-z$
s_1	$-C$		$-w$
q_2	B	$-A$	0
s_2		$-C$	$-w$
q_3		B	z
$\leqq$	$-v\,(x_1)$	$-\alpha\,v'(x_2)$	

In this two-period problem the discount factor α, where $0 < \alpha \leqq 1$, corresponds to the $e^{-\varrho}$ of Section 2. It is a policy parameter expressing, with regard to a constant program ($x_1 = x_2 = x$), the planners' relative preference $\frac{1}{\alpha}$ for first-period over equal second-period small increments to consumption. For such a program $\frac{1}{\alpha}$ corresponds to Irving Fisher's concept of "impatience", see Fisher [1930].

P_2 implies *two-period stationarity* by specifying that the same *given* capital stock

$$z_1 = z_3 \equiv z$$

shall be present at the beginning and at the end of the two-period horizon. The first and last set of primal constraints therefore specify

$$A x_1 \leqq z \leqq B x_2. \tag{4}$$

There is no reason why, in the optimal program $(\hat{x}_1, \hat{x}_2)$ corresponding to any given z, w, α, one should also have

$$A \hat{x}_2 \leqq z \leqq B \hat{x}_1, \tag{5}$$

that is, that z can also be regarded as the capital stock handed over from the first into the second period. However, since technology and resource availability are assumed the same in the two periods, analogy with the catenary property of the optimal path in Section 2 supports the conjecture that, for given w, α, a choice $\hat{z}(\alpha)$ of z that makes (5) true represents a self-preserving capital stock for the problem

$P_\infty = P_\infty(z, w; v, \alpha)$: *Maximize* $\sum_{t=1}^{\infty} \alpha^{t-1} v(x_t)$ *on the set of bounded programs*

$$\{(x_t) \,|\, 0 \leqq x_t \leqq \xi,\ t = 1, 2, \ldots, \text{ for some } \xi\}$$

subject to the primal constraints

$$-A x_t \geqq -z,\ \ -C x_t \geqq -w,\ \ B x_t - A x_{t+1} \geqq 0,\ \ t = 1, 2, \ldots,$$

extending those of P_2 to an infinite horizon.

More explicitly, in support of the foregoing observations, one can show the following statements (i), (ii), (iii).

(i) if z, w, α are such that a solution $(\hat{x}_1, \hat{x}_2)$ of P_2 satisfies (5), then both $(\hat{x}_1, z)$ and $(\hat{x}_2, z)$ are solutions $[\hat{x}(\alpha), \hat{z}(\alpha)]$ of

$P(\alpha)$: Given w, v, α, maximize $v(x)$ subject to $x \geqq 0$ and the primal constraints of

	x	$\geqq$
q_1	$-A$	$-z$
q_2	B	z
s	$-C$	$-w$
$\leqq$	$-v'(x)$	

where z is given a value $z = \hat{z}(\alpha) \geqq 0$ such that, for some q_1, q_2, s satisfying the corresponding dual constraints and completementary slackness condition, one has

$$q_2 = \alpha q_1 . \tag{6}$$

Conversely,

(ii) if $(x, z) = [\hat{x}(\alpha), \hat{z}(\alpha)]$ solves $P(\alpha)$, then (x, x) solves $P_2(z, w; v, \alpha)$ in such a way that (5) is satisfied by $(x, x) = (\hat{x}_1, \hat{x}_2)$.

Finally

(iii) $(x, z) \geqq 0$ solves $P(\alpha)$ if and only if both

(a) $Bx \geqq z$ and

(b) the constant program with $x_t = x$ for all t solves $P_\infty(z, w; v, \alpha)$.

Thus, the search for a capital stock z that is self-preserving under the infinite-horizon maximization P_∞, and for an associated vector x of activity levels, is identical with the search for solutions (x, z) of the single-period problem $P(\alpha)$.

Note that $P(\alpha)$ differs from the ordinary concave programming problem P_1 by two additional constraints,

$$\left.\begin{array}{l} z_1 = z_2 \equiv z \\ q_2 = \alpha q_1 \end{array}\right\} \tag{7}$$

one on the "constant terms" in the primal constraints, and one on the dual variables, *and* by the treatment of z as an unknown, to be chosen so that the additional constraints on the dual variables can be satisfied. The order of z equals the number of constraints on (q_1, q_2).

