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BEHAVIOR TOWARDS RISK WITH MANY COMMODITIES

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This paper investigates the restrictions on the indifference map that are implied by alternative assumptions about consumer behavior under uncertainty and, conversely, the restrictions on consumer behavior under uncertainty that are implied by alternative assumptions about the indifference map It is shown, for instance, that if the individual is to be risk neutral at all incomes and price ratios in a given neighborhood, the income-consumption curves must be linear, and if all income-consumption curves are linear, there exists a cardinal representation of utility which is linear in income The cases of constant relative risk aversion, constant absolute risk aversion, and quadratic utility functions are also investigated

INTRODUCTION

IN THE RECENT development of the theory of consumer behavior, there have been two seemingly disjoint strands: the ordinal utility theory of consumer's choice with many commodities and the Von Neumann-Morgenstern cardinal utility theory of consumer behavior under uncertainty (generally in one commodity economies, or equivalently, many commodity economies with fixed price ratios), where individuals maximize their expected utility. It has not been sufficiently realized just how intimately related these two theories are. It is the purpose of this paper to show what restrictions on the indifference map (e.g., on the income-consumption curves) are implied by alternative assumptions about consumer behavior under uncertainty and, conversely, what restrictions on consumer behavior under uncertainty are implied by alternative assumptions about the indifference map.

The importance of the results obtained here is threefold: First, they suggest further reasons for avoiding particular parameterizations of attitudes towards risk, which at present are extensively used in the literature (e.g., risk neutrality, quadratic utility functions, constant relative risk aversion). Second, they provide us a way, in principle, of ascertaining *some* information about an individual's attitudes towards risk by looking at his Engel curves (and, of course, conversely). Since the latter information is, for the most part, more accessible than direct information on attitudes towards risk from betting experiments,[2] this may prove to be important for empirical work in the field. Third, if the different commodities are interpreted as consumption in different periods, then the techniques developed here can be used to infer restrictions on consumer time preference from certain assumptions about saving behavior under uncertainty, and vice-versa.[3]

[1] I am indebted to R Pollak, A Klevorick, M Rothschild, K Shell, P A Samuelson, and R M Solow; I am responsible for all remaining errors My interest in this topic was stimulated by a question posed by R L Bishop P. A. Samuelson has simultaneously derived some of the results reported in Sections 1.2 and 1 3 in "Linearity of Engel's Curves and Gambler's Indifference," mimeo, M I T The research described in this paper was carried out in part under a grant from the National Science Foundation and from the Ford Foundation

[2] See, e g , F Mosteller and P Nogee, "An Experimental Measurement of Utility," *Journal of Political Economy*, Vol. 59 (1951), pp 371–401

[3] See, e g , P Pestieau, "Epargne et Consommation Dans L'Incertitude Un Modele A Trois Periodes," CORE Discussion Paper, No 6715, 1967, and J. Drèze and F Modigliani, "Consumers' Behavior under Uncertainty," CORE Discussion Paper (forthcoming)

1. RISK NEUTRALITY

1.1. *Risk Neutrality at Discretely Different Price Ratios*

It is well known that risk neutrality is equivalent to linearity in income of the Von Neumann-Morgenstern utility function for fixed prices. We shall now show how to construct utility functions which are linear in income for certain fixed price ratios.

On the x^1-x^2 graph pictured below, where x^1 and x^2 are the two commodities, we have marked off equal distances (of length, say, 1) along the x^1 axis. Through each of the marked points, we draw a line with slope p (the price ratio). We then mark off along the x^1 axis equal distances of length $\lambda < 1$, and through each of these points we draw a line with slope $p' > p$, so that the intersection of the nth p line with the nth p' line occurs in the positive quadrant for all n. Next, draw a family of indifference curves as follows: the nth utility curve is tangent to both the nth p line and the nth p' line. Then draw the lines connecting the points of tangency, which are the income-consumption curves. These in general will not be straight lines. Label each indifference curve with the number of the line to which it is tangent. Letting x^1 be the numeraire and letting y equal income in terms of the numeraire, it is clear that $U = y$ along the income-consumption curve marked CA in Figure 1, and $U = \lambda y$ along the income-consumption curve marked DB in Figure 1.

