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DIFFERENTIAL GAMES AND RELATED TOPICS

NONCOOPERATIVE EQUILIBRIA AND STRATEGY SPACES IN AN OLIGOPOLISTIC MARKET

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1. Introduction

In this paper we examine a nonsymmetric market structure which represents an oligopolistic market with quadratic payoff functions for the firms. The same market structure is used to set up two different games. In the first game the firms are assumed to name price as their strategy; in the second they name the amount offered for sale to the market. In the latter a market mechanism is assumed to exist. This mechanism determines the prices that clear the supplies offered.

The two game models correspond to those suggested by Cournot and by Bertrand and Edgeworth [1]. However they are generalized to account for product differentiation and for nonsymmetry with any number of players. The two solutions are then compared.

It is of interest to note that the economic background in both cases is identical. The only difference in the two models is that different strategic variables have been chosen. The equilibria are different, however when the players are symmetrical, then as $n \to \infty$ the noncooperative equilibria in both games approach the same limit.

The price model is applied to the automobile market in the United States.

2. The Price Game

The following notation is used:

Π_i = the net revenue for the ith firm.

p_i = the price charged by the ith firm.

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w_i = a weighting factor reflecting the nonsymmetry in the market of the product of the ith firm.

$\sum_{i=1}^{n} w_i = 1$.

$\bar{p}$ = the weighted average price ($\bar{p} = \sum_{j=1}^{n} w_j p_j$).

β = price sensitivity of overall demand.

V = the price at which demand is zero.

c_i = the unit average cost of the ith firm (average costs are assumed to be constant)

K_i = the fixed costs of the ith firm.

Let the demand for the product of the ith firm be given by:

$$(1) \qquad d_i = w_i \beta(\mathrm{V} - p_i - \gamma(p_i - \bar{p}))$$

The revenue of the ith firm may be expressed as:

$$(2) \qquad \Pi_i = (p_i - c_i)(w_i \beta(\mathrm{V} - p_i - \gamma(p_i - \bar{p}))) - K_i \,.$$

For a noncooperative equilibrium we must solve the set of equations resulting from taking the derivatives of (2) with respect to p_i and setting them equal to zero.

$$(3) \qquad \frac{\partial \Pi_i}{\partial p_i} = \beta w_i [-(2(1+\gamma) - \gamma w_i) p_i + \gamma \sum w_j p_j + \mathrm{V} + \{1 + \gamma(1 - w_i)\} c_i] = 0 \,.$$

We need to introduce extra notation in order to examine the solution of the equations (3) in matrix form.

Let $S = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix}$ a square matrix with 1 for every entry.

$1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ a column matrix with 1 for each entry.

$$W = \lfloor w_i \rceil = \begin{bmatrix} w_1 & & \bigcirc \\ & w_2 & \\ & & \ddots \\ \bigcirc & & w_n \end{bmatrix} \quad \text{a diagonal matrix with entries } w_i.$$

In equations (3) we divide by $\beta w_i \gamma$. For ease in notation set

$$\Delta = \frac{2(1+\gamma)}{\gamma}.$$

The equations (3) can be written in matrix notation as:

$$(4) \qquad (\Delta I - W - SW)p = \frac{V}{\gamma}\,\hat{1} + \left(\frac{\Delta}{2} I - W\right)c,$$

where

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

In order to solve (4) we wish to evaluate the inverse of $(\Delta I - W - SW)$, we write this as $[(\Delta W^{-1} - I) - S]\,W$. Let $\Delta W^{-1} - I = X$ and $[X - S]^{-1} = Y$. We may write δ_{ij} in terms of entries of X and Y as:

$$(5) \qquad \delta_{ij} = \sum_k (x_i \delta_{ik} - 1)\, y_{kj} = x_i y_{ij} - \sum_k y_{kj}$$

call $R_j = \sum_k y_{kj}$. As Y is symmetric we can write

$$(6) \qquad y_{ij} = \frac{R_j + \delta_{ij}}{x_i} = \frac{R_i + \delta_{ji}}{x_j}.$$

