A Consumption-Oriented Theory of the Demand for Financial Assets and the Term Structure of Interest Rates

It is reasonable to assume that individuals do not desire wealth for its own sake, but for the consumption that it provides. A long term (say $n$ period) bond is a perfectly safe asset in terms of consumption in the $n$th period, but a risky asset in terms of consumption in preceding and following periods. A one-period bond is safe in terms of consumption next period, but risky in terms of consumption in all following periods. A consol provides a perfectly safe income stream although its capital value is uncertain. Traditional theory, unfortunately, has focused on one-period capital valuations. If our hypothesis that individuals desire wealth for the consumption it provides is accepted, then it is not correct, in spite of common usage in economics dating at least back to Keynes, to consider long-term bonds as riskier than short-term bonds, and a theory of the demand for short-term bonds based on those considerations (such as that of Tobin [25]) may be misleading. The purpose of this paper is to provide an alternative, consumption-oriented theory of the demand for financial assets.

The theory of liquidity preference has traditionally focused on two questions:

1. Why do individuals hold short-term bonds (liquid assets) when they can get (on average) a higher rate of return on long-term bonds? Part I is devoted to answering this question. We show that the traditional answer that short-term bonds are safer is not really an adequate explanation. Indeed, individuals may hold long-term bonds even when they can get (on average) a higher rate of return from short-term bonds. Since the phenomenon of individuals sacrificing some returns (on average) to hold liquid (short-term) assets is known as liquidity preference, we may refer to the converse phenomenon—individuals sacrificing some returns (on average) to hold long-term bonds—as "liquidity

1 The author is indebted to M. Miller and O. von Weizsäcker for extremely helpful comments and suggestions. Both have been studying similar questions in the context of intertemporal models; Miller uses the quadratic utility function general equilibrium model, developed by Lintner [14], Sharpe [25], and Mossin [18]; von Weizsäcker uses a dynamic programming approach, assuming constant elasticity utility functions and particular parametrizations of the distribution functions. It is reassuring that the preliminary results of both authors seem in accord with the findings here. I would also like to acknowledge the many useful discussions with my colleagues at the Cowles Foundation and the comments of D. Hester, J. Mirels, and the referees. The research was supported by grants from the National Science Foundation and the Ford Foundation.

2 Although a number of authors have noted the fact that for some individuals long-term bonds may be the relatively safe asset, there have been few systematic investigations of the implications of this. See, however, the papers of Kalm [12] and Tobin [26].

Joan Robinson [21] has drawn a distinction between individuals (or institutions) who are concerned with income uncertainty and those who are concerned with capital uncertainty. But capital is, presumably, desired for the future consumption which can be derived from it. Thus, the appropriate distinction is between those who are concerned with consumption in the near future and those who are concerned with consumption in the more distant future. Moreover, since in general, individuals will be willing to forego some consumption in the near future for a sufficient increase in consumption in the distant future, individuals will be concerned about both "income" and "capital uncertainty" and will be willing to trade off one against the other. See also [6, 17]. For a fuller discussion of some similar models, see [16].
aversion". As we shall see, the common explanation for both phenomena is that individuals wish to avoid uncertainty in their consumption stream.

(2) Why is the liquidity preference schedule downward sloping? Tobin [25] has shown that, although in general it may not be, for low interest rates it always will be downward sloping. We will show that, if the term structure of interest rates remains unchanged, the case for a downward sloping schedule is even weaker. We also will show that whether expectations are elastic or inelastic makes little difference for these results.

Before turning, in Part III, to examination of these questions, we answer, in Part II, a question about which traditional theory has little to say: how the composition of portfolios changes as wealth changes. Finally in Part IV, we note how changes in uncertainty affect the demand for short- and long-term assets.

PART I. SPECULATION IN SHORT- AND LONG-TERM BONDS

After presenting our basic model in Section 1, we analyze in Section 2 the patterns of demand in a specific example—where consumption in different periods are perfect complements. Section 3 derives the necessary and sufficient conditions for expected utility maximization, while Section 4 examines the patterns of allocation in the general case. Finally, in Section 5 we attempt to provide some intuitive explanation of the results.

1. THE BASIC MODEL

We begin our investigation by considering an individual who has a given amount of wealth to invest, \( w_0 \). He can buy one-period bonds \(^1\); a bond which pays $1 next period costs \( p_1 > 0 \) this period, i.e. it yields a certain return of \( (1/p_1) - 1 \equiv r_1 \equiv g_1 - 1 \). Alternatively, he can buy two-period bonds; a bond which pays $1 at the end of two periods costs \( P > 0 \) this period; i.e. it yields a return over the two periods of \( (1/P) - 1 \equiv R \equiv G - 1 \). The price next period of a two-period bond purchased this period is just the price of a one-period bond next period, \( p_2 > 0 \), i.e. it simply depends on the "short-term" rate of interest next period. What \( p_2 \) will be is unknown to the individual at the time he has to make his original allocation. If he allocates a percentage, \( a \), of his portfolio to short-term bonds and the remainder to long-term bonds, at the end of the first period his net wealth is given by

\[
w_1 = w_0 \left[ \frac{a}{p_1} + \frac{(1-a)}{P} p_2 \right] = w_0 \left[ a g_1 + \frac{(1-a)}{g_2} G \right].
\]  

(1)

He then must allocate his wealth between consumption and investment in bonds. He chooses \( a \) to maximize expected utility of consumption over the two periods \(^2\): \( EU(C_1, C_2) \).

In the absence of uncertainty, \(^3\) market equilibrium would require

\[ g_1 g_2 = G \text{ (or } 1/p_1 = p_2/P). \]  

(2)

\(^1\) In this paper we will not be concerned with why individuals hold money rather than short-term bonds (demand deposits rather than savings deposits). The explanation of this must lie outside the portfolio analysis of this paper. We will assume that the one-period safe asset has a positive rate of return, since such an asset is always available to the individual; the question we are interested in is what determines individuals' demands for liquid (short-term) assets.

\(^2\) In order to make our analysis as close as possible to that of traditional monetary theory, we will follow the conventional practice of ignoring all non-monetary assets and other sources of income. (As Tobin has expressed it, "Liquidity preference theory takes as given the choices determining how much wealth is to be invested in monetary assets and concerns itself with the allocation of this amount among cash and alternative monetary assets." [25]). The assumption of no other sources of income is not crucial, provided that income is non-stochastic. (If \( Y_1 \) and \( Y_2 \) are the known incomes first and second period, and \( C_1, C_2 \) is consumption derived from selling financial assets the first (second) period, then the individual is interested in maximizing \( EU(C_1 + Y_1, C_2 + Y_2) \); but since \( Y_1 \) and \( Y_2 \) are fixed, this is equivalent to maximizing \( EU(C_1, C_2) \), where \( U \) differs from \( U \) by a simple change in origin. \( C_0 \), of course, no longer constrained to be non-negative, but such constraints play no crucial role in the ensuing analysis.)

\(^3\) And under the assumption that all individuals agree on the value of \( g_2 \) which they expect with certainty.
The long rate would have to equal the product of the short rates.

It has long been argued that in the presence of uncertainty, in order to induce individuals to hold long-term bonds, there would have to be a risk (or "liquidity") premium. As Hicks expressed it,

"If no extra return is offered for long lending, most individuals (and institutions) would prefer to lend short, at least in the sense that they would prefer to hold their money on deposit in some way or other." [10].

If the relevant holding period were two periods, this means that unless the long rate exceeds the product of the expected short rates,

$$ G > Eg_1g_2, \quad (2') $$

individuals would hold exclusively ("specialize in") short-term bonds.\(^1\) \(^3\) Similarly, if the relevant holding period were one period, this would mean that unless

$$ \frac{1}{p_1} < \frac{Ep_2}{P} \quad (2'') $$

individuals would hold exclusively short-term bonds.

In the subsequent sections of Part I, we will show that this presumption is not correct. In particular, we will show that if

Hypothesis A, \( G = Eg_1g_2 \),

individuals will not only not specialize in short-term bonds, but may even specialize in long-term bonds. Similarly, unless the individual is not very risk averse, if

Hypothesis B,

$$ \frac{1}{p_1} = \frac{Ep_2}{P} $$

it will be shown that he will buy some long-term bonds. Hypotheses A and B are alternative formulations of what has been called the assumption of market risk neutrality. Much of our discussion will be focused on these cases, not because we believe that the market in fact is likely to be described by them, but because they are convenient polar cases and, because they have been the centre of much of the discussion in recent literature, it is probably worth while exploring in some detail their implications.

2. AN EXAMPLE

Let us assume that the indifference curves between \( C_1 \) and \( C_2 \) are as drawn in Fig. 1, i.e. the utility function is of the form

$$ U[\min (C_1, C_2/(1+\delta))] $$

with \( U^\prime < 0 \). Then, given any \( w_1, \) \((1+\delta)C_1 = C_2 \), i.e.

$$ w_1 = C_1 + (C_2/g_2) = C_1(1+\delta + g_2)/g_2 $$

\(^1\) We will use the term "specialize" to refer to the situation where, if, say, individuals cannot sell long-term bonds short, they hold exclusively short-term bonds (\( a = 1 \)), or if they can sell long-term bonds short, \( a \leq 1 \).

\(^2\) The complete general equilibrium analysis would require us to take account of supplies of long- and short-term bonds. It has been argued, e.g. by Hicks, that firms will have a "strong propensity to borrow long" in order to hedge their future supplies of raw materials and loan capital. If this is true and lenders do demand a "liquidity premium" then (2') must hold. Here, however, our concern is primarily with the demand side of the market.

\(^3\) It should be emphasized that throughout, we are deriving properties of an individual's demand for monetary assets, and hence it is the individual's expectations of \( g_2 \) that are relevant. If we wish to verify from empirical data (see [15] or [16]) (2') (or hypothesis A or B below) we need to make a further assumption specifying how "on average" expectations are formed; alternatively, we may assume that "on average" the realized value of \( g_2 \) is equal to the expected value of \( g_2 \) of the representative individual. Note that this is a considerably weaker assumption than the assumption of perfect foresight.
or

\[ C_1 = w_1 \left( \frac{g_2}{1 + \delta + g_2} \right) = \frac{w_1}{1 + p_2(1 + \delta)}. \]  

(3)

Thus, \( \alpha \) will be chosen to maximize

\[ E \left\{ U \left[ \frac{(ag_1 g_2)}{1 + \delta + g_2} + (1 - a) \frac{G}{1 + \delta + g_2} \right] w_0 \right\}, \] i.e.

\[ EU'(C_1) \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) = 0. \]  

(4)

**Figure 1**
Since \( EU \) is concave in \( \alpha \), (4) is both a necessary and sufficient condition for an interior maximum. If \( \alpha \) is constrained to be between zero and one then if \( \alpha = 0 \), \( dEU/\alpha < 0 \), \( \alpha^* = 0 \), while if \( \alpha = 1 \), \( dEU/\alpha > 0 \), \( \alpha^* = 1 \).

