Increasing Risk II: Its Economic Consequences*

MICHAEL ROTHSCHILD

Harvard University, Cambridge, Massachusetts 02138

AND

JOSEPH E. STIGLITZ

Cowles Foundation, Yale University, New Haven, Connecticut 06520

Received April 27, 1970

I. INTRODUCTION

The first part of this paper [11] established the equivalence of the following three statements comparing random variables \( X \) and \( Y \) with the same expected value:

(a) “All risk averters—those with concave utility functions—prefer \( X \) to \( Y \).”

(b) “\( Y \) is equal to \( X \) plus some noise.”

(c) “\( Y \) has more weight in its tails than \( X \).”\(^1\)

The equivalence of these three apparently different concepts seemed to us a good argument that our definition was the natural definition of increasing risk. It also suggested a useful multiplicity of approaches to investigations of the effect of risk on economic decisions. This paper investigates two such approaches.

* Support from the National Science Foundation and the Ford Foundation is gratefully acknowledged.

\(^1\) Formal versions of (a), (b), and (c) are:

(a) \( EU(X) > EU(Y) \) for all concave \( Y \).

(b) There exists a random variable \( Z \) such that \( Y \) has the same distribution as \( X + Z \), where \( E[Z \mid X] = 0 \) for all \( X \).

(c) If the points of increase of \( F \) and \( G \), the distribution functions of \( X \) and \( Y \), are confined to the closed interval \([a, b]\), then if

\[
T(y) = \int_a^y (F(x) - G(x)) \, dx, \quad T(y) \geq 0 \text{ and } T(b) = 0.
\]
The first concerns the economic effects of increasing risk. Our paper contained some sharp criticism of the conventional mean-variance treatment of this problem. Mindful that counsels of perfection are best accompanied by demonstrations of the possibility of attaining virtue, we committed ourselves to demonstrating that our approach could yield “the kind of comparative static results that economists are interested in [16].” The next section of our paper is our attempt to deliver on that promise. If an individual’s utility is a function of some control parameter \( \alpha \), and a random variable \( \theta \), then the individual will choose \( \alpha \) to maximize his expected utility

\[
\max_\alpha \int U(\theta, \alpha) \, d\theta. \tag{1}
\]

The optimal \( \alpha \) must satisfy

\[
\int \frac{\partial U(\theta, \alpha)}{\partial \alpha} \, d\theta = EU_\alpha(\theta, \alpha) = 0. \tag{2}
\]

Let \( \alpha^* \) be the unique solution to (2), and assume that in the neighborhood of \( \alpha^* \), \( U \) is monotone decreasing in \( \alpha \). If \( U_\alpha(\theta, \alpha) \) is a concave function of \( \theta \), an increase in riskiness will decrease \( \alpha^* \). This is so because (a) states exactly that increasing risk decreases \( EU_\alpha(\theta, \alpha) \); lowering \( \alpha \) restores the first order conditions of (2). Similarly, if \( U_\alpha(\theta, \alpha) \) is convex in \( \theta \), increases in the variability of \( \theta \) raise \( \alpha^* \). If \( U_\alpha(\theta, \alpha) \) is neither convex nor concave, then the effect of an increase in risk is ambiguous.\(^3\)

\(^3\) As we emphasized in Part I [11], our approach yields only a partial ordering of distributions. The mean-variance approach gives rise to a complete ordering of distributions (with the same expected value) and, thus, often gives answers as to the effects of changes in uncertainty when our approach cannot. But this is an argument for rather than against our approach. The answers of mean-variance analysis are spurious; they hold only if the utility function or the class of distributions is arbitrarily restricted. Furthermore, mean-variance analysis does not seem to provide clues as to what restrictions must be imposed if its results are to hold. As we shall demonstrate repeatedly in what follows, the necessary restrictions are a byproduct of our method of analysis.

\(^4\) Earlier studies (see, e.g., [4,14]) have made comparisons between perfectly certain and risky situations. In a “certain” situation, we choose \( \alpha \) so that

\[
U_\alpha(X, \alpha) = 0.
\]

Whether \( \alpha \gtrless \alpha^* \), where \( \alpha^* \) is the solution to (1), depends simply on whether

\[
EU_\alpha(X, \alpha) \gtrless U(EX, \alpha).
\]

Jensen’s inequality allows us to make unambiguous statements whenever \( U_\alpha \) is concave or convex in \( X \); but this is the same condition under which we are able to make unambiguous statements for a wider class of problems.
The next section applies this idea. We try to determine under what conditions functions which arise in some simple economic models are concave or convex. The first three models concern savings and investment behavior while the next three treat the choice of technique and the level of output when production or demand conditions are uncertain. Our general conclusions are that the mean-variance approach gives misleadingly general results, and that the conditions for the relevant functions to be concave or convex can be usefully phrased in terms of the Arrow–Pratt concepts of relative and absolute risk aversion [1, 9]. When this approach fails to give unambiguous results of the effects of increases in riskiness, it is natural to ask what sorts of increases in riskiness do have determinate effects. We have found that the emphasis on the interplay between utility functions and changes in distribution functions which underlies the integral conditions of [11] often suggests a useful approach to this problem. An example is our discussion of the portfolio problem below.