If $i \neq j$ then $\dfrac{R_i}{x_j} = \dfrac{R_j}{x_i}$ hence

$$(7) \qquad R_i = \frac{R_j x_j}{x_i}.$$

But $R_j x_j$ is a constant which we may call Q and rewrite (7) as:

$$R_i = \frac{Q}{x_i} \ . \tag{8}$$

Returning to (6) we have:

$$y_{ij} = \frac{\frac{Q}{x_j}}{x_i} = \begin{cases} \frac{Q}{x_i x_j} & \text{for} \quad i \neq j \\ \frac{Q}{x_i^2} + \frac{1}{x_i} & i = j \end{cases} \tag{9}$$

call

$$Z = \begin{bmatrix} \frac{1}{x_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \frac{1}{x_n} \end{bmatrix} ,$$

$z_i = 1/x_i$ then we can write:

$$[X - S]^{-1} = QZSZ + Z \ . \tag{10}$$

Evaluating Q:

$$\frac{Q}{x_i} = R_i = \sum_j y_{ij} = \sum_j \frac{Q}{x_i x_j} + \frac{1}{x_i} \ ,$$

hence

$$Q = Q \sum_j \frac{1}{x_j} + 1 = \frac{1}{1 - \sum_j \frac{1}{x_j}} = \frac{1}{1 - \sum_j z_j} \ . \tag{11}$$

Now we have the inverse of $X - S$ hence the prices which can be written as

$$p = W^{-1}(QZSZ + Z)\left(\frac{V}{\gamma}\,\hat{1} + \left(\frac{\Delta}{2} I - W\right) C\right). \tag{12}$$

Remembering that $z_i = w_i/(\Delta - w_i)$ and $\Sigma\, z_j = 1 - 1/Q$ we may write a specific price p_i as:

$$p_i = \frac{z_i}{w_i} \left\{ \frac{V}{\gamma} \left(Q \sum z_j + 1\right) + \tfrac{1}{2} \left[(\Delta - 2w_i) c_i + Q \sum z_j (\Delta - 2w_j)\, c_j \right] \right\} \ , \tag{13}$$

which simplifies to

$$p_i = \frac{1}{\Delta - w_i}\left\{\frac{V}{\gamma} Q + \tfrac{1}{2}[(\Delta - 2w_i)c_i + Q\sum z_j(\Delta - 2w_j)c_j]\right\} . \tag{14}$$

There are two special cases and a check that should be made. When (1) all costs are the same, in other words $c_i = c$ and (2) market conditions are the same, in other words $w_i = w$. The check comes when we assume that both $c_i = c$ and $w_i = w$ and investigate the resultant simplification to see if it gives us the formula in the symmetric case.

2.1. A Special Case: Equal Costs

If we assume that $c_\iota = c$ but that the w_i are different equation (14) may be written as:

$$p_i = \frac{1}{\Delta - w_i}\left\{\frac{V}{\gamma} Q + \frac{c}{2}[(\Delta - 2w_i) + Q\sum z_j(\Delta - 2w_j)]\right\} . \tag{15}$$

We wish to sum the term $\sum z_j(\Delta - 2w_j)$. This can be written as:

$$\sum z_j(\Delta - w_j) - \sum z_j w_j , \tag{16}$$

or as

$$\Delta \sum z_j - 2\sum z_j w_j . \tag{17}$$

However by setting $z_j = w_j/(\Delta - w_j)$ in the first part of (16) we have immediately that $\sum z_j(\Delta - \omega_j) = 1$. Furthermore as $Q = 1/(1 - \sum z_j)$ then $z_j = (Q-1)/Q$ which may be substituted in the first part of (17). Letting

$$1 - \sum z_j w_j = \Delta\left(\frac{Q-1}{Q}\right) - 2\sum z_j w_j \tag{18}$$

or

$$\sum z_j w_j = \Delta\left(\frac{Q-1}{Q}\right) - 1 , \tag{19}$$

hence

$$\sum z_j(\Delta - 2w_j) = 2 - \Delta\left(\frac{Q-1}{Q}\right). \tag{20}$$

Substituting (20) into (15) we obtain:

$$
\begin{aligned}
p_i &= \frac{1}{\Delta - w_i}\left\{\frac{VQ}{\gamma} + \frac{c}{2}(\Delta - 2w_i + 2Q - \Delta(Q-1))\right\} \\
(21) \qquad &= \frac{1}{\Delta - w_i}\left\{\frac{V}{\gamma}Q + \frac{c}{2}(2\Delta - 2w_i + Q(2-\Delta))\right\} .
\end{aligned}
$$