If the individual wanted to avoid all uncertainty about his level of consumption (utility) he could have invested \( \bar{\alpha} \) in short bonds, where

\[
(1 + \delta)\bar{\alpha}g_1 = (1 - \bar{\alpha})G
\]

or

\[
\bar{\alpha} = G/\{g_1(1 + \delta) + G\}.
\]

Then, no matter what \( g_2 \) turns out to be,

\[
C_1 = \frac{C_2}{1 + \delta} = \frac{Gg_1w_0}{G + g_1(1 + \delta)} = \frac{w_0}{p_1 + P(1 + \delta)} \equiv \breve{C}_1,
\]

If an individual buys more long-term bonds than \( (1 - \bar{\alpha})w_0 \), he is speculating on the expectation that on average the short-term rate of interest will be sufficiently small next period—the price of the two-period bonds next period will be sufficiently high—that on average he can obtain a sufficiently higher return from holding the long-term bonds for one period and selling some of them for consumption the first period to compensate him for the uncertainty thereby introduced. Similarly if \( \alpha > \bar{\alpha} \), he is "speculating" on there being a high short-term rate next period. The conditions determining whether he speculates, and if so, in which bond, may easily be derived. We shall prove that

\[
a^* \leq \bar{\alpha} \quad \text{as} \quad E \left\{ \frac{g_1g_2 - G}{1 + \delta + g_2} \right\} = \frac{1}{(1 + \delta)p_2 + 1} \left\{ \frac{1 - p_2}{p_1} - \frac{p_2}{P} \right\} \leq 0,
\]

i.e. the individual will "speculate" on long-term bonds (short-term bonds) if

\[
E \left\{ g_1g_2 - G \right\}/(1 + \delta + g_2) < (>) 0.
\]

Note that whether he speculates in long- or short-term bonds does not depend on his attitudes towards risk (it depends only on his expectations of \( g_2 \) and on \( \delta \)) although the extent of his speculation clearly will.

To see this result, observe that if \( a^* = \bar{\alpha} \), \( U' \) is constant, so the first order condition (4) is satisfied if and only if \( E(g_1g_2 - G)/(1 + \delta + g_2) = 0 \). Recalling the definition of \( \breve{C}_1 \) from (6), we can rewrite (4) as

\[
E[U'(C_1) - U'(\breve{C}_1)](g_1g_2 - G)/(1 + \delta + g_2) + U'(\breve{C}_1)E(g_1g_2 - G)/(1 + \delta + g_2) = 0. \quad (4')
\]

If the last term is positive, the first term must be negative, and if the last term is negative, the first must be positive. But differentiating equation (3) with respect to \( g_2 \), we obtain

\[
\frac{dC_1}{dg_2} = \frac{w_0}{(1 + \delta + g_2)^2} \left[ a_g(1 + \delta) - (1 - \alpha)G \right] \equiv 0 \quad \text{as} \quad \alpha \equiv \bar{\alpha}.
\]

Thus \( C_1 \geq \breve{C}_1 \) as \( g_2 \geq G/g_1 \) if \( a^* > \bar{\alpha} \), and as \( g_2 \leq G/g_1 \) if \( a^* < \bar{\alpha} \). Since \( U'' < 0 \), \( U'(C_1) \geq U'(\breve{C}_1) \) as \( C_1 \leq \breve{C}_1 \). Thus, if the first term in equation (4') is to be negative, \( U'(C_1) \geq U'(\breve{C}_1) \) as \( g_2 \leq G/g_1 \), i.e. \( a^* \) must be greater than \( \bar{\alpha} \); and similarly, if the first term is to be positive, \( a^* < \bar{\alpha} \).

Let us consider two polar cases: if \( \delta = -1 \), we can write \( U = U(C_1) \), i.e. the individual cares only about consumption next period, while if \( \delta \) is very large \((\rightarrow \infty)\), the individual pays attention only to consumption second period: \( U = U(C_2) \). In the first case \( \bar{\alpha} = 1 \), and

\[
a^* \equiv 1 \quad \text{as} \quad \frac{1}{p_1} \equiv \frac{E}{P_2}.\]

\[
\]
The individual will speculate in long-term bonds if the one period expected return exceeds the safe return. Note that \( a^* = 1 \) if and only if hypothesis B is satisfied.

In the second case, \( a = 0 \), and

\[
a^* \geq 0 \text{ as } Eg_1g_2 \geq G.
\]

The individual will speculate in short-term bonds if (and only if) the long rate is less than the product of the expected short. Now \( a^* = 0 \) if and only if hypothesis A is satisfied.

In the more general case where both \( c_1 \) and \( c_2 \) are desired we observe that (using Jensen’s inequality)

If

\[
Eg_1g_2 - G \leq 0, \quad E \left( \frac{g_1g_2 - G}{1 + \delta + g_2} \right) < 0,
\]

while if

\[
E \left( \frac{1}{p_1} - \frac{p_2}{P} \right) \geq 0, \quad E \left( \frac{p_1}{1 + \delta} - \frac{p_2}{p_2 + 1} \right) > 0.
\]

This means that if

\[
\frac{G}{g_1} \geq Eg_2, \quad a^* < a.
\] (9a)

If the long rate is greater than or equal to the product of the expected short all individuals speculate in long-term bonds. On the other hand, if

\[
\frac{G}{g_1} \leq \frac{1}{Ep_2}, \quad a^* > a.
\] (9b)

If the long rate is less than or equal to the product of the short rate and the reciprocal of the expected price of short-term bonds next period, all individuals speculate in short-term bonds.

But we can say even more about the allocation: if the long rate is equal to the product of the expected short rates not only will all individuals hold some long-term bonds, they may even specialize in them. To see this, evaluate (4) when \( a = 0 \). Observe that when \( a = 0 \),

\[
\frac{dU'(C_1)/(1 + \delta + g_2)}{dg_2} = \left( - \frac{U''C_1}{U'} - 1 \right) \frac{U'}{(1 + \delta + g_2)^2} = (\rho - 1) \frac{U'}{(1 + \delta + g_2)^2} \geq 0 \text{ as } \rho \geq 1,
\] (10)

where \( \rho = -\frac{U''C_1}{U'} \), the elasticity of marginal utility, is analogous in this two-period model to the Arrow-Pratt measure of relative risk aversion for a one-period model. If \( Eg_1g_2 = G \), (10) implies that

\[
a^* \geq 0 \text{ as } \rho \geq 1.
\] (11a)

If individuals have relative risk aversion equal to unity, they simply buy long bonds, if they are more risk averse (\( \rho > 1 \)), they “hedge” more, buying some short-term bonds, but if \( \rho < 1 \), they sell short-term bonds (i.e., borrow short) and lend long.

To see this, we rewrite equation (4):

\[
E \left[ \frac{U'(C_1)}{1 + \delta + g_2} - \frac{U'(\hat{C}_1)}{1 + \delta + \hat{g}_2} \right] (g_1g_2 - G) + \frac{U'(\hat{C}_1)}{1 + \delta + \hat{g}_2} E(g_1g_2 - G) = 0,
\]

where \( \hat{g}_2 \equiv G/g_1 \) and \( \hat{C}_1 \) is defined above in (6). From (10) when \( g_2 > \hat{g}_2 \) if \( \rho \geq 1 \),

\[
[U'(C_1)/(1 + \delta + g_2)] - [U'(\hat{C}_1)/(1 + \delta + g_2)] \geq 0;
\] (11a) follows immediately.
Similarly, if hypothesis B is satisfied individuals may specialize in short-term bonds. Since, if $a = 1$
\[
     d \left( \frac{U'(C_i)}{(1+\delta)p_2+1} \right) = (\rho - 1) \frac{U'(1+\delta)}{((1+\delta)p_2+1)^2} \; \Xi 0 \text{ as } \rho \; \Xi 1,
\]
by analogous arguments to those used earlier,
\[
     a^* \; \Xi \; 1 \text{ as } \rho \; \Xi \; 1 \tag{11b}
\]
i.e. if $\rho = 1$, the individual buys only short bonds, if he is more risk averse, he buys both, if less risk averse, he sells long-term bonds short.

The above results suggest that if two individuals have identical indifference maps (the same $\delta$), both will speculate in the same asset, but the less risk averse will take a more speculative position. More precisely if $U^A[\min (C_1, C_2/(1+\delta))]$ is the utility function of one individual, and $U^B[U^A[\min (C_1, C_2/(1+\delta))]$ is that of a second individual, then the latter is more risk averse than the former if $U^{B\rho} < 0$. We have already noted (p. 326) that whether $a^* \; \Xi \; \hat{a}$ depends simply on expectations and $\delta$, and $\hat{a}$ depends simply on $\delta$, $g_1$ and $G$ (so $\hat{a}^A = \hat{a}^B$). We wish to show that $|a^{A\rho} - \hat{a}| > |a^{B\rho} - \hat{a}|$. Assume $a^{A\rho} < \hat{a}$, i.e. the first individual speculates in long-term bonds. Then, $EU^A' \left( \frac{(g_1 g_2 - G)}{1+\delta + g_2} \right) = 0$ when $a = a^{A\rho}$. We wish to evaluate $EU^B' U^A' \left( \frac{(g_1 g_2 - G)}{1+\delta + g_2} \right)$ at $a = a^{A\rho}$. Since, however, $a^{A\rho} < \hat{a}$,
\[
     dC_2/dg_2 < 0, \text{ so } \frac{dU^B'}{dg_2} = U^{B\rho} U^{A\rho} dC_2/dg_2 > 0, \text{ so } EU^B' U^A' \left( \frac{(g_1 g_2 - G)}{1+\delta + g_2} \right) > 0, \text{ and } \hat{a} > a^{B\rho} > a^{A\rho}.
\]
The other case may be handled analogously. More risk averse individuals do not necessarily hold more liquid assets; they do speculate less than less risk averse individuals.

3. MAXIMIZATION OF EXPECTED UTILITY

These results readily generalize to the case where $C_1$ and $C_2$ are smoothly substitutable for one another, as in Fig. 2. We assume that the individual's utility function is concave, i.e. $U^{11} < 0$, $U^{22} < 0$, $U^{11} U^{22} - U^{122} > 0$.$^1$

For any given value of $w_t$ and $g_2$, the individual allocates his income between the two periods in the usual way, i.e., he maximizes
\[
     U(C_1, C_2) \tag{12}
\]
subject to the budget constraint
\[
     C_1 + (C_2/g_2) = w_1. \tag{13}
\]
The necessary condition for an optimum is, of course, that
\[
     U_1 = g_2 U_2 \tag{14}
\]
and since the indifference map is assumed to be quasi-concave, this is sufficient.

