A Re-Examination of the Modigliani-Miller Theorem

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In their classic paper of 1958, Franco Modigliani and Merton H. Miller demonstrated that the cost of capital for a firm was independent of the debt-equity ratio [13]. Although much of the subsequent discussion has focused on the realism of particular assumptions [3], [7], there have been few attempts to delineate exactly the class of assumptions under which the M-M theorem obtains. In particular, five limitations of the M-M proof may be noted:

1. It depended on the existence of risk classes.
2. The use of risk classes seemed to imply objective rather than subjective probability distributions over the possible outcomes.
3. It was based on partial equilibrium rather than general equilibrium analysis.
4. It was not clear whether the theorem held only for competitive markets.
5. Except under special circumstances, it was not clear how the possibility of firm bankruptcy affected the validity of the theorem.

In Section I, we show in the context of a general equilibrium state preference model that the M-M theorem holds under much more general conditions than those assumed in their original study. The validity of the theorem does not depend on the existence of risk classes, on the competitiveness of the capital market, or on the agreement of individuals about the probability distribution of outcomes.

The two assumptions which do appear to be important for our proof are (a) individuals can borrow at the same market rate of interest as firms and (b) there is no bankruptcy. But it is these assumptions which appear to be the center of much of the criticism of the M-M analysis. In Section II, we show that the M-M results may still be valid even if there are limitations on individual borrowing, and in Section III, we show that the possibility of bankruptcy raises more serious problems, although the M-M theorem can still be shown to hold under somewhat more stringent conditions.

I. The Basic Theorem

Consider a firm whose gross returns, \( X \) (before paying bondholders but after paying all non-capital factors of production) are uncertain. We can consider \( X \) as a function of the state of the world \( \theta \). One dollar invested in a perfectly safe bond yields a gross return of \( r^* \), so that \( r^* - 1 \) is

\[ \text{Except that they must agree that there is zero probability of bankruptcy. See discussion in text.} \]

\[ \text{It should be clear that these assumptions are not completely independent. Presumably, one of the most important reasons individuals cannot borrow at the same rate as firms is that there is a higher probability of default.} \]
the market rate of interest. If there is any chance of bankruptcy, the nominal rate \( \hat{r} \) which the firm must pay on its bonds will depend on the number issued. If principal payments plus interest exceed gross profits, \( X \), the firm goes bankrupt, and the gross profits are divided among the bondholders. Thus the gross return on a dollar invested in the bonds of the firm depends on state \( \theta \)

\[
\hat{r}(\theta) = \begin{cases} 
\hat{r} & \text{if } \theta B \leq X(\theta) \\
\frac{X(\theta)}{B} & \text{if } \theta B \geq X(\theta).
\end{cases}
\]

Earnings per dollar invested in equity in state \( \theta \) are given by

\[
\sigma(\theta) = \begin{cases} 
[X(\theta) - \theta B]/E & \text{if } \theta B \leq X(\theta) \\
0 & \text{if } \theta B \geq X(\theta)
\end{cases}
\]

where \( E \) is the value of the firm's equity. The value of the firm is

\[
V = E + B.
\]

Individuals will be assumed to evaluate alternative portfolios in terms of their income patterns across the states of nature.

We now prove the following proposition.

Assume there is no bankruptcy and individuals can borrow and lend at the market rate of interest. If there exists a general equilibrium with each firm having a particular debt-equity ratio and a particular value, then there exists another general equilibrium solution for the economy with any firm having any other debt-equity ratio but with the value of all firms and the market rate of interest unchanged.

Proof: Let \( w^j \) be the \( j^{th} \) individual's wealth, \( E_i^j \), the value of his shares of the \( i^{th} \) firm, \( B_i \) the number of bonds he owns. Assume the \( i^{th} \) firm, whose value is \( V_i \), issues \( B_i \) bonds. The \( j^{th} \) individual's budget constraint may be written

\[
w^j = \sum_i E_i^j + B_i.
\]

If we let \( \alpha_i = E_i^j/E_i \), the share of the \( i^{th} \) firm's equity owned by the \( j^{th} \) individual, (4) becomes

\[
w^j = \sum_i \alpha_i E_i + B_i.
\]

Then his income in state \( \theta \) may be written

\[
Y^j(\theta) = \sum_{i=1}^n (X_i - r^* B_i) \alpha_i^j
\]

\[
+ r^* \left( w^j - \sum_{i=1}^n \alpha_i^j (V_i - B_i) \right)
\]

\[
= \sum_{i=1}^n X_i \alpha_i^j + r^* \left( w^j - \sum_{i=1}^n \alpha_i^j V_i \right).
\]