In fact, it is not difficult to see that we could construct a utility function for which utility is linear in income at any finite number of price ratios p, not arbitrarily close to each other.

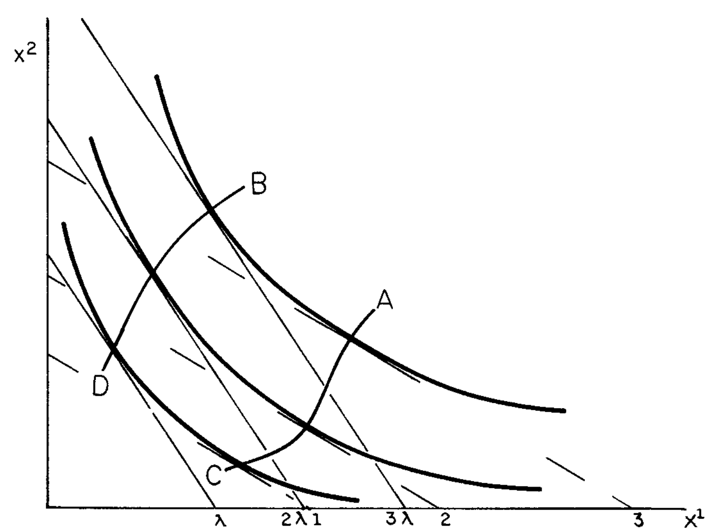

FIGURE 1

1.2 *Proof of Necessity of Linearity of Income-Consumption Curves for Risk Neutrality*

If, however, the individual is to be risk neutral at all prices and incomes in some open region, then all income-consumption curves must be straight lines. To see this, observe that since the individual is risk neutral at fixed prices, utility can be written as a linear function of income, $U = a'y + b'$. If the individual is to be risk neutral at another, nearby, set of prices, we can again write utility as a linear function of income, $U = a''y + b''$, where in general $a' \neq a''$, $b' \neq b''$. If this is to be true for all sets of prices in a given neighborhood, then it must be true that

$$U(y, p^1, \ldots p^n) = a(p^1, \ldots p^n)y + b(p^1, \ldots p^n). \tag{1}$$

But, considering U as a function of all its $(n + 1)$ arguments $(y, p^1, \ldots p^n)$, (1) is just the indirect utility function which expresses the level of utility as a function of income and prices.[4] The linearity of the indirect utility function means that the expenditure function, which gives the minimum level of expenditure E required to obtain a given level of utility $\bar{U}$ at given prices, is linear in $\bar{U}$:

$$E = \frac{\bar{U}}{a} - \frac{b}{a}. \tag{2}$$

The compensated demand curve for the ith commodity, x^i, is then linear in $\bar{U}$:[5]

$$x^i(p^1, \ldots p^n, \bar{U}) = \frac{\partial E}{\partial p^i} = -\frac{\bar{U}}{a}\frac{a_i}{a} - \frac{b_i}{a} + \frac{ba_i}{a^2} \tag{3}$$

where, as usual, $b_i = \partial b/\partial p^i$, $a_i = \partial a/\partial p^i$

The income-consumption curves may be derived by substituting (1) into (3), or directly from the fact that, if $U(y, p^1, \ldots p^n)$ is the indirect utility function,[6]

$$x^i = -\frac{(\partial U/\partial p^i)}{(\partial U/\partial y)} = -\frac{a_i}{a}y - \frac{b_i}{a} \tag{4}$$

These are linear in income, but need not go through the origin, i e., the indifference map need not be homothetic. If, however, an individual is risk neutral at all prices and incomes, all income-consumption curves must be straight lines through the origin, i e , the indifference map must be homothetic.[7]

[4] See H Houthakker, "Compensated Changes in Quantities and Qualities Consumed," *Review of Economic Studies*, Vol 19 (1950–51), pp 155–164