From (1), $w_t$ is just a function of $g_2$, for given values of $a$, $g_1$ and $G$. Thus $C_1$ and $C_2$ will be simply functions of $g_2$:
\[
     C_1 = C_1 [g_2; a, g_1, G] \tag{15}
\]
\[
     C_2 = (w_1 - C_1)g_2.
\]

$^1$ In the proof of certain propositions, more stringent conditions will need to be employed, e.g., $U_{12} \Xi 0$ ($C_1$ and $C_2$ are "complements"). We shall call attention to these additional restrictions at those points.
Figure 2
For future reference, we record the following facts about $C_1$ and $C_2$ as functions of $g_2$ and $w_1$: by totally differentiating equations (13) and (14), we obtain
\[
\begin{bmatrix}
U_{11} - g_2 U_{21} \\
g_2
\end{bmatrix}
\begin{bmatrix}
dC_1 \\ dC_2
\end{bmatrix}
= \begin{bmatrix}
U_2 \\
-w_0 a g_1 - C_1
\end{bmatrix} dg_2.
\]

We can solve for
\[
\frac{dC_1}{dg_2} = \frac{U_2 - (U_{12} - g_2 U_{22})(w_0 a g_1 - C_1)}{\Delta} \quad (16a)
\]
\[
\frac{dC_2}{dg_2} = \frac{-U_2 g_2 + (U_{11} - g_2 U_{21})(w_0 a g_1 - C_1)}{\Delta} \quad (16b)
\]

where
\[
\Delta = U_{11} - 2 g_2 U_{21} + g_2^2 U_{22} < 0,
\]

by the second order conditions for utility maximization. Similarly,
\[
\frac{dC_1}{dw_1} = g_2 U_{22} - U_{12}, \quad \frac{dC_2}{dw_1} = \frac{U_{11} - g_2 U_{21}}{\Delta} \quad (17)
\]

The problem of finding the optimal $a$ is to find that $a$ which maximizes
\[
EU(C_1, (w_1 - C_1)g_2) \quad (18)
\]

where $C_1$ and $w_1$ are functions of the random variable $g_2$ and the control variable $a$. For an interior maximum, we require
\[
E[(U_1 - g_2 U_2) dC_1 / da + U_2 (a g_2 - G) w_0] = 0.
\]

But from (14), this simply requires
\[
EU_2[g_2 g_2 - G] = EU_1 \left[ \frac{1}{P_1} - \frac{P_2}{P} \right] = 0. \quad (19)
\]

It is easy to verify that $EU$ is concave in $a$, so (19) is both necessary and sufficient. The maximum value of $U$ for any given value of $g_2$ and $w_1$ may be written $V(w_1, g_2)$. Since $U$ is concave in $C_1$ and $C_2$, $V$ is concave in $w_1$. But for any given $g_2$, $w_1$ is linear in $a$. The result is immediate. (More formally, let $\tilde{a}$ and $\bar{a}$ be two allocations, and
\[
\tilde{a} = \lambda \tilde{a} + (1 - \lambda)\bar{a}, \quad 0 \leq \lambda \leq 1.
\]

Then, for each $g_2$,
\[
\frac{w_1}{w_0} = \left[ \lambda \tilde{a} + (1 - \lambda)\bar{a} \right] g_1 + [\lambda (1 - \tilde{a}) + (1 - \lambda)(1 - \bar{a})] G g_2 = \lambda \frac{w_1}{w_0} + (1 - \lambda) \frac{w_1}{w_0}.
\]

\[
\lambda V(w_1, g_2) + (1 - \lambda) V(\bar{w}_1, g_2) \leq \lambda U(C_1^*, C_2^*) + (1 - \lambda) U(C_1^*, C_2^*)
\]

\[
\leq U(C_1^*, (1 - \lambda)C_1^* + (1 - \lambda)C_2^*) \leq U(C_1^*, C_2^*) \equiv V(\bar{w}_1, g_2).
\]

The first inequality follows from concavity of $U$; the second follows from the fact that
\[
(\lambda C_1^* + (1 - \lambda)C_1^*, \lambda C_2^* + (1 - \lambda)C_2^*)
\]

is feasible with $w_1 = \bar{w}_1$. Thus
\[
EV[w_0 (\tilde{a} g_1 + (1 - \tilde{a}) G g_2), g_2] \geq \lambda EV[w_0 (\tilde{a} g_1 + (1 - \tilde{a}) G g_2), g_2]
\]
\[
+ (1 - \lambda) EV[w_0 (\bar{a} g_1 + (1 - \bar{a}) G g_2), g_2].
\]

For most of our analysis, we will not restrict \( a \) to be between zero and one. If, however, no short sales are allowed, \( 0 \leq a \leq 1 \). Then, if at \( a = 0 \)
\[
EU_2[g_1g_2 - G] < 0
\]
an individual holds exclusively long-term bonds, while if at \( a = 1 \),
\[
EU_2[g_1g_2 - G] > 0
\]
he holds exclusively short-term bonds.

4. CHARACTERIZATION OF OPTIMAL ALLOCATION

(a) Risk neutral individuals. From (19) we immediately derive the result that if the individual is risk neutral, an interior solution requires the long rate to be equal to the product of the expected short rates:
\[
g_1g_2 = G.
\]
If the long rate is greater than the product of the expected short rates, he purchases only longs, and conversely.

(b) Patterns of speculation for risk averse individuals. For a risk averse individual, we can define a non-speculative ("hedging") position, \( \hat{a} \), in a manner analogous to that of our example (5): the allocation, which if the individual did not enter the market during period one, would maximize his utility, i.e. \( \hat{C}_1 \) is equal to his purchases of one-period bonds and \( \hat{C}_2 \) is equal to his purchases of two-period bonds, where
\[
\hat{g}_2 = G/g_1,
U_1(\hat{C}_1, \hat{C}_2) = \hat{g}_2U_2(C_1, \hat{C}_2),
\hat{C}_1 + \hat{C}_2/\hat{g}_2 = w_0\hat{g}_1,
\hat{C}_1 = \hat{a}w_0\hat{g}_1, \quad \hat{C}_2 = (1-\hat{a})w_0G.
\]
Unlike our previous example, where, if the individual completely hedges, his consumption is independent of \( g_2 \), here as \( g_2 \) varies, so do \( C_1 \) and \( C_2 \). Indeed, if \( g_2 > (>) \hat{g}_2 \), \( C_1 < (>) \hat{C}_1 \) and \( C_2 > (>) \hat{C}_2 \). Thus, if \( U_{12} \geq 0 \), \( U_2(C_1, C_2) - U_2(\hat{C}_1, \hat{C}_2) \geq 0 \) as \( g_2 \leq \hat{g}_2 \). From these facts and (19) it immediately follows that \(^1\)
\[
\begin{align*}
&\text{if } \quad \frac{Eg_1g_2}{P} \leq G, \quad \hat{a} < \hat{a}, \\
&\text{if } \quad \frac{E P_2}{P} \leq \frac{1}{P_1}, \quad \hat{a} > \hat{a}.
\end{align*}
\]
In words, if the long rate is greater than or equal to the product of the expected short rates, every one "speculates" on long-term bonds, while if the expected one-period rate of return from holding a two-period bond is less than the certain one-period return, everyone speculates on short-term bonds. These results may be put somewhat differently as follows.

\(^1\) If \( U_{12} < 0 \), these results will hold provided the variance of \( g_2 \) is small. Alternatively, if \( U_{12} < 0 \), if \( C_1 \) and \( C_2 \) are both superior (\( dC_i/dw_1 > 0 \)), \( \hat{a} < 1 \) if \( E g_1g_2 \leq G \) and \( \hat{a} > 0 \) if \( E \frac{P_2}{P} \leq \frac{1}{P_1} \), since under these assumptions
\[
\begin{align*}
sign(dU_2/dg_2)_{a=1} &= -\sign[U_2(U_{21} - g_2U_{22}) + (U_{11}U_{22} - U_{12}^2)C_2/g_2] < 0, \\
sign(dU_1/dg_2)_{a=0} &= \sign[U_3(U_{11} - U_{12}g_2) - (U_{11}U_{22} - U_{12}^2)C_1] < 0.
\end{align*}
\]
\[ \frac{G}{g_1} \geq Eg_2 \quad \text{Everyone speculates on long-term bonds,} \]

If \[ \frac{1}{E_1/g_2} < \frac{G}{g_1} < Eg_2 \]

\[ \frac{G}{g_1} \leq \frac{1}{E_1/g_2} \quad \text{Everyone speculates on short-term bonds.} \]

In contrast to our example, in the middle case, whether the individual speculates and so, whether on short- or long-term bonds, depends not only on his expectations, but on his utility function as well.

The proof of (21) is straightforward. Observe that if \( Eg_1g_2 - G \leq 0 \), at \( a = \hat{a} \),

\[ EU_2(g_1g_2 - G) = E[U_2(C_1, C_2) - U_2(\hat{C}_1, \hat{C}_2)](g_1g_2 - G) + U_2(\hat{C}_1, \hat{C}_2)E(g_1g_2 - G) \leq 0. \]

Since \( EU \) is concave in \( a \), this means that \( \hat{a} > a^* \). Similarly for the other case.

(c) Patterns of specialization. Under what conditions will the individuals specialize in long- or short-term bonds? Consider first the case of an additive utility function (i.e., \( U = U(C_1) + (1 - \delta)U(C_2) \) where \( \delta \) is the pure rate of time preference). Assume \( a = 0 \).

Then, since \( U_{12} = 0 \), we have from (16a)

\[ \frac{dC_2}{dg_2} = \frac{U_1 + U_{11}C_1}{(-\Delta)} \]

\[ = \frac{U_1}{-\Delta} (1 - \rho), \]

(23)

where \( \rho \), it will be recalled, is the Arrow-Pratt measure of relative risk aversion. Hence, if relative risk aversion is between 0 and 1, \( dC_2/dg_2 \) is positive,\(^1\) while if the individual is very risk averse, i.e., has relative risk aversion greater than unity, \( dC_2/dg_2 \) is negative.

If the utility function is logarithmic in consumption (the Bernoulli utility function) then \( dC_2/dg_2 \) is identically zero. Using this with (19), one immediately observes that, since

\[ EU_2(g_1g_2 - G) = E[U_2(C_2) - U_2(\hat{C}_2)](g_1g_2 - G) + U_2(\hat{C}_2)E(g_1g_2 - G), \]

(where \( \hat{C}_2 \) is defined in (20)), at \( a = 0 \),

\[ EU_2(g_1g_2 - G) \geq 0 \text{ if } E(g_1g_2 - G) \geq 0 \text{ and } \rho \geq 1, \]

\[ EU_2(g_1g_2 - G) \leq 0 \text{ if } E(g_1g_2 - G) \leq 0 \text{ and } \rho \leq 1. \]

Since \( EU \) is concave in \( a \), if at a particular value of \( a \), say \( a' \),

\[ dEU/da = EU_2(g_1g_2 - G) > 0, \quad a^* > a', \text{ if } dEU/da < 0, \quad a^* < a'. \]

(This fact will be used repeatedly in the following analysis.) This implies that

\[ \text{if } E(g_1g_2 - G) \leq 0 \text{ and } \rho \leq 1, \quad a^* \leq 0, \]

\[ \text{if } E(g_1g_2 - G) \geq 0 \text{ and } \rho \geq 1, \quad a^* \geq 0. \]

In particular, if there is no risk premium for long-term bonds, i.e., if hypothesis A is true, and if relative risk aversion is less than or equal to one, the individual specializes in long-term bonds, while if relative risk aversion is greater than one, the individual buys both kinds of bonds.