The paper concludes with a different demonstration of the usefulness of the results of [11]. Choice under uncertainty can be conceived of as the choice of a random variable. The selection of a portfolio, e.g., is the selection of a random stream of returns. We contrast this view with that discussed above by formulating the individual’s problem as one of choosing $\beta$ to maximize

$$\int U(x) \, dF(x, \beta).$$

That is, he chooses the most favorable random variable. Condition (b) gives us a criterion for determining which random variables risk averters prefer. The final section of the paper gives two examples of how this condition may be used.

2. THE EFFECTS OF INCREASING RISK

A. Savings and Uncertainty

We begin with an analysis of the effect of uncertainty in the rate of return on savings.\textsuperscript{4} Consider an individual who has a given wealth $W_0$, which he wishes to allocate between consumption today and consumption tomorrow. What he does not consume today he invests; at the end of the

\textsuperscript{4} A further important application of the concepts developed in [11] is to the problem of income inequality which is discussed at length in the article by A. B. Atkinson [2].

\textsuperscript{5} For a fuller discussion of this and related problems, see [4, 5, 7, 14].
period his investment yields the random return \( e \) per dollar invested. He wishes to allocate \( W_0 \) between the two periods to maximize two period expected utility:

\[
E[U(C_1) + (1 - \delta) U(C_2)] = U((1 - s) W_0) + (1 - \delta) EU(sW_0e),
\]

(4)

where \( s \) is the savings rate, and \( \delta \) the pure rate of time discount. We assume the man is risk averse so that his utility function satisfies

\[
U'' > 0; \quad U''' < 0.
\]

(5)

The necessary (and, because of (5), sufficient) condition for utility maximization is found by setting the derivative of (4) with respect to \( s \) equal to zero. The optimal savings rate must satisfy

\[
U'((1 - s) W_0) = E[U'(sW_0e)](1 - \delta) e.
\]

(6)

With its customary ambiguity, intuition suggests that increased uncertainty in the return on savings will either lower savings because “a bird in the hand is worth two in the bush” or raise it because a risk-averse individual, in order to insure his minimum standard of living, saves more in the face of increased uncertainty. Mean-variance analysis—equivalent to the assumption that \( U \) is quadratic—suggests that only the first argument has validity. If \( U(C) = aC - \frac{1}{2}bC^2 \), then the RHS of (6) is equal to

\[
(1 - \delta)(aE(e) - bsW_0E(e^2)),
\]

which decreases as \( e \) becomes riskier. A decrease in \( s \) will compensate for this disturbance of the first-order conditions as the LHS of (6) falls and the RHS of (6) rises as \( s \) decreases.

But this result is not general. Whether increasing variability increases or decreases \( s^* \) depends on whether \( eU'(sW_0e) \) is convex or concave in \( e \). Thus, increasing risk increases savings if

\[
2U''(C) + U'''(C) C > 0,
\]

(7)

and decreases it if

\[
2U''(C) + U'''(C) C < 0
\]

(8)

for all positive \( C \). Otherwise the effect of an increase in risk is ambiguous. Clearly, a nonpositive third derivative is sufficient for increasing risk to decrease savings. The Arrow–Pratt concept of relative risk aversion \( (R = -U''C/U') \) can be used to put these results in a somewhat different form. Since \( R' \) has the same sign as \(-(U''C + U'(1 + R))\), if \( R \) is non-increasing and greater than unity, (7) will hold, while if \( R \) is nondecreasing
and less than unity, (8) will hold. Thus for the class of constant relative risk aversion utility functions, \( U(W) = (1 - a) W^{1 - a}, \) \( a > 0, a \neq 1, \) whether or not increased risk increases or decreases savings depends on whether or not \( a > 1 \) or \( a < 1. \)

**B. A Portfolio Problem**

The conventional portfolio problem admits a similar, albeit somewhat more complicated analysis. Consider an individual who must allocate his portfolio between two assets: money, which yields a zero rate of return, and a risky asset which yields a random rate of return \( e. \) If he puts \( \alpha \) of his initial wealth \( W_0 \) in the risky asset, his terminal wealth is \( W(\alpha) = W_0(\alpha e + 1). \) The optimal portfolio mix is chosen to maximize the expected utility of terminal wealth, \( EU(W(\alpha)), \) where \( U, \) an increasing concave function, satisfies (5). The chosen \( \alpha \) is, therefore, the solution to

\[
H(\alpha) = W_0 EU' e = W_0 \int U'(W(\alpha)) e dF(e) = 0, \tag{9}
\]

where \( F \) is the distribution function of \( e. \) Note that (9) is a necessary and sufficient condition, as (5) implies that

\[
H'(\alpha) < 0. \tag{10}
\]

What happens to the demand for risky assets—which we identify with \( \alpha \)—as \( e \) becomes riskier? Once again, mean-variance analysis gives misleadingly general results which appear to confirm the common sense proposition that increasing the variability of the risky asset makes it less attractive to risk averse investors and reduces demand. If \( U \) is of the form \( U(W) = aW - \frac{1}{2}bW^2 \) then (9) becomes

\[
0 = W_0[(a - bW_0) E(e) - abW_0 E(e^2)].
\]

Thus, \( \alpha = (a - bW_0) E(e)/E(e^2) bW_0. \) As \( e \) becomes riskier, \( E(e^2) \) increases and \( E(e) \) remains constant; \( \alpha \) must increase.