[5] The expenditure function is the analog of the cost function in production theory For a proof that $\partial E/\partial p^i = x^i$, see, e g , P Samuelson, *Foundations of Economic Analysis*, Cambridge, 1948, Chapter IV

[6] See, e.g , R Roy, "La Distribution Du Revenu Entre Les Divers Biens," *Econometrica*, Vol 15 (1947), pp 205–225

[7] The point, of course, is that if we include the whole commodity space, any income-consumption curve which does not go through the origin cannot be linear throughout, at best, it can be a two-segmented broken line, a straight line in the interior and the segment of the relevant axis from the origin to the point where the interior straight line intercepts the axis For a fuller discussion of the implications of linearity of all income-consumption curves for the indifference map, see W M Gorman, "On a Class of Preference Fields," *Metroeconomica*, 13 (August, 1961), and "Community Preference Fields," *Econometrica*, 21 (January, 1953)

1.3 *Implications of Linear Income-Consumption Cures*

The converse of the proposition proved above also turns out to be true: if all income-consumption curves are linear in some open region, there exists a cardinal representation of utility which is linear in income. Gorman has shown[8] that if income-consumption curves are straight lines in an open region in commodity space, then there exist functions $g(p)$ and $h(p)$ such that $x^i = g^i(p) + Uh^i(p)$, where U is the utility index and p is the vector of prices. Multiplying by p^i and summing over all commodities, we obtain the expenditure function of the form in (2), which in turn implies that U is linear in y, for given p.

1.4 *Extensions to Concave and Convex Utility Functions*

If an individual has a utility function that is concave (or convex) in income at a given set of prices, p^*, for all values of $y \leqslant y^*$, then if all income-consumption curves are linear, his utility function will be concave (or convex) in y at any fixed set of prices p for all values of y such that $U(y, p) \leqslant U(y^*, p^*)$.[9] To see this, assume we performed a betting experiment in which it turned out that $U_{yy}(y, p^*) < 0$ for all values of $y \leqslant y^*$. One cardinal representation of the utility function is $\hat{U} = a(p)y + b(p)$. U and $\hat{U}$ must be related by a monotonic transformation F, such that $U(y, p^*) = F(\hat{U}(y, p^*))$, so $U_{yy}(y, p^*) = F''(a(p^*))^2$. In order to have $U_{yy}(y, p^*) < 0$ for $y \leqslant y^*$, we must have $F'' < 0$, for $\hat{U}(0, p^*) \leqslant \hat{U} \leqslant \hat{U}(y^*, p^*)$. But this implies that for all p, $U_{yy} < 0$, provided only that $U(y, p) \leqslant U(y^*, p^*)$.

2. CONSTANT RELATIVE AND ABSOLUTE RISK AVERSION

In the recent literature on uncertainty, two definitions of risk aversion have been used extensively:[10] absolute risk aversion,[11] $-U_{yy}/U_y$, and relative risk aversion, $-U_{yy}y/U_y$. In this section, we investigate the restrictions imposed on the indifference map if relative (or absolute) risk aversion is constant. The interest in this class of functions is due partly to the important role it has played

[8] W M Gorman, "On a Class of Preference Fields," op cit An alternative constructive proof, by P A Samuelson and myself follows (For a detailed discussion, see P A Samuelson, "Linearity of Engel's Curves and Gambler's Indifference," op. cit) If $x^i = \alpha^i(p) + \beta^i(p)y$, from the Slutsky equation, $(\alpha^i_j + \alpha^j\beta^i) + \beta^i_j y = (\alpha^j + \alpha^i\beta^j) + \beta^j_i y$ or $\beta^i_j = \beta^j_i$ and $\alpha^i_j + \alpha^j\beta^i = \alpha^j_i + \alpha^i\beta^j$ So β^i is an exact differential Define $\phi = \oint \Sigma\, \beta^i\, dp^i$ and $a = e^{-\phi}$ Then $\alpha^i a$ is an exact differential since $d\alpha^i a/dp^j = (\alpha^i_j - \alpha^i\beta^j)a = (\alpha^j_i - \alpha^j\beta^i)a = d\alpha^j a/dp^i$ Define $b = -\oint \Sigma\, \alpha^i a\, dp^i$ Then $\beta^i = -a_i/a$, and $-b_i/a = \alpha^i$, so $x^i = -(a_i y/a) - (b_i/a)$, which is identical to equation (4)