\(^1\) More accurately, \( dC_2/dg_2 \) is non-negative, since if relative risk aversion is below one, the indifference curves hit the axes, and the individual may not change his consumption \( C_2 \) for some changes in \( g_2 \).
An extension to non-additive utility functions is possible. We first observe that
\[
\left( \frac{dU_2}{dg_2} \right)_{\sigma = 0} = U_{21} \left( \frac{dC_1}{dg_2} \right)_{\sigma = 0} + U_{22} \left( \frac{dC_2}{dg_2} \right)_{\sigma = 0} = -\frac{U_2}{g_2} \frac{dC_1}{dw_1} + \left( \frac{U_{12} - U_{22}U_{11}}{\Delta} \right) C_1
\]
\[
= -\frac{U_2}{g_2} \frac{dC_1}{dw_1} \left[ 1 - \left( \frac{(U_{12} - U_{22}U_{11})}{\Delta} \frac{w_1}{U_2} \right) \frac{d\ln C_1}{dw_1} \right].
\]

The numerator of the second term in the brackets has a very natural interpretation.

Assume an individual is given an uncertain wealth, with mean \( \bar{w} \) and variance \( \sigma^2 \), with which he will buy consumption goods in the two periods. What is the certainty equivalent of the uncertain wealth? I.e., if \( V(w, g) \) is the maximum value of \( U \) attainable with wealth \( w \) and interest rate \( g \), for what value of \( x \) is
\[
EV(w, g) = V(x\bar{w}, g).
\]

Taking a Taylor Series expansion around \( \bar{w} \) we find
\[
\frac{1}{2} V_{11} \sigma^2 = \frac{1}{2} \frac{V_{11} \bar{w}}{V_1} \left( \frac{\sigma}{\bar{w}} \right)^2 = x - 1.
\]

In analogy to the Arrow-Pratt measure of relative risk aversion for a one-period model, we can define
\[
\hat{\rho} = \frac{V_{11} \bar{w}}{V_1} \]

as the measure of relative (wealth) risk aversion for a two-period model.\(^1\) But \( V_1 = U_1 \), so
\[
V_{11} = U_{11} \frac{dC_1}{dw_1} + U_{12} \frac{dC_2}{dw_1} = \frac{(U_{11}U_{22} - U_{12}g_2)^2}{\Delta} < 0
\]
if \( U \) is concave. Thus
\[
\hat{\rho} = -\bar{w}(U_{11}U_{22} - U_{12}g_2)/U_2 \Delta.
\]

Substituting in (25)
\[
\left( \frac{dU_2}{dg_2} \right)_{\sigma = 0} = -\frac{U_2}{g_2} \frac{dC_1}{dw_1} (1 - \rho/\eta_1)
\]
where \( \eta_1 = d\ln C_1/d\ln w_1 \) is the wealth elasticity of \( C_1 \). Thus we obtain as before,
\[
\text{If } E(g_1g_2 - G) \leq 0 \quad \text{and} \quad \hat{\rho} \leq \eta_1, \quad a^* \leq 0,
\]
\[
E(g_1g_2 - G) \geq 0 \quad \text{and} \quad \hat{\rho} \geq \eta_1, \quad a^* \geq 0.
\]

If the indifference map is homothetic, this reduces to (24). More generally, (28) says that if hypothesis A is true, then the individual specializes in short-term bonds if and only if two-period relative risk aversion is less than the wealth elasticity of \( C_1 \).

By analogous arguments we can show that if the utility function is additive, from (16a) using the facts that \( w_0[aq_1 + (1-a)g_2/G] = C_1 + (C_2/q_2) \) or \( C_2/q_2 = w_0g_1 - C_1 \) if \( a = 1 \),
\[
(dC_i/dg_2)_{a = 1} = [U_2 + U_{22}C_2]/\Delta = (U_2/\Delta)(1 - \rho).
\]

We can thus establish that
\[
\frac{E P_2}{P} \leq \frac{1}{p_1} \quad \text{and} \quad \rho \leq 1, \quad a^* \leq 1,
\]
\[
\text{If } \frac{E P_2}{P} \geq \frac{1}{p_1} \quad \text{and} \quad \rho \geq 1, \quad a^* \leq 1.
\]

\(^1\) There are other possible measures of risk aversion which may be defined by letting, for instance, \( g_2 \) be random.
THE DEMAND FOR FINANCIAL ASSETS

Under hypothesis B, if individuals have relative risk aversion less than unity they specialize in short-term bonds; if $\rho > 1$, they speculate on short-term bonds but hold some long-term bonds.

In the more general case, letting $\eta_2 = d\ln C_2/d\ln w_1$, the wealth elasticity of $C_2$,

$$E \frac{P_2}{P} \leq \frac{1}{p_1} \quad \text{and} \quad \rho \leq \eta_2, \quad a^* \geq 1,$$

If

$$E \frac{P_2}{P} \geq \frac{1}{p_1} \quad \text{and} \quad \rho \geq \eta_2, \quad a^* \leq 1. \quad (30)$$

(d) Comparison of individuals. The above results show that of two individuals, one with $\rho < 1$, the other with $\rho > 1$, the more risk averse does not necessarily demand

\[ \text{Figure 3} \]
more of the liquid asset; he does “speculate” more. In our example, we showed, more generally, that the less risk averse the individual, the more he speculates. Unfortunately, such a general proposition does not seem to hold for the case where $C_1$ and $C_2$ are not perfect complements. The reason for this is that utility is no longer monotonic in $g_2$. If, however, the range of $g_2$ is suitably restricted, then we can establish the desired result.

As before, if $U^A(C_1, C_2)$ is the utility function of the first individual, and if 

$$U^B(U^A(C_1, C_2))$$

is that of the second, the latter is less risk averse if $U^{B^*} > 0$. Assume, for instance, the first individual has allocated $a^{k_A} < \hat{a}$ to short-term bonds. We define $g'_2$ by

$$V \left( \frac{a^{k_A} g_1 + (1 - a^{k_A}) G}{g'_2} w_0, g'_2 \right) = V(g_1 w_0, g_2),$$

i.e., the maximum level of $g'_2$ at which the individual with allocation $a^{k_A}$ attains a level of utility less than or equal to that attained when $C_1 = \hat{C}_1$ and $C_2 = \hat{C}_2$. Assume $g_2 \leq g'_2$.

Expected utility maximization implies, at $a^B = a^{k_A}$, $EU^B_U(g_1 g_2 - G) = 0$. We wish to evaluate, at $a^B = a^{k_A}$, $EU^B_U(g_1 g_2 - G)$. But if $U$ is the level of utility attained when $g_2 = G/g_1$, then $U \geq \hat{U}$ as $g_2 \geq G/g_1$, so $EU^B_U(g_1 g_2 - G) > 0$, i.e. $a^{k_B} < a^{k_A} < \hat{a}$. A similar result obtains if $a^{k_A} > \hat{a}$, provided $g_2 \geq g''_2$, where $g''_2$ is defined by

$$V \left( \frac{a^{k_A} g_1 + (1 - a^{k_A}) G}{g''_2} w_0, g''_2 \right) = V(g_1 w_0, g_2).$$

One might have conjectured that the greater the elasticity of substitution between $C_1$ and $C_2$ the more the individual speculates. After all, the less he has to lose if he is wrong, since he can always substitute $C_1$ for $C_2$ (or vice versa) (see Fig. 4). But such a proposition does not seem to be true in general.
5. INTERPRETATION

The results presented in the above discussion run counter to much of the literature on the term structure of interest rates and liquidity preference. There are at least three reasons for the discrepancy between our view and the traditional one:

(a) Capital gains. Keynes rightly emphasized the importance of the capital gains or losses in long-term bonds from a change in the (short-term) interest rate. When the short-term interest rate falls, the price of a long-term bond increases: the holder of the bond experiences a capital gain. At the end of one period, the value of the two-period bond is $G/g_2$, so the expected return from holding a two-period bond for one period is $E \frac{G}{g_2} \geq \frac{G}{Eg_2}$ by Jensen’s inequality. Hence even if $G = g_1 Eg_2$, the expected return on a two-period bond held for one period would exceed that for a one-period bond; a risk averse individual would still buy long-term bonds, and a person who is only slightly risk averse would specialize in long-term bonds.\(^1\)

\(^1\) von Weizsäcker has emphasized this point in his analysis of the problem.
(b) Negative covariance with short-term interest rates. The value (yield) on long-
term bonds is negatively correlated with the short-term rate of interest. To see the im-
plications of this, consider a portfolio consisting only of short-term bonds. In this case
the level of utility increases with the rate of interest. Wealth at the end of the initial
period is \( w_0t_1 \) and the budget constraint rotates as \( g_2 \) increases as indicated in Fig. 5a.
Since the value of long bonds varies inversely with the (short term) rate of interest, the
long-term bonds act like "insurance", yielding a high return when utility is low and a low
return when utility is high. Thus, even when the long rate is equal to the product of the
expected short rates, the individual will always buy some long-term bonds.

In practice, the covariance effect is probably even stronger than suggested by our
model which is limited to two types of bonds, since the typical portfolio consists also of
stocks, the return to which appears to be positively correlated with the short-term rate
of interest and therefore negatively correlated with the price of long-term bonds.  

(c) Consumption patterns. The typical analysis of demand for bonds follows the
Keynesian tradition of separating out the savings decision from the portfolio allocation
problem. This is, as we have observed, not really legitimate. The two-period bond is
safe in terms of consumption two periods hence. Thus the variance in the one-period
return of a long-term asset does not mean that it is not a relatively safe asset from the
point of view of the individual's consumption pattern.

PART II. CHANGES IN WEALTH AND PORTFOLIO COMPOSITION

6. INTRODUCTION

Traditional models of liquidity preference have little to say on how the composition
of the portfolio ought to change as wealth increases.\(^2\) Recently, Arrow has shown in the
context of a one-period model with two assets, one of which is perfectly safe, that the
proportion of the portfolio allocated to the risky asset increases, is constant, or decreases
as the individual has decreasing constant, or increasing relative risk aversion. (I1); for
a diagrammatic exposition, see [24].)

Cass and Stiglitz, also using a one-period model, have noted two further properties
of the portfolio composition as a function of wealth:

(a) If there are two assets, one of which is safe, or if there are as many securities as
states of nature, then as wealth increases the variance of the rate of return increases,
remains constant, or decreases as there is decreasing, constant, or increasing relative risk
aversion. (b) The certainty equivalent rate of return for the optimal portfolio decreases
or increases as there is decreasing or increasing relative risk aversion.

Although in this model we have retained the assumption of two assets, it is no longer
unambiguous which is the safe asset, which the risky asset. We would therefore not
expect that the allocation to the short-term asset, even though it is conventionally con-
considered to be "safe", would increase if the individual is increasingly risk averse. We
might, however, expect that if the individual is decreasingly risk averse, as his wealth
increases, he takes a more speculative position, and that the other two properties when
appropriately modified for a two-period model, still obtain. Section 7 shows that this in
fact is the case if \( C_1 \) and \( C_2 \) are perfect complements, while Section 8 establishes some-
what weaker results for the general case.