This is not true in general. If the distribution of \( e \) is changed from \( F \) to

---

4 Arrow [1] has argued that normally relative risk aversion is increasing and—for low incomes—it is less than unity. Stiglitz [13] has questioned this assertion.

5 Since in this subsection, we shall always assume that \( W_0 \) is constant, it makes no difference whether we formulate the problem as Tobin does in terms of the rate of return \( r = (W - W_0)/W_0 \) or in terms of the value of terminal wealth. The interpretation of certain characteristics of the utility function (e.g., its elasticity) differs if utility is viewed as a function of \( r \) rather than of \( W. \) (In the former case, since \( r \) may be both positive and negative, \( U' \) cannot have constant elasticity.)
G, where \( F \) is less variable than \( G \), then the new allocation parameter \( \tilde{\alpha} \) satisfies \( \int U'(W(\tilde{\alpha}))\,e\,dG(e) = 0 \). Thus, \( \tilde{\alpha} \leq \alpha \), as \( \int U'(W(\alpha))\,e\,dSe \leq 0 \), where \( S = G - F \). If increasing risk is to decrease all risk-averse investors' demand for risky assets, then, if \( V(e) = U'(W(\alpha)) \) and if the points of increase of \( F \) and \( G \) are confined to \((a, b)\), we must have

\[
\int_a^b V(e)\,e\,dS(e) \leq 0,
\]

for all positive and decreasing \( V \) and all \( S \) satisfying the integral conditions of [11]; i.e., all \( S \) satisfying

\[
\int_a^b S(e)\,de = \int_a^b e\,dS(e) = 0 \quad (12a)
\]

and

\[
\int_a^t S(e)\,de \geq 0, \quad \text{for all} \quad t \in (a, b). \quad (12b)
\]

Using (12b) and the second mean value theorem of the integral calculus, we find the integral in (11) is equal to \([V(a) - V(b)]\,h(c)\) for some \( c \in (a, b) \) where \( h(c) = \int_a^c e\,dS(e) \). A sufficient condition for (11) therefore is that

\[
h(c) \leq 0 \quad (13)
\]

for all \( c \in (a, b) \). This is also a necessary condition. Suppose there is a \( \bar{c} \) such that \( h(c) > 0 \). For (11) to hold we must have \( \int_a^b V(e)\,dS(e) < 0 \) for all positive and decreasing \( V \). One such \( V \) is given by

\[
V(e) = \begin{cases} V_1 & \text{for} \quad a \leq e < \bar{c} \\ V_2 & \text{for} \quad \bar{c} \leq e \leq b, \end{cases}
\]

where \( V_1 > V_2 > 0 \). Then \( \int_a^b V(e)\,dS(e) = (V_1 - V_2)\,h(\bar{c}) > 0 \).

Still the presumption is that increasing risk decreases the demand for risky assets. It is possible both to exhibit increasing concave utility functions for which this is always true, and to show that no such utility function has the property that increasing risk always increases \( \alpha \). An increase in risk will always decrease demand for risky assets if \( Z(e) = eU'(W(\alpha)) \) is concave in \( e \). Since

\[
Z'(e) = [(1 - R + W_0A)\,U'' + (W_0A' - R')\,U']\,W_0A,
\]

where \( R = -U''W/U' \) and \( A = -U''/U' \), the measure of absolute risk
aversion, a sufficient condition for an increase in uncertainty to increase the allocation to the safe asset is that relative risk aversion be less than or equal to unity and nondecreasing, and that absolute risk aversion be nonincreasing. The Bernoulli or logarithmic utility function as well as all constant relative risk aversion functions where the degree of risk aversion is less than one satisfies these conditions; other members of the constant relative risk aversion class do not.

No risk averse investor will always increase his holdings of risky assets when their variability increases. To prove this, it suffices to exhibit an increase in risk which satisfies (13). One such is

\[ S(e) = \begin{cases} 
S_1 > 0 & \text{for } e_1 \leq e < 0 \\
S_2 > 0 & \text{for } 0 \leq e \leq e_2 \\
0 & \text{otherwise}
\end{cases} \]

where the \( e \), and \( S_1 \) are chosen to satisfy (12b). This kind of reasoning can be used to demonstrate that increases in the riskiness of any member of a rather important class of assets, bets, always lowers the demand for it.

A bet is an investment which can have only two outcomes. If all goes well, it pays off \( 1 + E(e) + \gamma \) per dollar invested. Otherwise, only \( 1 + E(e) - \lambda \) is returned. We call the gain from the bet \( \gamma \), and the loss, \( \lambda \), while \( E(e) \) is the expected rate of return. A "fair" bet is one for which \( E(e) = 0 \). As risk aversers will not take fair bets and as a bet for which \( E(e) - \lambda \geq 0 \) represents no risk at all, the portfolio problem is only of interest if

\[ E(e) - \lambda < 0 < E(e). \] (14)

If \( P \) is the probability of winning the bet then \( P\gamma - (1 - P) \lambda = 0 \) or \( \lambda = [P(1 - P)]\gamma \). All bets with the same rate of return may be represented by points in \( (P, \gamma) \) space (Fig. 1).