This can be rewritten in a better form as:

$$
(22) \qquad \underline{p_i = c + \frac{Q}{\Delta - w_i}\left(\frac{V-c}{\gamma}\right)} .
$$

2.2. A Special Case: A Symmetric Market but Unequal Costs

Let $w_i = w = 1/n$, then $z_i = (1/n)/(\Delta - 1/n)$ and $Q = (n\Delta - 1)/(n\Delta - 1 - n)$. We may write (14) as:

$$
\begin{aligned}
p_i &= \frac{n}{n\Delta - 1}\left\{\frac{V}{\gamma}Q + \tfrac{1}{2}\left[\left(\frac{n\Delta - 2}{n}\right)c_i + Q\sum\left(\frac{1}{n\Delta - 1}\right)\left(\frac{n\Delta - 2}{n}\right)c_j\right]\right\} \\
(23) \qquad &= \frac{nV}{(n\Delta - 1 - n)\gamma} + \left(\frac{n}{n\Delta - 1}\right)\left(\frac{n\Delta - 2}{2n}\right)\left[c_i + \frac{1}{n\Delta - 1 - n}\sum c_j\right] .
\end{aligned}
$$

Substituting in $\Delta = \dfrac{2(1+\gamma)}{\gamma}$ we obtain:

$$
(24) \qquad \underline{p_i = \frac{V}{2 + \left(\frac{n-1}{n}\right)\gamma} + \frac{1 + \gamma\left(\frac{n-1}{n}\right)}{2 + \left(\frac{2n-1}{n}\right)\gamma}\left[c_i + \frac{\gamma}{2 + \frac{n-1}{n}\gamma}\bar{c}\right]} .
$$

2.3. The Symmetric Case: Two Checks

In (22) we use the added condition of $w_i = w = 1/n$. This gives $z_i = 1/(n\Delta - 1)$ hence $Q = (n\Delta - 1)/(n\Delta - 1 - n)$. We obtain:

$$p_i = c + \frac{n\Delta - 1}{n\Delta - 1 - n}\left(\frac{n}{n\Delta - 1}\right)\left(\frac{V - c}{\gamma}\right)$$

(25)

$$= c + \frac{1}{2\frac{1+\gamma}{\gamma} - \frac{1}{n} - 1}\left(\frac{V - c}{\gamma}\right).$$

This simplifies to:

(26)
$$p_i = \frac{V + c\left(1 + \frac{n-1}{n}\gamma\right)}{2 + \frac{n-1}{n}\gamma}.$$

This is the symmetric solution as has been shown previously [2].

In (24) we use the added condition that $c_i = c$. This immediately gives

(27)
$$p_i = \frac{V}{2 + \left(\frac{n-1}{\gamma}\right)\gamma} + \frac{1 + \gamma\left(\frac{n-1}{n}\right)}{2 + \left(\frac{2n-1}{n}\right)\gamma}\left[\frac{c\left(2 + \frac{n-1}{n}\gamma + \gamma\right)}{2 + \frac{n-1}{n}\gamma}\right],$$

or

(28)
$$p_i = \frac{V + c\left(1 + \frac{n-1}{n}\gamma\right)}{2 + \frac{n-1}{n}\gamma}.$$

3. The Quantity Game

The demand structure in the price game was based upon the assumption that the consumers' preferences could be represented in aggregate by a quadratic utility function. From this function demands could be deduced. For the quantity or Cournot game the demand structure must be inverted, i.e. we need to express prices as functions of quantities offered for sale, not vice-versa. When there is product differentiation the generation of prices in the extended Cournot model poses an interesting problem closely related to general equilibrium and rationing analysis.

We assume that consumer preferences can be represented by a general quadratic utility function. Our somewhat strong special assumption is that to a first approximation there is no income effect between this class of goods and the remainder of the consumer's purchases. In terms of quantity the consumer's utility function is:

$$\frac{u(x)}{\lambda} = \mathrm{V}\sum_{i=1}^{n} x_i - \frac{1}{2\beta(1+\gamma)}\left\{\frac{\sum_{i=1}^{n} x_i^2}{w_i} + \gamma(\sum x_i)^2\right\}, \tag{29}$$

where λ = marginal worth of money.