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1 Miller in his work has emphasized this aspect of the problem.
2 Formulations of portfolio allocation models in which individuals maximize expected utility of the
rate of return implicitly either assume that the portfolio composition is unchanged as wealth changes
(i.e. constant relative risk aversion) or that the utility function changes as wealth changes.
7. WEALTH EFFECTS WITH PERFECT COMPLEMENTARITY

Implicit differentiation of (4) shows that

\[ \frac{da}{dw_0} \sim -E \rho \frac{U'(g_1 g_2 - G)}{1 + \delta + g_2}, \]

(31)

so that

\[ \frac{da}{dw_0} \equiv 0 \text{ as } (a-\hat{a})\rho' \equiv 0 \]

(32a)

or

\[ \frac{d}{dw_0} \left| a-\hat{a} \right| \equiv 0 \text{ as } \rho' \equiv 0, \]

(32b)

where

\[ \rho' = \delta \rho/\hat{C}_1 \]

i.e. if the individual speculates on long-term bonds, his elasticity of demand for long-term bonds is greater than unity (less than unity) if he has decreasing (increasing) relative risk aversion. Conversely, if he speculates on short-term bonds. In either case, he increases (decreases) his speculation (as measured by \( |a-\hat{a}| \)) as he has decreasing (increasing) relative risk aversion.

To verify (32), recall from (8) that \( d\hat{C}_2/dg_2 \equiv 0 \) as \( \alpha \equiv \hat{a} \), and observe that the right-hand side of (31) may be written as follows:

\[ -E[\rho(C_2-\hat{C}_2)] \frac{U'(g_1 g_2 - G)}{1 + \delta + g_2} = -\rho(\hat{C}_2)E \frac{U'(g_1 g_2 - G)}{1 + \delta + g_2}. \]

Using (32) it is easy to verify that the standard deviation of \( C_2 \) (or \( C_1 \)) relative to its mean (the coefficient of variation) increases or decreases with wealth as the individual has decreasing or increasing relative risk aversion. Defining \( EC_2 = \hat{C}_2 \), we obtain

\[ dE \left( \frac{C_2 - \hat{C}_2}{\hat{C}_2} \right)^2 = \frac{2\omega_0}{\hat{C}_2^3} E(C_2 - \hat{C}_2) \left[ \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) \hat{C}_2 - C_2 E \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) \right] \frac{da}{dw_0} (1 + \delta) \]

\[ = 2 \frac{da}{dw_0} \frac{\omega_0}{C_2^2} E(C_2 - \hat{C}_2) \left[ \frac{g_1 g_2 - G}{1 + \delta + g_2} - E \left( \frac{1}{1 + \delta + g_2} \right) g(1 + \delta)^2 \right] \]

\[ = \frac{G(1 + \delta)^2}{1 + \delta + g_2} E \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) \equiv 0 \text{ as } \rho' \equiv 0: \]

(33)

(33) follows from (32) and from that fact that since \( dC_2/dg_2 \equiv 0 \) as \( a \equiv \hat{a} \), and

\[ d \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} E \left( \frac{1}{1 + \delta + g_2} \right) - \frac{1}{1 + \delta + g_2} E \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) \right)/dg_2 \]

\[ = \frac{g_1}{(1 + \delta + g_2)^2} \left[ E \left( \frac{1 + \delta}{1 + \delta + g_2} \right) + E \left( \frac{g_2}{1 + \delta + g_2} \right) \right] > 0, \]

\[ E(C_2 - \hat{C}_2) \left[ \frac{g_1 g_2 - G}{1 + \delta + g_2} E \left( \frac{1}{1 + \delta + g_2} \right) - \frac{1}{1 + \delta + g_2} E \left( \frac{g_1 g_2 - G}{1 + \delta + g_2} \right) \right] \equiv 0 \text{ as } a \equiv \hat{a}. \]

\[ \text{1 The symbol } \sim \text{ means "is of the same sign as".} \]
It can also be shown that the certainty equivalent rate of return increases or decreases as $\rho' \geq 0$. We define the certainty equivalent rate of return, $\tilde{g}$, by

$$U(w_0(\tilde{g} + (1 + \delta + \tilde{g}))) = \min \{C_1, C_2(1 + \delta)\}, \tag{34}$$

i.e. that certain rate of return which would yield the same utility as yielded by the optimally chosen portfolio. Straightforward differentiation shows that

$$\frac{dg}{dw_0} \sim \frac{U'(w_0\tilde{g})}{1 + \delta + \tilde{g}} = \left( EU' \left( \frac{dC_1}{dw_0} \right) \right) + w_0EU' \left( \frac{dC_1}{da} \right) \left( \frac{da}{dw_0} \right)$$

$$= \frac{U'(w_0\tilde{g})}{1 + \delta + \tilde{g}} - EU'C_1 \leq 0 \text{ as } \rho' \geq 0, \tag{35}$$

since $\frac{dC_1}{dw_0} = -\frac{C_1}{w_0}$ and $EU' \frac{dC_1}{da} = 0$ (from (4)), while the sign of $U'(w_0\tilde{g} + (1 + \delta + \tilde{g}) - EU'C_1$ follows immediately from the results of Pratt [20].

8. WEALTH EFFECTS IN THE GENERAL CASE

The first set of results—on the wealth effects of the proportion of wealth held in liquid assets—carry over in a straightforward manner to the more general case, although under slightly more stringent conditions. Implicit differentiation of (19) yields

$$\frac{da}{dw_0} \sim E_{w_1} \left( U_{11} \frac{dC_1}{dw_1} + U_{12} \frac{dC_2}{dw_1} \right) \left( \frac{1}{p_1} - \frac{p_2}{P} \right)$$

$$= -E\rho U_1 \left( \frac{1}{p_1} - \frac{p_2}{P} \right) \tag{36}$$

This clearly equals zero if $\rho$ is constant. (It should be clear that if the utility function is additive and $\rho$ is constant then $\rho$ is constant.) If

$$U = C_1^{\alpha_1} + (1 + \delta)C_2^{\alpha_2} + \tilde{\rho} = \frac{U_{11}U_{22}w_1g_2^2}{(U_{11} + U_{22}g_2^2)U_1} = \left( \frac{U_{11} + U_{22}g_2^2}{U_1} \right) \left( \frac{1}{p_1} - \frac{p_2}{P} \right) \left( \frac{U_{11} + U_{22}g_2^2}{U_1} \right) \left( \frac{1}{p_1} - \frac{p_2}{P} \right)$$

$$= (n-1)(C_1^{-1}C_2^{-1}w_1g_2) = n - 1.)$$

If $\rho$ is viewed as a function of $C_1$ and $C_2$ such that whenever $U$ increases, $\rho$ increases—a natural generalization of increasing relative risk aversion—then $\rho(C_1, C_2) \leq \rho(C_1, C_2)$ as $\left( \frac{1}{p_1} - \frac{p_2}{P} \right)(a^* - \tilde{a}) \leq 0$ (as $(g_1, g_2 - G)(a^* - \tilde{a}) \leq 0$), provided the range of $g_2$ is sufficiently restricted ($g_2^* \leq g_2 \leq g_2^*$, where $g_2^*$ and $g_2^*$ are defined above). This implies that if there is increasing relative risk aversion, the individual decreases his allocation to the bond in which he is speculating. Conversely, if there is decreasing relative risk aversion.

It may be worth while looking in some more detail at the case of additive utility functions. Then, it is easy to show that

$$\frac{da}{dw_0} \sim E - \rho(C_2)\eta_2U_2(g_1g_2 - G) = -E\rho(C_1)\eta_1U_1 \left( \frac{1}{p_1} - \frac{p_2}{P} \right) \tag{37}$$
If \( \eta_1 = \eta_2 = 1 \), then \( \frac{da}{dw_0} = 0 \), since an additive utility function is homothetic if and only if it has constant elasticity. (To see this, observe that if

\[
U(C_1, C_2) = U(C_1) + (1 - \delta)V(C_2),
\]

the marginal rate of substitution between \( C_1 \) and \( C_2 \) is given by \( \frac{(1 - \delta)V'(C_2)}{U'(C_1)} \). If the utility function is homothetic this must be constant along a ray through the origin: \( \frac{(1 - \delta)V'(C_2)}{U'(\lambda C_1)} \) must be independent of \( \lambda \). Thus

\[
\frac{V'(C_2)C_2}{V'(C_2)C_2} - \frac{U'(C_1)C_1}{U'(C_1)} = 0, \text{ for all } C_1, C_2.
\]

More generally if \( a > \hat{a} \), if the variance of \( g_2 \) is small, and if \( \eta_2 \) is constant and positive, then (37) may be approximated by a Taylor series expansion around \( \theta_2 \):

\[
- \left( \rho'(\theta_2) \frac{dC_2}{dg_2} \eta_2 \right) g_1 EU_2(g_2 - \theta)^2.
\]

Since if \( a > \hat{a}, \frac{dC_2}{dg_2} \eta_2 > 0 \) (by (16b)), \( \frac{da}{dw_0} \leq 0 \) as \( \rho' \geq 0 \).

Similarly, if \( \eta_1 \) is constant and positive, and the variance of \( p_2 \) is small, then expanding (37) in a Taylor series around \( \theta_2 = \frac{P}{p_1} \), we obtain

\[
\left( \rho'(\theta_1) \frac{dC_1}{dp_2} \eta_1 \right) \frac{U_1}{p_1 \hat{p}_2} E(\hat{p}_2 - p_2)^2.
\]

From (16a), \( \frac{dC_1}{dp_2} > 0 \) if \( a < \hat{a} \), so

\[
\frac{da}{dw_0} \leq 0 \text{ as } \rho' \leq 0.
\]

The other two characterizations of how portfolios change as wealth changes do not carry over simply to the case where \( C_1 \) and \( C_2 \) are not perfect complements. For instance, while the coefficient of variation of \( C_1 \) and of \( C_2 \) are identical in that case, they are not in general. Although it is possible to prove that the coefficient of variation of either \( C_1 \) and/or \( C_2 \) changes with wealth in the expected manner, such a result is of limited interest.  

\[1\] If, for example, the indifference map is homothetic, and the range of \( g_2(p_2) \) is sufficiently restricted,

\[
\frac{dE((C_1 - \bar{C}_1)(C_2 - \bar{C}_2))^2}{dw_0} = \frac{dE((C_1 - \bar{C}_1)(C_2 - \bar{C}_2))^2}{da} \frac{da}{dw_0} + \frac{dE((C_1 - \bar{C}_1)(C_2 - \bar{C}_2))^2}{da} \frac{da}{dw_0}.
\]

But

\[
\frac{dC_1}{da} = C_1 \frac{1}{a[p_1 + (1 - a)p_2]} \frac{1}{dp_2} \left( \frac{(1/p_1 - p_2)^2}{a[p_1 + (1 - a)p_2]} \right) < 0 \quad \text{and} \quad \frac{dC_2}{da} = C_2 \frac{1}{dp_2} \left( \frac{1}{a[p_1 + (1 - a)p_2]} \right) < 0 \quad \text{and} \quad \frac{dC_1}{da} \frac{dC_2}{da} < 0
\]

is positive for small \( p_2 \), equals zero at a unique point and is then negative. Hence, if \( a < \hat{a} \), so

\[
E(C_1 - \bar{C}_1) \frac{dC_1}{da} C_1 - C_1 \frac{dC_1}{da} < 0
\]

and \( \frac{da}{dw_0} \leq 0 \) as \( \rho \) increases or decreases with utility; thus the coefficient of variation of \( C_1 \) increases or decreases as \( \rho \) decreases or increases with \( U \). Similarly, if \( a > \hat{a} \).
PART III. EFFECTS OF CHANGES IN THE INTEREST RATE

9. STATEMENT OF THE PROBLEM

In this part we consider the effects of a change in interest rates on the relative demands for short- and long-term bonds.