Figure 1 permits a graphic depiction of the difference between the use of our criterion and the variance criterion to rank risky prospects. The point \( B \) represents a typical bet with a possible gain of \( \gamma_B \) and a probability of attaining it of \( P_B \). A straightforward application of the definition of a mean preserving spread, or of the integral conditions of [11], shows that a bet \( C \) with gain \( \gamma_C \) and loss \( \lambda_C \) is riskier than \( B \) if and only if

\[ \gamma_C > \gamma_B \] (15a)

and

\[ \lambda_C > \lambda_B . \] (15b)

Clearly, all bets to the right of \( B \) satisfy (15a). The solid line in Fig. 1
represents the locus of all bets with loss \( \lambda_B \). The slope of this line is 
\[ \frac{d\gamma}{dP}_{\lambda=\lambda_B} = -\frac{\gamma}{P(1-P)}. \] Only bets above it satisfy (15b). Thus points in regions III and IV are riskier than \( B \). Similarly, those in I and VI are safer. Those in regions II and V cannot be compared with \( B \) by our criterion.

![Diagram illustrating the locus of bets with loss \( \lambda_B \).](image)

**Fig. 1**

The variance criterion ranks all the bets in Fig. 1. The variance of a bet is
\[ \sigma^2 = P\gamma^2 + (1 - P)\lambda^2 = \gamma^2((1 - P))/P(1 - P). \] The dotted line in the figure is the locus of all bets with the same variance as \( B \). All bets above it have greater variance than \( B \); all below it less. Its slope is

\[ \frac{d\gamma}{dP}_{\sigma=\sigma_B} = -\frac{1}{2} \frac{\gamma}{P(1-P)} = \frac{1}{2} \frac{d\gamma}{dP}_{\lambda=\lambda_B}. \]

The dotted line cuts the broken line from below at \( B \) and lies entirely in regions II and V. There are bets in each region which have greater variance than \( B \) but which are not riskier than \( B \).

It is now quite straightforward to use condition (13) to demonstrate
that if \( C \) is riskier than \( B \) a risk averter will always hold more of \( B \) than \( C \) in his portfolio. Define \( e_i, i = 1, 2, 3, 4, \) as follows:

\[
e_1 = E(e) - \lambda_C \quad e_2 = E(e) - \lambda_B \\
e_3 = E(e) + \gamma_B \quad e_4 = E(e) + \gamma_C.
\]

Then, combining (14) and (15) we see that

\[
e_1 < e_2 < 0 < e_3 < e_4. \tag{16}
\]

Since \( B \) and \( C \) have the same expected return

\[
(1 - P_C) e_1 - (1 - P_B) e_2 - P_B e_3 + P_C e_4 = \sum_{i=1}^{4} S_i e_i = 0. \tag{17}
\]

For this case, (13) becomes

\[
h_k = \sum_{i=1}^{k} S_i e_i \leq 0; \quad k = 1, 2, 3, 4. \tag{18}
\]

Since \( e_1 < 0 \) and \( S_1 = (1 - P_C) > 0 \), \( h_1 < 0 \); (17) states that \( h_4 = 0 \); furthermore, \( h_k = h_4 - P_C e_4 = -P_C e_4 < 0 \). It remains to show that \( h_3 \leq 0 \), or that

\[
(1 - P_C) e_3 - (1 - P_B) e_2 = -(P_C e_4 - P_B e_3) \leq 0. \tag{19}
\]

But, either \( P_B \leq P_C \) or \( P_B > P_C \). If \( P_B \leq P_C \), then (16) implies the RHS of (19) is negative. If \( P_B > P_C \), then \((1 - P_B) < (1 - P_C)\) which with (16) implies the LHS of (19) is negative.

C. A Combined Portfolio-Savings Problem

Levhari and Srinivasan\(^8\) [5] have recently analyzed the optimal portfolio and savings decision of an individual who wishes to maximize the expected value of the discounted utility of consumption,

\[
E \sum_{i=0}^{\infty} (1 - \delta)^i U(C_i), \tag{20}
\]

\(^8\) The comparison is, of course, only between a portfolio consisting of \( B \) and a safe asset and of \( C \) and a safe asset.

Another class of changes which satisfies (13) is given by the partial ordering \( \preceq_m \) mentioned in Part I [11]. If \( Y \) is a random variable with mean \( \bar{Y} \) and \( X \) is a random variable with the same distribution as the mixture \( \alpha Y + (1 - \alpha) \bar{Y}, 0 \leq \alpha \leq 1 \), then we write \( X \preceq_m Y \). It is straightforward to show that if \( X \preceq_m Y \), then the difference between the c.d.f.'s of the two distribution functions satisfies (13). We have analyzed the case of bets at length because of its greater importance and because of the intrinsic interest of the diagrammatic representation of different concepts of increasing risk which is given in Fig. 1 below.

\(^8\) Henceforth \( LS \). For other discussions of dynamic portfolio problems, see [4].
where $\delta$ is the discount rate and $C_t$ is consumption at time $t$, subject to the stochastic constraints
\begin{equation}
W_{t+1} = (W_t - C_t) r_t; \quad W_t \geq C_t \geq 0. \tag{21}
\end{equation}

$W_t$ is the individual's wealth at $t$ and $r_t - 1$ is the stochastic rate of return on the investment of $(W_t - C_t)$ in period $t$. At time $t$, the individual may invest either in asset 1 with a rate of return of $r_1^t - 1$, or in asset 2 with a rate of return of $r_2^t - 1$. If $\alpha$ is the fraction invested in the first asset and $(1 - \alpha)$ is the fraction invested in the second, then $r_t^i$ is given by
\begin{equation}
r_t = \alpha r_1^t + (1 - \alpha) r_2^t. \tag{22}
\end{equation}