In the simple case of quantity duopoly with an undifferentiated product, price is easily determined by the demand relation

$$p = f\left(\sum_{i=1}^{n} q_i\right). \tag{30}$$

When there are differentiated products we must solve a limited general equilibrium model to determine the set of prices which will just clear all markets. A detailed analysis of this problem and its relationship to rationing and general equilibrium has been given by Levitan [3].

3.1. The Cournot Equilibrium

Let x = the vector $(x_1, x_2, ..., x_n)$ of quantities offered by the firms.
p = the vector $(p_1, p_2, ..., p_n)$ of prices.

$$I = \begin{bmatrix} 1 & \cdots & & 0 \\ 0 & 1 & & \\ \vdots & & 1 & \\ & & & \ddots \\ 0 & & & 1 \end{bmatrix}.$$

c = the vector $(c_1, c_2, ..., c_n)$ of costs of the firms. β, V and γ are parameters.

We may express demand in terms of price by equation (31).

$$x = \beta W(\mathrm{V}\hat{1} - ((1+\gamma)I - \gamma SW)p), \tag{31}$$

hence

$$p = [(1+\gamma)I - \gamma SW]^{-1}\left(\mathrm{V}\hat{1} - \frac{1}{\beta} W^{-1}x\right)$$

$$= \frac{1}{\gamma} W^{-1}\left(\frac{1+\gamma}{\gamma}\; W^{-1} - S\right)^{-1}\left(\mathrm{V}\hat{1} - \frac{1}{\beta} W^{-1}x\right)$$

$$= \frac{1}{1+\gamma}\,(I + \gamma SW)\left(\mathrm{V}\hat{1} - \frac{1}{\beta} W^{-1}x\right),$$

or

(32) $$p = \mathrm{V}\hat{1}\; \frac{1}{\beta(1+\gamma)}\,(W^{-1} + \gamma S)x\,.$$

The ith payoff may be expressed as

(33) $$\Pi_i = x_i(p_i - c_i)$$

$$= x_i\left(\mathrm{V} - \frac{1}{\beta(1+\gamma)}\left(\frac{x_i}{w_i} + \gamma \sum x_j\right) - c_i\right).$$

Differentiating with respect to quantity we obtain:

(34) $$\frac{\partial \Pi_i}{\partial x_i} = \mathrm{V} - \frac{1}{\beta(1+\gamma)}\left(\frac{x_i}{w_i} + \gamma \sum x_j\right) - c_i - \frac{x_i}{\beta(1+\gamma)}\left(\frac{1}{w_i} + \gamma\right).$$

Setting (34) equal to zero we have:

(35) $$\mathrm{V} - c_i - \frac{1}{\beta(1+\gamma)}\left(\left(\frac{2}{w_i} + \gamma\right)x_i + \gamma \sum x_j\right) = 0\,.$$

Writing this in matrix notation:

(36) $$\frac{\gamma}{\beta(1+\gamma)}\left(\frac{1}{\gamma}\,(2W^{-1} + \gamma I) + S\right)x = \mathrm{V}\hat{1} - c\,.$$

We note that

(37) $$(Z^{-1} + S)^{-1} = (Z - qZSZ)\,,$$

where

$$q = \frac{1}{1 + \sum z_i} \ .$$

In this case (equation (36)).

$$z_i = \frac{\gamma w_i}{2 + \gamma w_i} \ .$$

Specifically

$$q = \frac{1}{1 + \gamma \sum \frac{w_j}{2 + \gamma w_j}} \ .$$

Solving for x we obtain:

(38) $$x = \frac{\beta(1+\gamma)}{\gamma} (Z - qZSZ)(\mathrm{V}\hat{1} - c) \, ,$$

and

(39) $$p = \mathrm{V}\hat{1} - \frac{1}{\gamma} (W^{-1} + \gamma S)(Z - qZSZ)(\mathrm{V}\hat{1} - c) \, .$$

Multiplying out the factors on the right of (39)