One of the difficulties in analyzing the effects of an increase in the short-term rate of interest is the determination of what happens, as a result of the change, to (a) the price of long-term bonds and (b) expectations about future short-term rates of interest. As we note below the familiar liquidity preference analysis has made confusing, if not contradictory, assumptions.

In the ensuing discussion, we shall focus on two questions:

(a) What role does inelasticity of expectations have in the determination of the demand for short-term assets and its elasticity with respect to changes in the short-term rate? Keynesian theory suggests that this inelasticity of expectations is an important aspect of the speculative demand for money: if expectations about future interest rates fall as the interest rate today falls, a capital loss from holding a long-term bond would be no more likely at a low interest rate than at a high one. But even if expectations are completely inelastic, it is not clear that the traditional argument for the negative slope of the liquidity preference schedule is correct; the usual textbook analysis may be sketched as follows: as the interest rate falls, the price of long-term bonds rises—if the long-term bond is a consol, the price is equal to $1/r_t = 1/(g_1 - 1)$. But since expectations about future short rates are unaffected by what happens today, the expected capital loss from holding the consol increases, so the demand for them decreases, and the demand for short-term bonds increases. But why should the price of a consol rise in proportion to the fall in $g_1$? Only if future expected short rates also fall as $g_1$ falls. But the second part of the argument requires that expectations about future expected short rates be unaffected by the change in short-term interest rates.

We shall show that in fact if expectations are perfectly inelastic and prices of long-term bonds adjust so the term structure remains unchanged (in the sense defined below, p. 341) the demand schedule may be upward or downward sloping, and an increase in the elasticity of expectations may increase or decrease the elasticity of the demand schedule for the liquid assets.

To consider this question of the role of the elasticity of expectations we employ a modification of the usual expectations model:

$$p_2 = (up_1 + (1-u)p^*)v,$$

where $v$ is a random variable with mean one. Thus a change in the interest rate today (or price of short-term bonds) shifts the probability distribution of $p_2$. $u$ is the measure of the elasticity of expectations. If $u = 1$, doubling $p_2$ doubles the mean value of $p_2$. $u = 0$ represents the case of complete inelasticity of expectations: expectations about $p_2$ do not depend on $p_1$ at all. $p^*$ may be interpreted as the long-run “normal” price of short-term bonds.

(b) What is the relation between the maturity structure of the debt and the term structure of interest rates?

In the late fifties and early sixties, there was extensive discussion of whether the government could or should change the term structure of interest rates by changing the maturity structure of the government debt. The “expectations” school took the position that the maturity structure could only affect the term structure by affecting expectations
of future short-term rates. The following results suggest the opposite conclusion: that even if expectations of future short-term rates are unaffected by, e.g. a given change in the maturity structure, the term structure will change.

There is no single definition of what is meant by an unchanged term structure. We employ the following definition: The term structure will be said to be unchanged if the ratio of the price of long-term bonds to short-term bonds is a constant proportion of the expected price of short-term bonds next period.

\[ P/P_1 = \lambda E P_2. \]  
\[ \lambda = 1 \] is one version of the expectations hypothesis (hypothesis B above, p. 323).  

10. EXPECTATIONS ELASTICITIES, THE TERM STRUCTURE, AND CHANGES IN INTEREST RATES

Let us return to our example of Section 2. We consider first the case where the "term structure" (as defined above) remains unchanged. Then, straightforward differentiation shows that

\[ \frac{da}{dp_1} \sim E \left[ \frac{\rho}{p_1} - (1-\rho) \left( \frac{\mu(1+\delta)}{(1+\delta)p_2 + 1} \right) \right] \frac{U'(1-v/\lambda)}{(1+\delta)p_2 + 1}. \]  

(39)

In the polar case where \( u = 0 \) (perfectly inelastic expectations),

\[ \frac{da}{dp_1} \equiv 0 \text{ as } (a^* - \hat{a}) \rho' \equiv 0, \]  
\[ \text{(40a)} \]

or

\[ \frac{d}{dp_1} \Big| a - \hat{a} \Big| \equiv 0 \text{ as } \rho' \equiv 0, \]  
\[ \text{(40b)} \]

i.e. the individual decreases or increases his speculation as he has decreasing or increasing relative risk aversion. Note that an increase in \( p_1 \) (decrease in \( g_1 \)) with inelastic expectations is equivalent to a decrease in \( w_0 \) if the term structure remains unchanged (see Fig. 6).

Whether inelasticity of expectations makes the demand for liquid assets more or less elastic depends on the sign of \( E(1-\rho) \left( \frac{\mu(1+\delta)}{(1+\delta)p_2 + 1} \right) \frac{U'(1-v/\lambda)}{(1+\delta)p_2 + 1} \). If \( \rho \leq 1 \) and \( \rho'(a - \hat{a}) > 0 \) this is negative, i.e. individuals with more elastic expectations have less elastic (with respect to changes in the short-term rate of interest) demands while if \( \rho \geq 1 \) and \( \rho'(a - \hat{a}) < 0 \) just the opposite is true.

Note that if \( \rho \) is constant, if expectations are perfectly inelastic, \( da/dp_1 = 0 \), while if \( \mu > 0 \),

\[ \frac{da}{dp_1} \equiv 0 \text{ as } \rho \equiv 1. \]

More generally, we can establish that

\[ \frac{da}{dp_1} > 0 \text{ if } (a^* - \hat{a}) \rho' > 0, \text{ and } \rho \leq 1 \]

\[ \frac{da}{dp_1} < 0 \text{ if } (a^* - \hat{a}) \rho' < 0 \text{ and } \rho \geq 1. \]

1 An alternative definition, yielding similar results, is the following: The term structure is unchanged if the ratio of the long rate to the product of the expected short rates is constant: \( \frac{g_1g_2}{G} = \lambda' \); \( \lambda' = 1 \) is an alternative version of the expectations hypothesis (hypothesis \( \lambda \)).
Thus when there is some elasticity of expectations, whether \( a \) increases or decreases depends not only on whether the individual is speculating on long-term or short-term bonds, and on whether risk aversion is increasing or decreasing, but also on the degree of risk aversion.

Finally, turning to the question of the effect of a change in the term structure on the demand for short-term bonds, we will establish that the following are sufficient conditions for \( da/d\lambda > 0 \):

(a) \( \rho < 1 \);

(b) Variance of \( v \) is sufficiently small, and \( \lambda \) is near one.

If either of these conditions are satisfied, then increasing the short rate relative to the long increases the relative proportion of short-term assets in the portfolio.

To see (a), we have from (4)

\[
\frac{da}{d\lambda} \approx E \rho \frac{v/\lambda^2 (1-a)}{(a + (1-a)v/\lambda) (1+\delta)p_2 + 1} U' \frac{(1-v/\lambda)}{\lambda^2 ((1+\delta)p_2 + 1)} + \frac{U'v}{\lambda^2 ((1+\delta)p_2 + 1)}
\]

(41)

(b) follows from the fact that if \( E(1-v/\lambda)^2 \) is sufficiently small, the first term of (41) is smaller in absolute value than the second.

For the more general case, the analysis proceeds along lines parallel to that of our example. If expectations are completely inelastic, and the term structure remains unchanged, an increase in \( g_1 \) (decrease in \( p_1 \)) is equivalent to an increase in \( w_0 \) (see Section 8), e.g. \( da/dg_1 (- da/dp_1) \) is positive if there is increasing relative risk aversion and the individual is speculating in long-term bonds, or decreasing relative risk aversion and the individual is speculating in short-term bonds. Similarly, individuals with \( \mu > 0 \) may have more or less elastic demands for liquid assets. Finally, if \( \rho \) is less than unity or the variance in \( p_2 \) is small and \( \lambda \) is near one, an increase in the long rate relative to the short will
always increase the relative allocation to long-term bonds, since
\[
\frac{da}{d\lambda} \sim E - \frac{dU_1}{dw_1} \left(1 - \frac{1}{\lambda}\right) w_0 + \frac{U_1}{\lambda^2} - \frac{1}{\lambda^2} \frac{U_1}{\lambda^2} = E(1 - \beta + \left(\beta w_0 / w_1 p_1 \right)) U_1 v / \lambda^2.
\]

PART IV. UNCERTAINTY

1. CHANGES IN UNCERTAINTY

In the analysis thus far, we have assumed that the distribution of \( p_2(g_2) \) is given or changes in some systematic way when \( p_1 \) changes. In fact, however, the probability distribution of \( p_2(g_2) \) is likely to change markedly—and, perhaps, in a less than systematic manner—as the economy swings from boom to depression. Traditional Keynesian analysis suggests that the demand for the short-term asset—the "safe" asset in the Keynesian model—should increase as uncertainty increases. In our consumption-oriented theory, we have repeatedly observed that when the individual speculate in long-term bonds, short-term bonds act like the relatively safe asset, and when he speculates in short-term bonds, it is the long-term bond which acts like the relatively safe asset. We might, accordingly, expect that an increase in uncertainty leads to a decrease in the demand for the asset in which the individual is speculating. This would mean that, if depression situations are periods of greater uncertainty, the demand schedule for short-term assets shifts to the left if individuals are speculating on short-term assets, to the right if they are speculating on long-term assets. The importance of ascertaining whether in fact this is the case should be clear. If the liquidity preference schedule changed radically with changes in the degree of uncertainty, some doubt would be cast on the relevance of estimates of these schedules based on time series data which express the demand for liquid assets solely as a function of interest rates, income, and wealth. For instance, if in depression situations the conventional presumption that increases in uncertainty lead to an increased demand for liquid assets were correct, then, because of the demand curve for short-term assets would shift to the right in such situations, it might appear as if there were a liquidity trap, even if there were none.

We have, however, already established that certain qualitative properties of the demands for long- and short-term assets do not depend on any property of the distribution of \( g_2(p_2) \) except its mean. For instance, when the long rate is equal to the product of the expected short (Hypothesis A), all individuals speculate in long-term bonds, and individuals with relative risk aversion less than unity sell short-term bonds short. Indeed, for the case of the Bernoulli utility function (constant relative risk aversion of unity) we have established certain quantitative characteristics of the portfolio allocation which depend only on the mean of the distribution; if Hypothesis A is true, individuals hold exclusively long-term bonds, if Hypothesis B is true, they hold exclusively short-term bonds. We would like to establish some more general propositions about the precise effects of a change in the degree of uncertainty. Unfortunately in the general case whether an increase in uncertainty increases, decreases, or leaves unchanged the allocation between short- and

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1 For a general discussion of alternative definitions of increasing uncertainty and their economic implications, see [22].

2 When, for instance, the long rate exceeds the product of the expected short, we may say that the individual holds long-term bonds for speculative purposes or that he holds short-term bonds for precautionary reasons. We might expect that whether the demand for short-term assets increases or decreases depends on whether they are held for precautionary or speculative reasons. Cf. Kahn's discussion [12].