5. Two Fixed-point Problems Equivalent to $P(\alpha)$

While $P(\alpha)$ thus appears to be a highly condensed equivalent of the search for a solution of P_∞ that is constant in time, it can be further condensed in either of two ways to result in two fixed-point problems, both equivalent to $P(\alpha)$ and hence to each other, one in the space of the primal variables x, the other in that of the dual variables $q_1 \equiv q$. To state the two problems, $P'(\alpha)$, $P''(\alpha)$, we assume that the technology matrix includes free disposal activities for all resources and capital goods,

$$\begin{matrix} (x) \\ \begin{bmatrix} -A \\ B \\ -C \end{bmatrix} \end{matrix} = \begin{matrix} (\bar{x}) \quad (x_A) \quad (x_B) \quad (x_C) \\ \begin{bmatrix} -\bar{A} & -I & 0 & 0 \\ \bar{B} & 0 & -I & 0 \\ -\bar{C} & 0 & 0 & -I \end{bmatrix} \end{matrix}$$

As a result, the inequality in the primal constraints in all problems considered can be changed to equality.

$P'(\alpha)$: *Given w, v, α, choose z such that a maximizer $x = \hat{x}(z)$ of $v(x)$*

	x	$=$
q	$-A + \alpha B$	$-(1 - \alpha) z$
s	$-C$	$-w$
$\leqq$	$-v'(x)$	

satisfies $Bx = z$.

$P''(\alpha)$: *Given w, v, α, choose $q^* \geqq 0$ such that maximization of $v(x) - (1 - \alpha) q^* Bx$ subject to the primal constraints of*

	x	$=$
q	$-A + B$	0
s	$-C$	$-w$
$\leqq$	$-v'(x) + (1 - \alpha) q^* B$	

permits $q = q^*$.

Interpretation of $P''(\alpha)$: If we define $z(x) \equiv Bx$ and interpret it as the end-of-period capital stock resulting from an activity vector x, the first set of constraints implies that also $Ax = z(x)$, thus requiring single-period stationarity. The maximand can then be written as

$$v^*(x) \equiv v(x) - q^* z(x) + \alpha q^* z(x).$$

Imagine a public agency that announces a price vector $q_1 = q^*$ at which the capital stock $z(x)$ is to be bought from it at the beginning of the period, while *at the same time* selling it back, for delivery at the end of the period, to that agency at present-value prices $q_2 = \alpha q^*$. Accepting these terms, the planners maximize a modified objective function $v^*(x)$ obtained from the social preference indicator $v(x)$ by adding the (negative) net gain (i. e., subtracting the positive loss) on the two simultaneous transactions with the agency. The stipulation $q = q^*$ ensures that the price vector q^* (constant over time before discounting) announced by the agency corresponds to the price vector q for the same capital goods, implicit in the maximization. A similar economic interpretation can be given to $P'(\alpha)$.

Proofs of equivalences of $P(\alpha)$, $P'(\alpha)$, $P''(\alpha)$: These are obtained by writing out the dual constraints and the complementary slackness conditions for each problem, and establishing equivalences by elementary implications.

$P'(\alpha)$ as a fixed point problem: Let $P'(z; \alpha)$ denote the problem differing from $P'(\alpha)$ in two respects, (a) that z is given instead of unknown, and (b) that the requirement $Bx = z$ is dropped. Let $\hat{x}(z, \alpha)$ denote the solution set of $P'(z; \alpha)$ as a set function of z. Then the solution $x = \hat{x}(\alpha)$ of $P'(\alpha)$ can be seen as a fixed point of the mapping $x \rightarrow \hat{x}(Bx, \alpha)$. The Kakutani fixed point theorem assures the existence* of a solution to $P'(\alpha)$.

$P''(\alpha)$ *as a fixed point problem:* Here the mapping connects q^* with the set $q(q^*)$ of values (normally a unique value) of the dual vector q associated with the problem $P''(q^*, \alpha)$ obtained from $P''(\alpha)$ by treating q^* as given and dropping the requirement $q = q^*$.

6. Non-unique Solutions

Nothing in the foregoing would suggest uniqueness of a solution $[\hat{x}(\alpha), \hat{z}(\alpha)]$ of $P(\alpha)$. A simple and straightforward example of the existence of more than one solution has been given by Gale [1968] and Sutherland [1967] in terms of a strictly concave utility function $U(z_1, z_2)$ of single-period initial and terminal scalar capital stocks z_1, z_2. While it may not be possible to obtain their example from the expression suggested by the present model,

$$U(z_1, z_2) = u[f(z_1) - z_2],$$

where $f(z)$ is one-period output of the single good from capital stock z and labor force $w = 1$, assumed increasing and strictly concave, similar examples with a strictly concave utility function $u(y)$ of a consumption vector produced by a technology of activity analysis type can also be constructed. For any α for which more than one solution of $P(\alpha)$ exists, particular interest attaches to that (those) solution(s) corresponding to a choice of z giving the highest constrained maximum of $v(x)$.