(40) $$p = \mathrm{V}\hat{1} - \frac{1}{\gamma} (W^{-1} + \gamma S - qW^{-1}ZS - \gamma q \, \hat{1}\hat{1}^{\mathrm{T}} Z \hat{1}\hat{1}^{\mathrm{T}})(Z)(\mathrm{V}\hat{1} - c) \, ,$$

however,

$$\hat{1}^{\mathrm{T}} Z \hat{1} = \frac{1-q}{q} \ ,$$

hence

(41) $$p = \mathrm{V}\hat{1} - \frac{1}{\gamma}(W^{-1} + \gamma S - qW^{-1}ZS - \gamma(1-q)S)(Z)(\mathrm{V}\hat{1} - c) \, ,$$

or:

(42) $$p = \mathrm{V}\hat{1} - \frac{1}{\gamma} (W^{-1} + q(\gamma I - W^{-1}Z) S) (Z) (\mathrm{V}1 - c)$$

which simplifies to:

$$p = \mathrm{V}\left[\hat{1} - \frac{1}{\gamma}(W^{-1}Z\hat{1} + q(\gamma I - W^{-1}Z)SZ\hat{1}\right]$$

$$+\frac{1}{\gamma}(W^{-1} + q(\gamma I - W^{-1}Z)S)ZC, \tag{43}$$

writing $S = \hat{1}\,\hat{1}^{\mathrm{T}}$ in (43) gives us $SZ\hat{1} = \hat{1}\,\hat{1}^{\mathrm{T}}Z\hat{1} = \hat{1}((1-q)/q)$ hence

$$p = \mathrm{V}[\hat{1}(1-(1-q)) - \frac{1}{\gamma}W^{-1}Z\hat{1}(\hat{1}-(1-q))] \tag{44}$$

$$+\frac{1}{\gamma}W^{-1}ZC + \frac{q}{\gamma}(\gamma I - W^{-1}Z)SZC\,,$$

or

$$p = q\mathrm{V}\left[\hat{1} - \frac{1}{\gamma}W^{-1}Z\hat{1}\right] + \left\{\frac{c_i}{2+\gamma w_i}\right\} + q(\gamma I - W^{-1}Z)\sum\frac{w_j c_j}{2+\gamma w_j}\,, \tag{45}$$

hence

$$p_i = q\mathrm{V}\left\{\frac{1+\gamma w_i}{2+\gamma w_i}\right\} + \left\{\frac{c_i}{2+\gamma w_i}\right\} + q\gamma\left\{\frac{1+\gamma w_i}{2+\gamma w_i}\right\}\sum\frac{w_j c_j}{2+\gamma w_j}\,, \tag{46}$$

or

$$p_i = q\left(\mathrm{V} + \gamma\sum\frac{w_j c_j}{2+\gamma w_j}\right)\left\{\frac{1+\gamma w_i}{2+\gamma w_i}\right\} + \left\{\frac{c_i}{2+\gamma w_i}\right\}. \tag{47}$$

We may check some special cases.

(A) Suppose $\gamma = 0$. This implies $q = 1$,

$$p = \tfrac{1}{2}\mathrm{V}\hat{1} + \tfrac{1}{2}c$$

$$= \tfrac{1}{2}(\mathrm{V}\hat{1} + c)\,,$$

which is the monopoly solution.

(B) Suppose $\gamma \to \infty$. This implies $z_i \to 1$, hence $q \to 1/(n+1)$.

$$p \to \frac{1}{n+1}(\mathrm{V} + \sum c_j) + 0$$

$$= \frac{\mathrm{V}}{n+1} + \frac{n}{n+1}\bar{c}\,,$$

where $\bar{c}$ is the average cost.

(C) Suppose $w_i = 1/n$, this formula becomes

$$(48) \qquad p_i = \frac{n+\gamma}{2n+(n+1)\gamma}\left(V + \frac{\gamma n}{2n+\gamma}\,\bar{c}\right) + \frac{nc_i}{2n+\gamma}\,.$$

We now compare the noncooperative equilibrium obtained from regarding price as the independent variable or quantity as the independent variable. For $n = 1$ we see below from the two general symmetric market formulae that we obtain the same result. Similarly for $n \to \infty$ we obtain the same limit (when all $c_i = \bar{c}$).