It is readily apparent that in the special case where (a) the $r_t^i$ are independently and identically distributed in time\footnote{This assumption allows us to omit the superscript on $r_t^i$ in most of the following discussion.} and (b) the utility function has constant relative risk aversion
\begin{equation}
U(C) = \frac{C^{1-a}}{1-a} \quad a > 0 \quad a \neq 1 \quad \text{or} \quad U(C) = \ln C, \tag{23}
\end{equation}
the savings decision, the determination of $C_t$ (given $W_t$) is independent of the portfolio decision (the choice of $\alpha$). Moreover, the savings rate $(C/W_t)$ is a constant, independent of $W_t$, i.e., optimal behavior entails following the rule of saving a constant proportion of wealth
\begin{equation}
C_t = (1 - s) W_t \tag{24}
\end{equation}
for all $W_t > 0$. LS show that for any choice of $\alpha$, the optimal savings ratio $s(\alpha)$\footnote{We assume $a \neq 1$. The case of $a = 1$ needs special treatment detailed below.} must satisfy
\begin{equation}
s(\alpha)^* = (1 - \delta) E(r^{1-a}), \tag{25}
\end{equation}
where
\begin{equation}
r_t = \alpha r_1 + (1 - \alpha) r_2. \tag{26}
\end{equation}

They also show that, for any initial wealth $W_0$, when $s$ is chosen to satisfy Eq. (25), the expected discounted utility of an optimal savings program is given by
\begin{equation}
V(W_0, \alpha) = \frac{1}{1 - a} E \Sigma (1 - \delta)^t C_t^{1-a} = \frac{(1 - s(\alpha))^{-a} W_0^{1-a}}{1 - a}. \tag{27}
\end{equation}
It follows from Eqs (25) and (26) that $s$ is a function of $\alpha$, and so is the RHS of Eq. (27). Hence, $\alpha$ is selected to maximize (27). For an interior solution, $0 < \alpha < 1$, we must have

$$0 = \frac{\partial V}{\partial \alpha} = \frac{\partial V}{\partial s} \frac{\partial s}{\partial \alpha},$$

where

$$\frac{\partial V}{\partial s} = \frac{a(1 - s)^{(1 + \alpha)} W_0^{1 - \alpha}}{1 - \alpha} \geq 0 \quad \text{as} \quad a \leq 1.$$

It can be shown that in the neighborhood of $\alpha^*$, where $\partial s/\partial \alpha = 0$, $(1 - a) \partial^2 s/\partial \alpha^2 < 0$ for $a \neq 1$, so that the only critical points of (27) are relative maxima, implying a single critical point. Thus, the condition $\partial s/\partial \alpha = 0$ determines $\alpha^*$.

It is natural to ask what effect an increase in variability of the return of one of the assets will have on portfolio choice and on savings. LS have shown that, under special circumstances, an increase in the variance of one asset, with the mean held constant, decreases the proportion invested in that asset. To see whether this is true in general, we note that for $a \neq 1$, $\partial s/\partial \alpha = 0$ if, and only if,

$$E(r_1 - r_2)(\alpha r_1 + (1 - \alpha) r_2)^{-\alpha} = 0. \tag{28}$$

The effect of an increase in variability will be unambiguous if, and only if, $H(r_1) = (r_1 - r_2)(\alpha r_1 + (1 - \alpha) r_2)^{-\alpha}$ is convex or concave in $r_1$. But

$$H''(r_1) = -\left[\alpha(1 - \alpha) r_1 + r_2(\alpha(1 + a) + 2(1 - \alpha))\right] \\
\times [\alpha(\alpha r_1 + (1 - \alpha) r_2)^{-\alpha - 2}].$$

This expression is negative for $a < 1$ (assuming $\alpha > 0$), so that an increase in the variability of $r_1$ reduces the demand for this risky asset. The same result holds if $a = 1$ although the argument is more complicated. For $a > 1$, the sign of $H''(r_1)$ is ambiguous and an increase in variability could have the opposite effect.\footnote{LS's argument is incorrect here as their evaluation of $\partial^2 S/\partial \alpha^2$ is in error.}

The effects of a change in variability on the savings rate are much easier to analyze than the effects on portfolio allocation. By (25), what happens to $s$ depends on what happens to

$$E(\alpha r_1 + (1 - \alpha) r_2)^{1 - \alpha}, \tag{29}$$

\footnote{This generalizes the result obtained for the lognormal distribution in [5].}
where \( \alpha \) is chosen optimally. But, since, using Eq. (28),
\[
\frac{\partial E(\alpha r_1 + (1 - \alpha) r_2)^{1-a}}{\partial \alpha} = (1 - a) E(r_1 - r_2)(\alpha r_1 - (1 - \alpha) r_2)^{-a} = 0,
\]
we need consider only what happens to (29) at any fixed value of \( \alpha \). But,
\[
\frac{\partial^2 (\alpha r_1 + (1 - \alpha) r_2)^{1-a}}{\partial (r_i)^2} = \alpha^2 (1 - a) a [\alpha r_1 + (1 - \alpha) r_2]^{-(1+a)} \geq 0 \quad \text{as} \quad a \leq 1,
\]
from which it immediately follows that an increase in the variability of \( r \) increases the savings rate if \( a < 1 \) and decreases it if \( a > 1 \).