3 As Kahn expressed it, "Sufficient has been said to demonstrate the unsuitability of thinking of a schedule of liquidity preference as though it could be represented by a well-defined curve or by a functional relationship expressed in mathematical terms or subject to econometric processes." [12, p. 250].
long-term bonds depends on the utility function, the original probability distribution, and how the distribution changes. For one simple parametrization of a reduction in uncertainty, the allocation will always be unchanged. Clearly a mixture of a distribution with a given mean and the improper distribution with all its mass at the given mean is less risky than the original distribution:

\[ F(x, \mu) = \mu F(x, 1), \quad x < \bar{x}, \]
\[ F(x, \mu) = \mu F(x, 1) + 1 - \mu, \quad x \geq \bar{x}. \]  

(42)

Note that the above change in the distribution leaves the mean unchanged, but as \( \mu \) increases, the variance increases.

Now assume that Hypothesis A obtains, so that the first order condition for utility maximization may be written

\[ \int U_2(\bar{g}_2 - g_2) dF(g_2, \mu) = \mu \int U_2(\bar{g}_2 - g_2) dF(g_2, 1) = 0 \]

where \( \bar{g}_2 \) is the mean of \( g_2 \), and \( F \) is the distribution function of \( g_2 \). It is obvious that the derivative of the above expression with respect to \( \mu \) is identically zero, so that \( da/d\mu = 0 \).

Similarly, if Hypothesis B obtains, then the first order conditions for utility maximization may be written

\[ \int U_1(\bar{p}_2 - p_2) dG(p_2, \mu) = \mu \int U_1(\bar{p}_2 - p_2) dG(p_2, 1) = 0 \]

where \( \bar{p}_2 \) is the mean of \( p_2 \) and \( G \) is the distribution function of \( p_2 \). Again, it is clear that \( da/d\mu = 0 \).

12. UNCERTAINTY AND UTILITY

Does uncertainty make an individual worse off? Although it is by no means clear what is the most meaningful way to compare certain and uncertain situations (see [3]), the answer generally provided to this question is that it does. Indeed, one of the purposes of the concept of certainty equivalence is to measure the extent to which he is worse off: the amount of wealth an individual would be willing to forego to avoid the uncertainty. Some twenty-five years ago, Hart argued against this view:

"... the mere fact that the expectation value is raised by embracing flexibility is enough to overthrow the standard assumption that measures for meeting uncertainty necessarily involve sacrificing part of the income expectation in order to gain security. The more important devices for meeting uncertainty ... contribute to flexibility and tend to raise income expectations." (p. 115).

Our model provides another example of Hart's point: if we compare the level of utility attained when \( g_2 \) is known for certain (and so must equal \( G/g_2 \)) and when \( g_2 \) is random one is better off in the latter situation than in the former. Under the certain situation, the individual chooses the consumption bundle \((\bar{C}_1, \bar{C}_2)\) (defined above, eq. 20). In the uncertain situation, the individual could have set \( a = \bar{a} \) (also defined in eq. 20). Then, since

\[ C_1 + \frac{C_2}{g_2} = w_0 \left( \bar{a} g_1 + (1-\bar{a}) \frac{G}{g_2} \right) = \bar{C}_1 + \frac{\bar{C}_2}{g_2}, \]

no matter what \( g_2 \) is, \((\bar{C}_1, \bar{C}_2)\) is feasible. When \( 1/g_2 \neq g_1/G \), he could consume some bundle other than \((\bar{C}_1, \bar{C}_2)\) and increase his level of utility (if \( C_1 \) and \( C_2 \) are not perfect complements). Finally, he can always increase his expected utility still further by choosing \( a \) optimally. Thus, if \( a* \) is the optimal allocation,

\[ EU(C_1(g_2, a*), C_2(g_2, a*)) \geq EU(C_1(g_2, \bar{a}), C_2(g_2, \bar{a})) \geq U(\bar{C}_1, \bar{C}_2). \]
PART V. EXTENSIONS OF THE ANALYSIS

So far, we have considered a two-period model, in which at the end of the first period, the individual could only buy a safe one-period bond. Now, we assume that each period the individual can choose between a one- and two-period bond. The analysis of the individual's behaviour under these circumstances requires first an analysis of consumer savings behaviour under uncertainty.

13. SAVINGS UNDER UNCERTAINTY: A DIGRESSION

We consider an individual with given wealth, $w_1$. He wishes to maximize his expected utility:

$$EU(C_1, C_2) = EU((1-s)w_1, sw_1(\alpha g + (1-\alpha)\bar{g})),$$

(44)

where $\bar{g}$ is the (random) rate of return on the risky asset (from the single period point of view), and $g$ is the return on the safe asset, $s$ the proportion of wealth saved and $\alpha$ the proportion of savings put into the safe asset. Both $\alpha$ and $s$ must be chosen optimally. The first order conditions are 1

$EU_1 = EU_2(\alpha g + (1-\alpha)\bar{g})$, \hspace{1cm} (45a) $EU_2(g - \bar{g}) = 0$, \hspace{1cm} (45b)

(a) Comparison of savings rate under certainty and uncertainty. There are at least two stories of how uncertainty affects savings: (a) a risk averse individual, in order to ensure his “minimum standard of living” saves more in the face of uncertainty; (b) a risk averse individual is discouraged from saving for the future by the uncertainty of the return. The former story seems to be preferred by those who have studied consumption behaviour, to help them explain the seemingly higher rate of savings of groups facing greater uncertainty [4, 5]. But as the following analysis suggests, it is by no means clear that this will in general be the case. As we noted above (p. 344), it is not obvious what is the most meaningful way to compare certain and uncertain situations. For our purposes, we compare the savings rate with (for simplicity) a single risky asset (with mean $\bar{g}$) with one where there is only a safe asset, with return equal to $\bar{g}$. Then it should be clear that the savings rate under uncertainty, $s$, will be greater or less than that under certainty, $s^*$, as

$$U_2(s^*w_1, \bar{g}) \leq EU_2(s^*w_1, g)$$

i.e., as $U_2g$ is a convex (concave) function of $g$. But

$$\frac{dU_2g}{dg} = U_2 \left(1 + \frac{U_{22}}{U_2} C_2\right) = U_2(1 - \rho(C_2)),$$

(46)

so

$$\frac{d^2U_2g}{dg^2} = (U_{22} C_2/g)(1 - \rho(C_2)) - U_2 \rho'(C_2)C_2/g.$$  \hspace{1cm} (47)

Hence, $s > s^*$ if relative risk aversion is greater than one and there is decreasing or constant relative risk aversion; while $s < s^*$ if relative risk aversion is less than one and there is increasing or constant relative risk aversion. Note if $\rho$ is constant at unity, savings is unaffected by uncertainty. (See also [7, 8, 13, 19].)

(b) The effect of an increase in “uncertainty” may be similarly analyzed. It is easy to show that if $U_2g$ is a concave (convex) function of $g$, and if we increase the variance in the manner described in Section 10, then savings are reduced (increased). (For a more general discussion of this, see [22].)

1 Throughout this and the next section, we will assume the utility function is additive:

$$U(C_1, C_2) = U(C_1) + (1-s)U(C_2).$$
(c) Effects of increased wealth on average propensity to save. First, we must consider how \( s \) changes with \( w \) in this model in the absence of uncertainty. The first order condition is then simply \( U_1 - U_2g = 0 \). This defines an implicit relation between \( s \) and \( w \), which yields
\[
\frac{ds}{dw_1} \sim -\rho(C_1) + \rho(C_2).
\]
But since \( C_1 \preceq C_2 \) as \( g(1-\delta) \geq 1 \), we obtain the result that
\[
\frac{ds}{dw_1} \sim \rho'[g(1-\delta) - 1]. \quad (48)
\]
If there is constant relative risk aversion or \( g(1-\delta) = 1 \), then the average propensity to save does not change with wealth; if, however, there is increasing relative risk aversion and \( C_2 > C_1 \), or decreasing relative risk aversion and \( C_2 < C_1 \), then the average propensity to save will be rising.

In the case of uncertainty,
\[
\frac{ds}{dw_1} \sim -U_1 \rho(C_1) + E\rho(C_2)[U_2g].
\]
Provided \( Eg(1-\delta) \neq 1 \), for sufficiently small variance this can be approximated by the first terms in a Taylor series expansion, so that (48) still is true.

14. TOWARDS A MORE GENERAL MODEL

We now assume that at the end of the first period the individual can buy either a one-period bond, which yields a return \( g_2-1 \), or a two-period bond, at a price of \( G_2 \), which yields, at the end of the second period a return of \( (G_2/g_3) - 1 \), where \( g_2 \) is the one-period bond rate for the third period. In order to make his initial allocation decision, he now must form expectations not only on \( g_2 \), but also on \( G_2 \) and on \( g_3 \) (conditional on \( g_2 \)). This is, clearly, a considerably more complicated problem. In this section, all we wish to do is to sketch how our analysis may be extended to this case.

The individual wishes to maximize
\[
E \left\{ E \left\{ U \left[ (1-s)w_1, sw_1 \left[ a_2g_2 + (1-a_2)\frac{G_2}{g_3} \right] \right] \right\} \right\}, \quad (49)
\]
where the inside expectation is over all possible values of \( g_3 \) given \((g_2, G_2)\) and the outside expectation is over all sets of \((g_2, G_2)\); where
\[
w_1 = w_0(a_1g_1 + (1-a_1)G_1/g_2);
\]
and where \( a_2 \) is the allocation in the second period, \( a_1 \) in the first, between one- and two-period bonds. Straightforward differentiation with respect to \( a_2 \), using the first order conditions for \( a_2 \) and \( s \), yields
\[
E\{E[U_2|g_2g_1-G_1]\} = 0, \quad (50)
\]
which is identical to the corresponding earlier result.

There are two special cases to which attention should be drawn: If \( C_1 \) and \( C_2 \) are perfect complements or if \( g_2 \geq G_2E(1/g_3) \), and if no short sales are allowed, then \( a_2 = 1 \), and the analysis with two assets available the second period is completely identical to that of Parts I-IV. The former case follows directly from the conditions for expected utility maximization; the latter case follows from an analysis of the patterns of specialization, to which we now turn.
THE DEMAND FOR FINANCIAL ASSETS

First we analyze the patterns of specialization in the second period of his life. We show that in the second period of his life, if \( g_2 \leq G_2E(1/g_3) \) (and \textit{a fortiori}, if \( g_2Eg_3 = G_2 \), under hypothesis A), he never specializes in short-term bonds, while if \( g_2 > G_2E(1/g_3) \) he does specialize in short-term bonds. Letting \( a_2 = 1 \), we observe that

\[
EU_2(sw_1g_2)(g_2 - (G_2/g_3)) = U_2E(g_2 - (G_2/g_3)) \leq 0 \text{ as } E(g_2 - (G_2/g_3)) \leq 0.
\]

Now assume \( a_2 = 0 \). Then

\[
EU_2(sw_1G_2/g_3)(g_2 - G_2/g_3) = E \frac{U_2}{g_3} \{g_2g_3 - G_2\}.
\]

But \( \frac{dU_2/g_3}{dg_3} = U_2 \frac{1}{g_3^2} \{\rho - 1\} \). Thus

- if \( \rho \geq 1 \) and \( E(1/g_3) \geq G_2 \), \( a_2 \geq 0 \),
- if \( \rho \leq 1 \) and \( E(1/g_3) \leq G_2 \), \( a_2 \leq 0 \).