$$(49) \qquad p_i\,(\text{Edgeworth}) = \frac{n}{2n+(n-1)\gamma}\left(V + \left\{\frac{n+(n-1)\gamma}{2n+(2n-1)\gamma}\right\}\gamma\bar{c}\right) + \frac{(n+(n-1)\gamma)\bar{c}}{2n+(2n-1)\gamma}\,,$$

$$(50) \qquad p_i\,(\text{Cournot}) = \frac{n+\gamma}{2n+(n+1)\gamma}\left(V + \frac{\gamma n}{2n+\gamma}\,\bar{c}\right) + \frac{n}{2n+\gamma}\,\bar{c}\,.$$

For $n = 1$,

$$p_i(\text{Edgeworth}) = \tfrac{1}{2}\left(V + \frac{\gamma}{2+\gamma}\,\bar{c}\right) + \frac{\bar{c}}{2+\gamma} = \tfrac{1}{2}(V+\bar{c})\,,$$

$$p_i\,(\text{Cournot}) = \frac{1+\gamma}{2(1+\gamma)}\left(V + \frac{\gamma}{2+\gamma}\,\bar{c}\right) + \frac{\bar{c}}{2+\gamma} = \tfrac{1}{2}(V+\bar{c})\,.$$

For $n \to \infty$

$$p_i\,(\text{Edgeworth}) = \frac{1}{2+\gamma}\left(V + \frac{1}{2}\left(\frac{1+\gamma}{1+\gamma}\right)\gamma\bar{c}\right) + \frac{\bar{c}}{2+\gamma}$$

$$= \frac{V}{2+\gamma} + \bar{c}\,\frac{\gamma+1}{\gamma+2}\,.$$

$$p_i\,(\text{Cournot}) = \frac{1}{2+\gamma}\left(V + \frac{\gamma}{2}\,\bar{c}\right) + \frac{\bar{c}}{2+\gamma}$$

$$= \frac{V}{2+\gamma} + \bar{c}\left(\frac{\gamma+1}{\gamma+2}\right)\,.$$

We note that for $\gamma \to \infty$ these both give $p_i = \bar{c}$.

Returning to (49) and (50) for $c_i = c$ we may observe that for $n > 1$ and $\gamma > 0$:

$$p_c > p_e \,.$$

This follows from observing that

$$\frac{n+\gamma}{2n+(n+1)\gamma} > \frac{n}{2n+(n-1)\gamma}\,,$$

then observing that price is a weighted average of costs c_i and V, however $V \geqslant \bar{c}$ and V appears with a larger weighting in (50).

4. A Simulation of an Automobile Market

If the automobile industry were in noncooperative equilibrium we could use the noncooperative solution together with industry information to calculate parameter estimates for the automobile industry. A very crude set of parameters estimates can be obtained from the 1965 figures for the three major automobile companies. The calculations given are merely meant to be suggestive of an approach and not a careful econometric estimate of the automobile industry. The game constructed will be somewhat like the industry.

Owing to the lack of unconsolidated figures, several approximations and simplifications will be made. In particular we consider only those firms and their world-wide competition. We lump all automotive units such as autos, trucks and tractors. We know that civilian nonauto products and defense accounted for \$1.9 billion for General Motors or approximately 10% of sales. Rather than break out the multiproduct features explicitly we implicitly include them by inflating the price of an automotive unit so that we make the crude approximation that there is a constant ratio in multiproduct sales. Furthermore the distribution system is not accounted for explicitly. The firms obtain wholesale prices but the cars are sold at retail.

The first table gives sales (not corrected for total income which is slightly different), total assets and before tax profits.

Table 1

	Sales ($\times 10^6$)	Assets ($\times 10^6$)	Profits ($\times 10^6$)
General Motors	20,734	11,479	2,126
Ford	11,537	7,596	710
Chrysler	5,300	2,934	233

Variable costs are assumed to correspond to the item "costs of products sold" on the earnings statements in the annual reports. Depreciation, amortization, administrative expenses, debt servicing and various pension and retirement payments are assumed to define fixed costs. Taxes are reported separately, they include foreign taxes.

Table 2

	Variable costs ($\times 10^6$)	Fixed costs ($\times 10^6$)	Taxes ($\times 10^6$)
General Motors	15,250	1,559	1,966
Ford	8,853	1,401	596
Chrysler	4,121	746	213

A crude indication of the physical size of the corporations is given by the value placed on plant, equipment, property and special tools. These figures, of course, are highly influenced by accounting practices and especially when land values have increased may grossly underestimate the worth of the capital investment.