The case of the logarithmic utility function \( (a = 1) \) needs some special attention. We can write the functional equation for this case as
\[
V(W_o) = \ln(1 - s) W_o + (1 - \delta) EV(s W_o r).
\]
(30)
The solution of Eq. (30) is\(^{14}\)
\[
V(W_o) = \frac{1}{\delta} \ln(1 - s) W_o + \frac{1 - \delta}{\delta^2} [\ln(s) + E \ln r]
\]
(31)
and the optimal value of \( s \) is given by\(^{15}\)
\[
s = 1 - \delta
\]
(32)
\(^{14}\) \( V(W_o) \) may be calculated directly as follows: If \( s \) is a constant, utility from a single realization of \( r_i \) \((i = 1, \text{by convention})\) is
\[
V(W_o) = \ln(1 - s) W_o + (1 - \delta) \ln(1 - s) W_o r^t + (1 - \delta)^2 \ln(1 - s) W_o r^t r^t + \cdots
\]
\[
= \sum_{t=0}^{\infty} (1 - \delta)^t \ln(1 - s) W_o r^t + \sum_{t=0}^{\infty} (1 - \delta)^t i \ln s + \sum_{t=0}^{\infty} (1 - \delta)^t \ln \Pi_{t=0}^t r^t.
\]
Thus,
\[
EV(W_o) = \frac{1}{\delta} \ln ((1 - s) W_o) + \frac{(1 - \delta)}{\delta^2} \ln s + \frac{(1 - \delta)}{\delta^2} E \ln r.
\]
The second term is evaluated by observing that if
\[
F(x) = \frac{1}{1 - x} = \sum_{i=0}^{\infty} x^i, F'(x) = \sum_{i=0}^{\infty} i x^i = \frac{1}{(1 - x)^2}.
\]
Since the \( r^t \) are independently and identically distributed, \( E \ln \Pi_{t=0}^t r^t = i E \ln r \). Hence,
\[
E \sum_{i=0}^{\infty} (1 - \delta) \ln \prod_{t=0}^t r^t = E \sum_{i=0}^{\infty} (1 - \delta)^t i \ln r = \frac{1 - \delta}{\delta^2} E \ln r.
\]
\(^{15}\) If \( s \) is chosen to maximize (31), then \( 1/(1 - \delta) = (1 - \delta)/\delta^2 \) from which (32) follows.
independent of \( r \). To find the optimal portfolio allocation (the optimal value of \( \alpha \)), we maximize \( V(W) \) with respect to \( \alpha \); i.e., from Eq. (31), we must maximize \( E \ln r \). Thus, \( \alpha \) satisfies

\[
E \left\{ \frac{r_1 - r_a}{\alpha r_1 + (1 - \alpha) r_a} \right\} = 0.
\]

(33)

Since

\[
\frac{d^2 E \ln r}{d \alpha^2} = -E \left\{ \frac{(r_1 - r_a)^2}{\alpha r_1 + (1 - \alpha) r_a} \right\} < 0,
\]

(33) is both necessary and sufficient for an interior solution. Since

\[
\frac{r_1 - r_a}{\alpha r_1 + (1 - \alpha) r_a}
\]

is a concave function of \( r_1 \), an increase in the variability of \( r_1 \) always increases the proportion of the portfolio allocated to \( r_a \) but leaves the savings rate unaffected.

D. A Firm's Production Problem

Consider a firm whose output \( Q \) next period is uncertain (e.g., a public utility which must meet all demands at a fixed price). It wishes to minimize the expected cost of producing \( Q \). \( Q \) is produced by a two-factor concave production function \( Q = F(K, L) \), where \( K \) is, say, capital, a factor which cannot be varied in the short run, and \( L \) is, say, labor, the variable factor. What happens to expected costs as \( Q \) becomes more variable? If \( r \) is the cost of capital and \( w \) that of labor, expected costs are given by

\[
E[rK + wL(K, Q)] = rK + wE[L(K, Q)],
\]

(34)

where \( L(K, Q) \) is the labor required to produce the given output \( Q \) with capital \( K \). Since \( F \) is concave, it is easy to show that \( L(K, Q) \) is convex in \( Q \), for any given \( K \). Hence, an increase in variability of \( Q \) always leads to an increase in expected cost.

A somewhat more difficult problem is: What happens to the optimum level of \( K \)? Not surprisingly, the answer depends on the elasticity of substitution between \( K \) and \( L \). We choose \( K \) to minimize expected costs. From (34), the first order conditions may be written

\[
\frac{r}{w} = E \frac{\partial L(Q, K)}{\partial K},
\]

\[10\] See [10] for a more complete analysis of this problem.
i.e., the factor-price ratio must be equal to the average marginal rate of substitution. Let us assume that the production function has constant elasticity of substitution. Then\textsuperscript{17}

\[
Q = (\delta K^\rho + (1 - \delta) L^\varrho)^{1/\varrho}
\]

\[
\frac{\partial L}{\partial K} = \frac{\delta}{1 - \delta} \left( \frac{K}{L} \right)^{\varrho - 1} = \frac{\delta}{1 - \delta} \left( \frac{Q^\rho - \delta K^\rho}{1 - \delta} \right)^{(1-\varrho)/\varrho} K^{\rho - 1}
\]

\[
\frac{\partial^2 (\partial L/\partial K)}{\partial Q^2} = \left[ \frac{\delta K^{\rho - 1}}{(1 - \delta)^{1/\varrho}} (1 - \rho) Q^{\rho - 2}(Q^\rho - \delta K^\rho)^{(1-\varrho)/\varrho} \right]
\times (-\rho Q^\rho + (1 - \rho) \delta K^\rho).
\]

A sufficient condition for convexity is that $\rho \leq 0$, i.e., that the elasticity of substitution be less than or equal to unity. Thus, if the elasticity of substitution is less than or equal to unity, the optimal level of $K$ increases with an increase of variability in $Q$.