These results are perfectly consistent with those of Section 4, if we observe that both short- and long-term bonds are being held for consumption next period; that is, if we define a non-speculation portfolio analogously to (20), \( a_2 = 1 \).

We can compare the savings rate with that which prevailed in the earlier parts of our study. Let \((C'_2, C'_3)\) be the consumption bundle if there were only a one-period bond available second period. Then

Under Hypothesis A,

\[
EU_2(C_2) \frac{G_2}{g_3} \leq U_2(C'_2) \frac{G_2}{Eg_3} = U_2(C'_2)g_2 = U_1(C'_2) \quad \text{as} \quad \frac{d^2(U_2/g_3)}{dg_3^2} \leq 0.
\]

Under Hypothesis B,

\[
EU_2(C_2) \frac{G_2}{g_3} \leq U_2(C'_2)E \frac{G_2}{g_3} = U_2(C'_2)g_2 = U_1(C'_2) \quad \text{as} \quad \frac{d^2(U_2/g_3)}{d(1/g_3)^2} \leq 0.
\]

Thus, if \( \rho < 1 \), and there is constant or increasing relative risk aversion and constant or decreasing absolute risk aversion, the savings rate is decreased under Hypothesis B and increased under Hypothesis A.

In order to proceed with the analysis for the pattern of specialization with respect to \( a_1 \), further assumptions about the nature of the stochastic process describing short-term rates of interest must be made. Two assumptions are conventionally employed. The first is that the distribution of \( g_2 \) is independent of time. The second is that \( Eg_1 = g_1^{-1} \) (i.e., the Martingale assumption). We shall consider a more general stochastic process which includes both as special cases \(^1\); if \( g^* \) is the normal short-term rate of return, and if \( v \) is a random variable with mean one, then

\[
g_{t+1} = ug_t + (1-u)g^*, \quad Eg_{t+1} = (ug_t + (1-u)g^*).
\]

The first case is the one for which \( u = 0 \), the second for which \( u = 1 \). The Keynesian hypothesis that when interest rates are low, they are more likely to increase (return to the normal level) is represented by a low value of \( u \).

Under hypothesis A, then, we have

\[
G_t = g_tEg_{t+1} = [ug_t + (1-u)g^*]g_t
\]

and

\[
\frac{G_t}{g_{t+1}} = \frac{g_t}{v},
\]

which does not depend on \( u \).

\(^1\) This is a simple generalization of the expectations model presented above, p. 340.
We now observe that if \( a_1 = 0 \)

\[
C_2 = sw_0(Eg_2)g_1 \left( a_2 + \frac{Eg_3}{g_3} (1-a_2) \right), \quad C_1 = (1-s)w_0 \frac{Eg_2}{g_2} g_1. \tag{51}
\]

Substituting this into the first-order conditions for expected utility maximization (45), we obtain the result that

\[
\begin{bmatrix}
-\frac{U_{11}}{1-s} - \frac{EU_{22}C_2}{s} \\
\frac{E}{s} \frac{U_{22}C_2}{s} \left( 1 - \frac{Eg_3}{g_3} \right) - EU_{22}sw_0g_1Eg_2 \left( 1 - \frac{Eg_3}{g_3} \right)^2
\end{bmatrix}
\begin{bmatrix}
ds \\
da_2
\end{bmatrix}
= \begin{bmatrix}
-\frac{U_{11}}{1-s} - \frac{EU_{22}C_2}{s} \\
\frac{E}{s} \frac{U_{22}C_2}{s} \left( 1 - \frac{Eg_3}{g_3} \right) - EU_{22}sw_0g_1Eg_2 \left( 1 - \frac{Eg_3}{g_3} \right)^2
\end{bmatrix}
\begin{bmatrix}
g_2 \\
0
\end{bmatrix}
\tag{dEg_2}
\]

Using this with the facts that

\[
\frac{dEU_2}{dg_2} = \frac{dEU_2}{ds} \frac{ds}{dg_2} + \frac{dEU_2}{da_2} \frac{da_2}{dg_2}, \quad \frac{dEU_2}{ds} = \frac{EU_{22}C_2}{s},
\]

and

\[
\frac{dEU_2}{da_2} = EU_{22}g_1Eg_2 \left( 1 - \frac{Eg_3}{g_3} \right),
\]

we obtain

\[
\frac{dEU_2}{dg_2} \sim -(1-\rho) \left\{ EU_{22}C_2 \frac{E}{s} \left[ U_{22} \left( 1 - \frac{Eg_3}{g_3} \right)^2 \right] - E \left[ \frac{EU_{22}C_2}{s} \left( 1 - \frac{Eg_3}{g_3} \right) \right] EU_{22} \left( 1 - \frac{Eg_3}{g_3} \right) \right\}
\]

\[
= -(1-\rho)sw_0(Eg_2)g_1 \left\{ EU_{22}EU_{22} \left( 1 - \frac{Eg_3}{g_3} \right)^2 - EU_{22} \left( 1 - \frac{Eg_3}{g_3} \right)^2 \right\}. \tag{52}
\]

Using Schwartz's inequality, the term in brackets is clearly positive, so \( \frac{dEU_2}{dg_2} \sim -(1-\rho) \).

Thus, the results obtained earlier (Section 4), that \( a \geq 0 \) as \( \rho \geq 1 \) still obtain. Similarly, if we set \( a_1 = 1 \),

\[
C_1 = (1-s)w_0g_1,
\]

\[
C_2 = sw_0g_1g_2 \left[ a_2 + (1-a_2) \frac{Eg_3}{g_3} \right].
\]

Then

\[
\frac{dEU_2}{dg_2} = EU_{22}C_2 \frac{E}{g_2} + EU_{22}C_2 \frac{ds}{dg_2} + EU_{22}(sw_0g_1g_2) \left( 1 - \frac{Eg_3}{g_3} \right) \frac{da}{dg_2}
\]

\[
\sim \left[ \frac{U_{11}}{1-s} - E \left( \frac{U_{22}g_2Eg_3}{g_3} \right) \right] \left( EU_{22}C_2 EU_{22} \left( 1 - \frac{Eg_3}{g_3} \right)^2 \right)
\]

\[
- EU_{22}C_2 \left( 1 - \frac{Eg_3}{g_3} \right) EU_{22} \left( 1 - \frac{Eg_3}{g_3} \right), \tag{53}
\]

As before, the term in brackets may be shown to be positive. Thus, at \( a_1 = 1 \)

\[
E(EU_2 g_2 - G) < 0,
\]

so the individual does not specialize in short-term bonds.

The comparative statics analysis proceeds in the same manner, with qualitative results analogous to those of the preceding parts. Finally, we note that this sequential decision-making problem serves to illustrate one further point: In Keynesian analysis there is a
separation between the savings-consumption decision, the decision of what proportion of one's wealth to put into safe assets ("money") and what proportion into risky, and the decision of in what proportions different risky assets should be held. Recently, some effort has gone into attempts to determine the conditions under which such separations can be justified. It has been shown, for instance, that when asset returns are statistically independent and stationary over time and utility functions are quadratic (constant elasticity will do as well) these decision can in fact be separated; the proportions of different risky assets in the portfolio will be constant [8, 26] and the portfolio allocation may be done myopically. In our model, even if the individual has a constant elasticity additive utility function, the decisions cannot be separated; the individual cannot allocate his portfolio myopically (expectations about $g_3$ and $G_3$ affect the allocation the first period) and portfolios will not be constant over time. The point is that if there are multiperiod bonds, returns will not in general be statistically independent over time; moreover which is the safe security and which the risky depends on one's horizon, and as the individual grows older this is likely to change.

15. CONCLUDING COMMENTS

This paper has attempted to present a theory of the demand for financial assets based on consumption valuations rather than on capital valuations. We have already mentioned some directions in which this theory needs to be extended. First and foremost, we need an analysis of the supply side of the market: what determines the maturity structure of the bonds offered by, for instance, corporations? Second, account must be taken of the stochastic nature of income in future periods. Third, the analysis needs to be extended to many periods and to more than two assets. Until these studies are completed, only limited statements about the equilibrium in the financial markets can be made. Subject to these qualifications, the major conclusions of this study may be summarized as follows:

(1) This model provides both an explanation of why individuals hold short-term assets even when they could get, on average, a higher return from holding long-term bonds and an explanation of why individuals hold long-term bonds even when they could get, on average, a higher return from holding short-term bonds.

If the long rate equals or exceeds the product of the expected short rates, all individuals speculate in long-term bonds, and individuals who are not very risk averse hold exclusively long-term bonds.\(^1\) If the price of the long-term bond equals or exceeds the ratio of the expected price of a short-term bond next period to the price of a short-term bond this period, everyone "speculates" in short-term bonds, and if the individual is not very risk averse, he holds exclusively short-term bonds. In all other cases, some individuals speculate in long-term bonds, some in short.\(^2\)

(2) It is not correct to treat the long-term bond as a risky asset and the short-term bond as a safe asset. Since, from different horizons each are safe and from different horizons each are risky, at times each may act "more" like a safe asset than the other.

\(^1\) For a definition of speculation in this model and a precise statement of the conditions under which this and all subsequent propositions discussed in these concluding comments are valid, see the previous sections.

\(^2\) These observations suggest a possible indirect test of the expectations hypothesis. If only individuals who may be said, on the basis of other observations, to be not very risk averse ($\rho \leq 1$) specialize in long-term bonds, then the long rate is probably less than the product of the expected short rates. If only the individuals who are not very risk averse specialize in short-term bonds, then the price of the long-term bonds is probably less than the ratio of the expected price of the short-term bond next period to the price of the short-term bond this period. In this analysis one ought probably to treat permanent life insurance policies, annuities, etc., at least partially, as ownership of long-term bonds. It should be clear that the fact that certain institutions, e.g. insurance companies, specialize in long-term assets, and other institutions (banks) specialize in short-term assets is not inconsistent with our theory. For the demands of these institutions depend on the supply of funds to them by individuals, and the determination of the relative supplies of individuals is dependent, at least in part, on the considerations that we have been discussing.
The asset in which the individual is speculating does, however, act very much like a risky asset. For instance, (a) the less risk averse the individual, the more he demands that asset; (b) the demand for that asset decreases with wealth if there is increasing relative risk aversion, increases if there is decreasing relative risk aversion.

(3) If, when short-term interest rates change, prices of long-term bonds adjust so that the term structure of interest rates is unchanged, the proportion of assets in short-term bonds may either increase or decrease; in the case of constant relative risk aversion, demands remain unchanged. In other words, changes in the maturity structure of the debt (the ratio of short- to long-term bonds) will in general change the term structure, even when expectations of future rates are unchanged.

(4) The elasticity of expectations does not play an important role in the determination of the qualitative properties of the demand schedules for liquid or long-term assets. An individual with inelastic expectations may have a more or a less elastic demand schedule for liquid assets than an individual with elastic expectations.

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