Table 3

	Property and plant ($\times 10^6$)	Special machinery ($\times 10^6$)
General Motors	4,161	455
Ford	2,574	446
Chrysler	887	180

World sales of automotive equipment including trucks and tractors is given in table 4.

Table 4

	Sales ($\times 10^3$)	Percentage
General Motors	7,278	52.2
Ford	4,595	32.9
Chrysler	2,077	14.9
	13,950	100.0

From the above information on the basis of the assumption of linear costs we may write:

$$p_1 = 20{,}734/7.278 = 2{,}849 \qquad c_1 = 15{,}250/7.278 = 2{,}095$$
$$p_2 = 11{,}537/4.595 = 2{,}511 \qquad c_2 = 8{,}853/4.595 = 1{,}927$$
$$p_2 = 5{,}300/2.077 = 2{,}552 \qquad c_3 = 4{,}121/2.077 = 1{,}984\,.$$

We assume that the demand for the automotive products of any firm i is given by:

$$(51) \qquad q_i = \beta w_i [\mathrm{V} - p_i - \gamma(p_i - \textstyle\sum w_j p_j)] \,, \quad i = 1, 2, 3\,.$$

From (15) we have

$$(52) \qquad p_i = \frac{1}{\Delta - w_i}\left\{\frac{\mathrm{V}}{\gamma} Q + \tfrac{1}{2}\left[(\Delta - 2w_i) c_i + Q\textstyle\sum z_j(\Delta - 2w_j)\, q\right]\right\}\,.$$

We know that $w_1 + w_2 + w_3 = 1$ or $w_3 = 1 - w_1 - w_2$ thus we have five undetermined parameters β, α ($\mathrm{V} = \alpha/\beta$ hence we determine α rather than V), γ, w_1 and w_2.

In (52)

$$(53) \qquad \mathrm{V} = \alpha/\beta\,, \quad \Delta = \frac{2(1+\gamma)}{\gamma}\,, \quad z_i = \frac{1}{x_i}\,,$$

$$x_i = \Delta\frac{1}{w_i} - 1\,, \quad \text{and} \quad Q = \frac{1}{1 - \sum_j z_j}\,.$$

From the three equations of the form

$$(54) \qquad q_i = \beta w_i (\mathrm{V} - (1+\gamma) p_i + \gamma \textstyle\sum p_j w_j)\,,$$

we obtain by subtraction:

$$(55) \qquad \frac{\frac{q_1}{w_1} - \frac{q_2}{w_2}}{p_2 - p_1} = \frac{\frac{q_1}{w_1} - \frac{q_2}{w_3}}{p_3 - p_1} = (1+\gamma)\beta$$

from which we derive

$$(56) \qquad w_2 w_3 q_1 (p_3 - p_2) + w_1 w_3 q_2 (p_1 - p_3) + w_1 w_2 q_3 (p_2 - p_1) = 0\,.$$

Let us call $q_i(p_j - p_k) = z_i$, we may rewrite (56) as

(57) $$z_1 w_2 w_3 + z_2 w_1 w_3 + z_3 w_1 w_2 = 0 .$$

We know that $w_1 + w_2 + w_3 = 1$, hence

(58) $$w_3 = 1 - w_1 - w_2 .$$

Substituting in (57) we obtain

(59) $$z_1 w_2(1 - w_1 - w_2) + z_2 w_2(1 - w_1 - w_2) + z_3 w_1 w_2 = 0 ,$$

giving

(60) $$-z_2 w_1^2 - z_1 w_2^2 + (z_1(1 - w_1) - z_2 w_1 + z_3 w_1) w_2 + z_2 w_1 = 0 .$$

Dividing (60) by $-z_1$ we obtain:

(61) $$w_2^2 - \left(1 + \frac{z_3 - z_1 - z_2}{z_1} w_1\right) w_2 - \frac{z_2}{z_1} w_1(1 - w_1) = 0 ,$$

set

$$\frac{z_3 - z_1 - z_2}{z_1} = r_1 , \quad \frac{z_2}{z_1} = r_2 .$$

We may rewrite (61) as:

(62) $$w_2^2 - (1 + r_1 w_1) w_2 - r_2 w_1(1 - w_1) = 0 .$$

We solve this equation to obtain w_2 as a function of w_1. We then search through successive values of w_1 until we obtain a positive root. Returning to equation (56) we may express β as a function of w_1 and γ.