To show that for other production functions $K$ may decrease with an increase in variability of output, consider the extreme case of a constant elasticity of substitution production function with infinite elasticity:

\[
Q = bK + aL.
\]

If the capital stock is given by $K$, expected costs are given by

\[
rK + \frac{w}{a} \int_{bb}^{\infty} (Q - bK) dG(Q),
\]

where $G(Q)$ is the distribution function for $Q$. Expected cost minimization requires (for an interior solution)

\[
r = \frac{wb}{a} (1 - G(bK)) = 0,
\]

so that

\[
K = \frac{G^{-1}(1 - (ar/\rho b))}{b}.
\]

Whether $K$ increases or decreases depends solely on whether $G^{-1}(1 - (ar/\rho b))$ increases or decreases (see Fig. 2) or, equivalently, whether the probability that $Q$ will be greater than $bK$ (the “capacity” of the original capital stock) increases or decreases.

\textsuperscript{17} If $L > 0$; $\partial L/\partial K = 0$ otherwise.
E. A Multi-stage Planning Problem

Consider a simple economy in which the final consumption good is produced by labor and an intermediate commodity $y$:

$$Q = P(L_2, y),$$

while $y$ is produced by labor alone:

$$y = M(L_1).$$

The economy faces an overall labor constraint $L$, so

$$L_1 + L_2 = L.$$ 

In the absence of uncertainty, maximization of $Q$ simply requires

$$P_1 = P_2 M'.$$

Assume that there is uncertainty associated with the production of $y$:

$$y = M(L_1) + e,$$

where $e$ has mean zero and distribution function $F$. We wish to maximize $EQ$; we require

$$E[P_1 - P_2 M'] = 0.$$ 

This problem was posed to us by M. Weitzman.
If $e$ becomes more variable, what happens to the allocation of labor between the two sectors depends on the sign of

$$P_{12} - M'P_{22}.$$

Assume that $P$ is a constant elasticity of substitution production function: $P = (\delta L_2^\rho + (1 - \delta) y^\rho)^{1/\rho}$. Then

$$P_{122} = \frac{A}{L_2} ((1 - \rho) \delta L_2^\rho + \rho(1 - \delta) y^\rho)$$

$$P_{222} = \frac{A((\rho - 2) \delta L_2^\rho + (1 + \rho)(1 - \delta) y^\rho)}{y},$$

where

$$A = \delta(1 - \rho)(\rho - 1) L_2^\rho y^{\rho - 2}(\delta L_2^\rho + (1 - \delta) y^\rho)^{(1 - 2\rho)/\rho} < 0.$$  

If $1 \geq \rho \geq 0$, i.e., the elasticity of substitution is greater than or equal to unity, $P_{122} \leq 0$ and $P_{222} \geq 0$, so $L_2$ decreases and $L_1$ increases; more labor is allocated to the “earlier” stage of production (to producing $y$).

Consider the other extreme case, where $Q$ is produced by a fixed-coefficients production function, $Q = \min(L_2, y)$. Then

$$E(Q) = \int_{L_2 - M(L_2)}^{L_2 - M(L_2)} [M(L_2) + e] dF(e) + L_2(1 - F(L_2 - M(L_2)))$$

$$= \int_{L_2 - M(L_2)}^{L_2 - L_2 - M(L_2)} [M(L_2) + e] dF(e) + (L_2 - L_2)(1 - F(L_2 - L_2 - M(L_2)))$$

so that maximization of $E(Q)$ requires

$$[M'(L_2) + 1] F(L_2 - L_2 - M(L_2)) = 1.$$  

The second-order conditions are satisfied, since $M'F - f(M' + 1)^2 < 0$, where $f$ is the density function corresponding to $F$; hence, there is a unique maximum. Whether $L_1$ increases or decreases depends solely on whether $F(L_2 - L_1 - M(L_1))$ increases or decreases, i.e., whether the probability that (at the old allocation) the $y$ constraint will be binding is increased or decreased; either is clearly possible. Note that if $y$ is also produced by a constant returns-to-scale production function

$$y = L_1,$$

then the optimal value of $L_1$ is simply given by

$$F(L_2 - 2L_1) = \frac{1}{2}.$$  

So what happens to $L_1$ depends completely on whether the median of $e$ increases or decreases. 
F. Choice of Output Level for a Competitive Firm

In the examples considered so far, the conditions we have obtained under which it is possible to make unambiguous statements about the effects of increases in variability have been essentially identical to those obtained earlier in comparisons between safe and risky situations. There are, however, problems in which the latter comparisons can be made under conditions weaker than the former. In the following example, we can, for instance, make unambiguous statements even when the first-order condition is neither concave nor convex.