[9] If the indifference curves are convex, $U(y, p) \leqslant U(y^*, p^*)$ if the consumption bundle purchased at (y, p) lies in the budget set of (y^*, p^*)

[10] These measures are due to K J Arrow, "Aspects of the Theory of Risk-Bearing," Yrjo Jahnsson Lectures, Helsinki, 1965, and J W Pratt, "Risk-Aversion in the Small and in the Large," *Econometrica*, Vol. 32 (January-April, 1964), for other uses of these measures, see A Klevorick, *Capital Budgeting Under Risk*, Princeton University Press (forthcoming), J Mossin, "Taxation and Risk Taking An Expected Utility Approach," *Economica*, Vol 35, Feb, 1968, pp 74–83, J Stiglitz, "The Effect of Income, Wealth, and Capital Gains Taxation on Risk-Taking," *Quarterly Journal of Economics*, May, 1969, and J Drèze and F Modigliani, op cit

[11] U is a function of both y and p $U = U(y, p)$ Arrow and Pratt assumed in effect that p is constant Thus $U_y = (\partial U/\partial y)$, the marginal utility of income at fixed prices

in the development of the theory of consumer behavior under uncertainty,[12] partly because of its continued use in modern portfolio analysis,[13] and partly because this is the class of utility functions satisfying the Pfanzagl "consistency" axiom that an individual exhibit the same betting behavior at all levels of income.[14]

2.1 *Bernoulli Utility Function*

If relative risk aversion is equal to unity, the indirect utility function is logarithmic in income

$$U(y, p) = w(p) \ln y + v(p). \tag{5}$$

Using again the duality properties of the indirect utility function, we immediately derive the income-consumption curves (demand curves)

$$x^i = -\frac{U_i}{U_y} = -\frac{w_i}{w} y \ln y - \frac{v_i}{w} y. \tag{6}$$

Again if this is to hold globally, the indifference curves must be homothetic. For both large and small y, if $w_i/w \neq 0$, $x^i \approx (w_i/w) y \ln y$. If $w_i/w > 0$, then for large y, $x^i < 0$, and if $w_i/w < 0$, for small y, $x^i < 0$. Thus, $w_i/w = 0$, and the income-consumption curves must be straight lines through the origin. Hence, the indifference curves must be homothetic.

The converse proposition is also true: if the income-consumption curves (demand curves) are of the form

$$x^i = \alpha^i y \ln y + \beta^i y, \tag{7}$$

then there exists a Bernoulli representation of the utility function; we present a constructive proof. From the symmetry of the Slutsky terms,

$$\begin{aligned} \left(\frac{\partial x^i}{\partial p_j}\right)_{\bar{U}} &= [\alpha^i_j + \alpha^j(\alpha^i + \beta^i)] y \ln y + [\beta^i_j + (\alpha^i + \beta^i)\beta^j] y + \alpha^i \alpha^j y (\ln y)^2 \\ &= \left(\frac{\partial x^j}{\partial p_i}\right)_{\bar{U}} = [\alpha^j_i + \alpha^i(\alpha^j + \beta^j)] y \ln y \\ &\quad + [\beta^j_i + (\alpha^j + \beta^j)\beta^i] y + \alpha^j \alpha^i y (\ln y)^2, \end{aligned} \tag{8}$$

[12] See, e g, D Bernoulli, "Exposition of a New Theory on the Measurement of Risk," *Econometrica*, Vol 22 (January, 1954) Translated into English by Dr Louise Sommer from "Specimen Theoriae Novae de Mensura Sortis," *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tomus V, 1738, pp 175–192