(63) $$\beta(w_1, \gamma) = \frac{\frac{q_1}{w_1} - \frac{q_2}{w_2}}{(p_2 - p_1)(1 + \gamma)} .$$

Using this in equation (54) to eliminate β we may solve for V.

(64) $$\mathrm{V}(w_1, \gamma) = \frac{q_1}{\beta w_1} + (1 + \gamma) p_1 - \gamma \sum p_j w_j .$$

This now leaves us the problem of estimating w_1 and γ which we do by the Chebychef Criterion of minimizing the maximum of the absolute value of the ratio of the deviation of predicted from observed prices $|(p_i - \hat{p}_i)/\hat{p}|$.

Using the crude aggregated data for p_i and c_i obtained from the yearly reports we have:

$p_i = 2{,}849$

V = 533,678	$\beta = 26.3$	$\gamma = 1{,}988$
$w_1 = .75$	$w_2 = .17$	$w_3 = .08$

maximum deviation = .0080

These estimates appear to be somewhat startling as can be seen from the β which implies that a $100 cut by all firms would result in the sale of 2,600 more automobiles! We noted previously however, that the aggregation used to obtain the observed average prices does not appear to be reasonable owing to the multiproduct nature of the firms. General Motors especially has an important part of its business (and hence costs and sales) in markets other than vehicles. We suspect that a more detailed gathering of statistics would somewhat lessen the differences in observed aggregate average prices which apparently have General Motors prices (and costs) considerably above the others. We reduce this difference (of approximately $325) by $50 and $100 in order to view the effect on our parameter estimation. Setting p_1 = $2,799 and then p_i = $2,749 we obtain:

$p_1 = 2{,}799$

V = 6,039.3	$\beta = 4{,}198.5$	$\gamma = 9.28$
$w_1 = .70$	$w_2 = .20$	$w_3 = .10$

maximum deviation = .0085

$p_i = 2{,}749$

V = 3,951	$\beta = 10{,}889$	$\gamma = 2.2$
$w_1 = .65$	$w_2 = .24$	$w_3 = .11$

maximum deviation = .0124

A quick crude check of the above shows that when $p_1 = 2{,}799$

$$e = \frac{p\,\Delta q}{q\,\Delta p} \simeq .84$$

which seems to be somewhat low. When $p_1 = 2{,}749$

$$e \simeq 2.1\ ,$$

this appears to be somewhat high. It is evident that for specific econometric use of our method we need better statistics on prices and costs than the ones we have used. However by means of a computer program not given here, using slightly different estimates the joint maximum, price noncooperative equilibrium efficient prices and consumers' surplus are calculated.

	Weights w_i	Average costs
General Motors	0.65	1870 *
Ford	0.236	1927
Chrysler	0.114	1984

* Our crude calculations give the average costs of General Motors higher than the others. We believe this to be due to the crudeness of our aggregation (2,095). For this calculation we reduce General Motors' costs.

These figures give $\beta = 5764$ and $\gamma = 4.6$

	p_1	p_2	p_3	P_1	P_2	P_3	φ	Total surplus
Joint max.	3486	3515	3543	10156	3272	1394	39730	22233
Actual	2667	2511	2552	6528	2683	1180	54077	27492
Price								
Noncoop.eq.	2703	2572	2565	6813	2560	1133	54625	27813
Efficient Pt.	1870	1927	1984	0	0	0	64637	29644

The p_i are prices, P_i are net variable revenues (no overheads are subtracted and the φ is the consumer utility.

We make no pretense at accuracy; the calculations are offered only to suggest the relative sizes of the different solutions. For example it appears that the actual market is very close to and slightly more inefficient than the price noncooperative equilibrium.

The relative efficiencies of the solutions are approximately in the ratio of 100:94:92.5:75 ranging from the efficient solution to total cooperation by the firms. The loss to the public is, of course greatest when the firms collude.

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