Consider a competitive firm which must decide today on the level of output tomorrow, although the price p of output Q is uncertain. It wishes to maximize expected utility of profits \( \pi \), where \( U \) is concave and

\[
\pi = pQ - C(Q),
\]

where \( C(Q) \), the cost function, is convex. A necessary and sufficient condition for an optimum is that

\[
\frac{E(U'p)}{EU'} = C'(Q^*).
\]  

(35)

If the producer is risk neutral, or if there is no variability in \( p \), profit maximization requires that price equals marginal cost, \( Ep = C'(Q) \). \( Q^* \geq Q \) as \( EU'p/\pi EU' \geq Ep \), i.e., as \( [E(U' - EU')(p - E(p))] \geq 0 \). But since \( U' < 0 \), \( U'(p) \geq U'(E(p)) \), as \( p \geq E(p) \); so

\[
E[(U' - EU')(p - E(p))] = E[U' - U'(E(p))(p - E(p))] < 0.
\]

Hence, there is always less output under uncertainty than under certainty.

Not surprisingly, the comparative statistics of the behavior of this firm differ significantly from that of the competitive firm with no uncertainty:

(a) In the absence of uncertainty, a proportional profit tax at rate \( t \) leaves output unchanged. Here it will increase or decrease output as relative risk aversion is increasing or decreasing. It is easy to see that \( dQ/dt \) has the same sign as \( ERU'(p - C'(Q^*)) \); from (35) this last quantity has the same sign as \( R' \) (once again, \( R = -U''u/U' \) is the measure of relative risk aversion).

(b) In the absence of uncertainty, a (uniform) upward shift in the total cost curve leaves output unchanged. Here, if

\[
C(Q, \tau) = C(Q) + \tau,
\]

24 For a discussion of the case of constant absolute risk aversion, see [6]. Except under very stringent conditions, how one ought to describe the behavior of a competitive firm in the presence of uncertainty remains an open question.
then 
\[ \frac{dQ}{d\tau} \geq 0, \quad \text{as} \quad -EU''(p - C') = EA(p - C') U' \geq 0, \]
where \( A = -U''/U' \) is the measure of absolute risk aversion. But \( EA(p - C') U' \) has the same sign as \( A' \). Thus output increases or decreases as absolute risk aversion is increasing or decreasing.

3. CHOOSING PROBABILITY DISTRIBUTIONS

The following examples show how our definition of variability and our basic theorem on the equivalence of the three alternative approaches discussed in [11] may be applied to prove some general theorems about situations where one must choose a probability distribution from among a set of possible probability distributions:

A. Diversification theorem.\(^{20}\)

Assume an individual can purchase shares of two\(^{21}\) securities whose value next period (per dollar invested) is described by identical but independent distributions. How should he allocate his given initial wealth, i.e., how should he choose \( b \) to maximize\(^{22}\)

\[ EU(W) = EU((be_1 + (1 - b) e_2) W_0), \]

where \( U \) is a concave function? We prove that independent of the utility function, \( b \) should be set at \( \frac{1}{2} \). We can write

\[ y_b = (be_1 + (1 - b) e_2) W_0 = y_{1/2} - (b - \frac{1}{2})(e_1 - e_2) W_0. \]

Note that

\[ E(e_1 - e_2 \mid y_{1/2}) = 0, \]

i.e., the expected value of the difference between two identically distributed independent random variables, given only that their sum be a particular number, is zero. Since \( y_b \) has the same distribution as \( y_{1/2} \) plus a random variable whose expectation conditional on \( y_{1/2} \) is zero, by Theorem 2 of [11], \( y_b \) is more variable than \( y_{1/2} \) for all \( b \neq \frac{1}{2} \), i.e., \( y_{1/2} \) is preferred to \( y_b \) by all individuals with concave utility functions.

B. The Rao–Blackwell Theorem.\(^{23}\)

Suppose a sample of random variables \( x = (x_1, \ldots, x_n) \) is generated by

\(^{20}\) See [12] for an alternative proof and general discussion of this theorem.
\(^{21}\) The generalization to the case of \( n \) securities is straightforward.
\(^{22}\) We assume that \( E(e_1) \) exists and is finite so \( EU \) exists and is finite.
\(^{23}\) Concepts and notation are borrowed from Ferguson [3].
distribution depending on an unknown parameter \( \theta \). An estimating procedure for \( \theta \) is a mapping \( d(x) \) from the sample \( x \) to the real line. Often a statistician tries to find an estimating procedure which minimizes the expected value of a convex loss function \( L(d(x)) \). The Rao–Blackwell theorem states that for any estimator \( d(x) \), and any convex \( L \), if there is a sufficient statistic \( T \) for \( \theta \), there is an estimator \( d^*(x) \) at least as good as \( d(x) \) in the sense that \( EL(d^*(x)) \leq EL(d(x)) \).

To prove this, it follows from Theorem 2 of Part I that it is clearly necessary and sufficient to find a \( d^*(x) \) such that \( d^*(x) \leq_{\alpha} d(x) \).

For every \( T \) let \( d^*(x) = E(d(x) \mid T) \). Then consider the r.v. \( z \) defined by \( d(x) = d^*(x) + z \). Clearly, \( E(z \mid T, x) = E(z \mid d^*(x)) = 0 \) and \( d^*(x) \leq_{\alpha} d(x) \) as was to be shown.

REFERENCES