[13] See, e g, N Hakansson, "Optimal Investment and Consumption Strategies for a Class of Utility Functions," unpublished Ph D dissertation, UCLA, 1966, and the papers by Arrow, Mossin, and Stiglitz cited in footnote 10 Related work using these parameterizations include J A Mirrlees, "Optimum Accumulation under Uncertainty," December, 1965 (unpublished); D Levhari and T N Srinivasan, "Optimal Savings under Uncertainty," *Review of Economic Studies*, April, 1969, pp 153–164, and E J Phelps, "The Accumulation of Risky Capital," *Econometrica*, Vol. 30, No 4 (1962), pp 729–43

[14] J Pfanzagl, "A General Theory of Measurement Applications to Utility," *Naval Research Logistics Quarterly*, Vol 6, No 4, December, 1959

it is easy to show that $\alpha_j^i = \alpha_i^j$ and $(\alpha^i\beta^j - \alpha^j\beta^i + \beta_j^i - \beta_i^j) = 0$ so that $\Sigma_i \alpha^i\, dp^i$ is an exact differential. Thus, there exists a function A, such that $\partial A/\partial p^i = \alpha^i$; in fact,

$$A = \oint \Sigma\alpha^i\, dp^i. \tag{9}$$

If we let $w = e^{-A}$, then $\alpha^i = -w_i/w$. Similarly, $\Sigma\beta^i e^{-A}\, dp^i$ is an exact differential, since

$$\frac{\partial \beta^i e^{-A}}{\partial p^j} = (\beta_j^i - \alpha^j\beta^i)e^{-A} = (\beta_i^j - \alpha^i\beta^j)e^{-A} = \frac{\partial \beta^j e^{-A}}{\partial p^i}. \tag{10}$$

Thus, we can define

$$v = \oint \Sigma\beta^i e^{-A}\, dp^i \tag{11}$$

so $v_i = -\beta^i e^{-A} = -\beta^i w$. Thus $x^i = -(w_i/w)y \ln y - (v_i/w)y$, which is identical to (6). Hence, there exists a Bernoulli representation of the utility function.

2.2 *The General Case*

If relative risk aversion is constant, but not equal to unity, then the indirect utility function can be shown to be

$$U(y, p) = w(p)y^{v(p)+1} + z(p), \qquad v(p) \neq -1, \tag{12}$$

where $v(p)$ is the degree of relative risk aversion along a given income consumption curve. If $v < 0$, the individual is risk averse; if $v > 0$ the individual is a risk lover. The income-consumption curves implied by (12) are slightly more general than (6).

$$x^i = -\frac{w_i y}{w(v+1)} - \frac{v_i}{(v+1)} y \ln y - \frac{z_i y^{-v}}{w(v+1)}. \tag{13}$$

If the degree of relative risk aversion is not only constant along a given income-consumption curve, but is the same for all income-consumption curves, then the income-consumption curves must be of the form (since then v is constant)

$$x^i = -\frac{w_i y}{w(v+1)} - \frac{z_i y^{-v}}{w(v+1)}. \tag{14}$$

If (13) is to hold globally, the indifference map must be homothetic.[15]

Unfortunately, not every indifference map whose demand curves are of the form

$$x^i(y, p) = \alpha^i(p)y + \beta^i(p)y \ln y + \gamma^i(p)y^{-v(p)} \tag{15}$$

has a constant relative risk aversion representation. For (15) to be derivable from

[15] This is true provided all relative prices are finite and nonzero If $v < -1$, and $z_i \neq 0$ for large y, $x^i \approx -z_i y^{-v}/w(v+1)$, thus $z_i/w(v+1) \leqslant 0$ But, since z is homogeneous of degree zero, $\Sigma z_i p^i = 0$ (This follows also from the budget constraint.) So $z_i = 0$, $\forall i$. If $v > -1$, and $z_i \neq 0$ for small y, $x^i \approx -z_i y^{-v}/w(v+1)$, so again, if $x^i \geqslant 0$, $z_i/w(v+1) \leqslant 0$, so $z_i = 0, \forall i$ In either case, $x^i = -(w_i y/w(v+1)) - (v_i/(v+1))y \ln y$ But this is identical in form to (6), which it has been shown can only hold globally if the indifference map is homothetic.