7. Possible Applications

Algorithms for the computation of solutions $[\hat{x}(\alpha), \hat{z}(\alpha)]$ of $P(\alpha)$ or $P'(\alpha)$ are suggested by the catenary property of solutions to P_2 and by the fixed point aspect of $P'(\alpha)$.

---

* This was observed with regard to P_∞ for a somewhat more general model by Sutherland (1966, 1967).

The model may provide a basis for estimating efficiency prices in which cost of capital use enters along with cost of labor and other resources. This can perhaps be done by using a continuing state as a first approximation to what is in reality a changing state. At a higher level of precision, a version of the present model of a continuing state could be imposed as the terminal stage of a dynamic model in which optimization over time results in non-proportional growth during earlier periods. (Similar proposals have been made in other contexts by Alan S. Manne [1970].)

The model might also provide an alternative approach to the theory of aggregation of production relationships. It is conceivable that the efficiency prices of the present model, and their variation under moderate virtual changes in α, $u(y)$, in the technical coefficients and in resource availabilities, may suggest definitions for aggregates of resource inputs, capital inputs and product outputs, and appropriate forms for aggregate production functions relating these aggregates with a locally good degree of approximation.

References

D. Cass: Optimum Growth in an Aggregative Model of Capital Accumulation. The Review of Economic Studies *32* (1965), p. 233–240.

D. Cass: Optimum Growth in an Aggregative Model of Capital Accumulation: A Turnpike Theorem. Econometrica *34* (1966), p. 833–850.

I. Fisher: The Theory of Interest (1930), reprinted by Augustus M. Kelley, New York (1961).

D. Gale: A Mathematical Theory of Economic Development. Bull. Am. Math. Assoc. *74* (1968), p. 207–223.

M. Inagaki: Utility Maximization over Infinite Time: A General Existence Theorem. Netherlands Economic Institute, Division of Balanced International Growth, Publ. No. 34/66 (Feb. 1966), and Utility Maximization: An Explicit Solution, Discussion Paper (May 1966).

S. Kakutani: A Generalization of Brouwer's Fixed Point Theorem. Duke Mathematical Journal *8* (1941), p. 457–459, reprinted in Newman (1968).

T. C. Koopmans: On the Concept of Optimal Economic Growth, in: The Econometric Approach to Development Planning, North Holland Publ. Co. and Rand McNally (1966), a re-issue of Pontificiae Academiae Scientiarum Scripta Varia *28* (1965), p. 225–300.

T. C. Koopmans [1967 a]: Objectives, Constraints and Outcomes in Optimal Growth Models. Econometrica *35* (1967), p. 1–15.

T. C. Koopmans [1967 b]: Intertemporal Distribution and "Optimal" Economic Growth. Ch. 5 of W. Fellner et al., Ten Economic Studies in the Tradition of Irving Fisher, Wiley, New York (1967), p. 95–126.

H. W. Kuhn and A. W. Tucker: Nonlinear Programming, in: J. Neyman, Ed., Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Univ. of California Press (1951), p. 481–492.

J. A. Mirrlees: Optimum Growth When Technology is Changing. Review of Economic Studies *34* (1967), p. 95–124.

P. Newman, Ed.: Readings in Mathematical Economics *1*, Value Theory, Johns Hopkins Press, Baltimore (1968).

F. P. Ramsey: A Mathematical Theory of Saving. The Economic Journal *38* (1928), p. 543–559.

P. A. Samuelson: A Catenary Turnpike Theorem Involving Consumption and the Golden Rule. The American Economic Review *60* (1965), p. 486–496.

W. R. Sutherland: On Optimal Development Programs when Future Utility Discounted. ORC-18, Operations Research Center, Univ. of California, Berkeley (Aug. 1966) and Doctoral Dissertation, Brown University, Providence, R. I. (1967).

W. R. Sutherland: On Optimal Development in a Multi-Sectoral Economy: The Discounted Case Review of Economic Studies *37* (1970), p. 585–589.

A. W. Tucker: Linear and Nonlinear Programming. Operations Research *5* (1957), p. 244–257.

Address of author: Prof. Tjalling C. Koopmans, Cowles Foundation, Yale University, Box 2125, Yale Station, New Haven, Conn. 06520, U. S. A.

Printed in Austria