a constant relative risk aversion utility function,

$$(16) \qquad \beta^i = \frac{-v_i}{v+1}.$$

But none of the usual restrictions on demand curves (e.g. symmetry of the Slutsky terms) guarantee that (16) will be true. If, however, the demand curves are of the form (15), and (16) is satisfied, then there exists a constant relative risk aversion representation of the utility function [16]

2.3 *Constant Absolute Risk Aversion*

Similar results obtain if there is constant absolute risk aversion. The indirect utility function is of the form

$$(17) \qquad U(y, p) = a(p)e^{c(p)y} + b(p), \qquad ac > 0,$$

so the income-consumption curves are of the form

$$(18) \qquad x^i = -\frac{a_i}{ac} - \frac{c_i y}{c} - \frac{b_i e^{-cy}}{ac}.$$

For this to hold globally, if the individual is risk averse ($c < 0$), the indifference map must be homothetic. But if $c > 0$, the indifference map need not be homothetic although it is necessary that $a_i = -b_i$ in order for (18) to hold globally.

If the income-consumption curves are of the form

$$(19) \qquad x^i = \alpha^i + \beta^i y + \gamma^i e^{\lambda y},$$

then there exists a constant absolute risk aversion representation of the utility function if and only if

$$(20) \qquad \beta^i = \frac{\lambda_i}{\lambda}.$$

3. QUADRATIC UTILITY FUNCTION

Another utility function which has been extensively used in the literature is the quadratic utility function, for which the indirect utility function is of the form

$$(21) \qquad U(y, p) = a(p)y^2 + b(p)y + c(p), \qquad a \neq 0.$$

If this is to hold for all prices and incomes in some open region, the demand curves must be of the form

$$(22) \qquad x^i = -\frac{a_i y^2 + b_i y + c_i}{2ay + b}.$$

[16] The proof proceeds as in the previous case; using (16) and the symmetry of the Slutsky terms, we can define $v + 1 = \exp[-\oint \Sigma \beta^i \, dp^i]$, $w = \exp[-\oint \Sigma \alpha^i (v+1) \, dp^i]$, and $z = -\oint \Sigma \gamma^i w(v+1) \, dp^i$ Then $w_i/w = -\alpha^i(v+1)$, $z_i = -\gamma^i w(v+1)$ Substituting in (15), we obtain an expression which is identical to (13)

Note that (22) is a slight generalization of the Törnqvist family of demand curves.[17] He investigated the three special cases where (i) a and c are constants (independent of p); (ii) a is a constant and (iii) c is a constant. In order for (22) to be consistent with a homothetic indifference map, $c_i = (bb_i/2a) - (b^2 a_i/4a^2)$, or $c = b^2/4a$.

4. EMPIRICAL TESTS

Unfortunately, except for the linear income-consumption curves and the Törnqvist family of demand curves, the forms suggested in this note have not been estimated. The studies of Houthakker and Prais,[18] among others,[19] suggest that linear income-consumption curves do not fit the data as well as the other forms tested. Thus, under the hypothesis that the Engel curves we are estimating are in fact those of an individual consumer, we can conclude that the individual is not neutral to risk, at least not at all price ratios near those which actually prevailed.

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[17] L Tornqvist, "Efterfrogån på jordbruksprodukter och dess kanslighet for pris- och inkomstforandringar," av Herman Wold, *Ekonomisk Tidskrift* 43 (June, 1941) See also H Wold and L. Jureen, *Demand Analysis*, John Wiley and Sons (New York, 1953)

[18] S J Prais and H S Houthakker, *The Analysis of Family Budgets*, Cambridge University Press (Cambridge, 1955)

[19] Wold and Jureen report that some of the results of the Tornqvist family are significantly better than with linear income-consumption curves However, since the Törnqvist family was never tested against a more general specification, all that can be said is that such results are not inconsistent with income-consumption curves of the form (22), and hence with a quadratic utility function See H Wold and L Jureen, op cit, Part 5