If, as \( B_i \) changes, \( V_i \) remains unchanged, the individual's opportunity set does not change, and the set of \( \alpha_i \), which maximizes the individual's utility is unchanged. If

\[
\sum_i \alpha_i^j = 1
\]

before, i.e., demand for shares equaled supply of shares, it still does. The total net demand for bonds is

\[
\sum_j \left( w^j - \sum_i \alpha_i^j (V_i - B_i) \right) + \sum_i B_i
\]

\[
= \sum_j w^j - \sum_i V_i.
\]

If the market was in equilibrium initially,

\[
\sum_j w^j - \sum_i V_i = 0,
\]

i.e., excess demand equalled zero. If as the debt equity ratio changes, all \( V_i \) remain unchanged, excess demand remains at zero.

\[\]
An alternative way of seeing this is the following. We may rewrite (6) as

\[(6') \quad Y^i(\theta) = \sum_i \epsilon_i(\theta) E_i^j + r^* \left( w^j - \sum_i E_i^j \right).\]

Assume now that the first firm, say, issues no bonds. If we let carets denote the values of the various variables in this situation, the opportunity set is given by

\[(6'') \quad \hat{Y}^i(\theta) = \sum_i \hat{\epsilon}_i(\theta) \hat{E}_i^j + \hat{r}^* \left( \hat{w}^j - \sum_i \hat{E}_i^j \right).\]

Assume \( r^* = \hat{r}^* \), \( E_i = \hat{E}_i, \hat{E}_1 \geq 2 \). Then from (2), \( \epsilon_i(\theta) = \hat{\epsilon}_i(\theta), \hat{E}_1 \geq 2 \). If \( \hat{E}_1 = E_1 + B_1 \), then the opportunity sets described by (6') and (6'') are identical. To see this, assume that for each dollar of equity he owned in the first firm in the initial situation, the individual borrows \( B_1 / E_1 \) in addition to \( B^i \)

so \( \hat{B}^i = B^i + E_1 B_1 / E_1 \).

With the proceeds of the loan he increases his holdings of equities in the first firm, so

\[(7) \quad \hat{E}_1^j = E_1^j + E_1 B_1 / E_1 = E_1 \left( V_1 / E_1 \right).\]

His income in state \( \theta \) is then given by

\[(8) \quad \hat{Y}^i(\theta) = \frac{X_i E_1^j}{E_1} + \sum_{i=2}^n \epsilon_i(\theta) E_i^j + r^* \left( \hat{w}^j - \sum_i \hat{E}_i^j - E_1^j V_1 / E_1 \right) = \left( \frac{X_i - r^* B_1}{E_1} \right) E_1^j + \sum_{i=2}^n \epsilon_i(\theta) E_i^j + r^* \left( \hat{w}^j - \sum_i \hat{E}_i^j \right)\]

which is identical to (6').

Since his opportunity set has not been changed as a result of the change in the debt-equity ratio of the firm, if he was maximizing his utility in the initial situation, the optimal allocation in the new situation is identical to that in the initial situation with the one modification given above.

We now need to show that the markets for the firm's equities and the market for bonds will clear. Summing (7) over all individuals, we obtain

\[\sum_j E_j^i = V_1 / E_1 \sum_i E_j^i.\]

Thus the demand for equities has increased by a factor \( V_1 / E_1 \). But since \( \hat{E}_1 / E_1 = V_1 / E_1 \), the supply has increased by exactly the same proportion, so if demand equalled supply before it also does now. Similarly, the increase in the demand for bonds by individuals equals \((B_1 / E_1) \sum E_j^i = B_1 \). But this exactly equals the decrease in the demand for bonds by the first firm.

It should be emphasized that in this proof, \( X_i(\theta) \) is subjectively determined; moreover no assumptions about the size of firms, the source of the uncertainty, and the existence of risk classes have been made. The only restriction on the individual's behavior is that he evaluates alternative portfolios in terms of the income stream they generate. The two crucial assumptions were (a) all individuals agree that for all firms \( X_i(\theta) > r^* B \) for all \( \theta \) (see Section III); and (b) individuals can borrow and lend at the market rate of interest. This assumption is considerably weaker than the assumption of a competitive capital market, since no assumption about the number of firms has been made: the market rate of interest need not be invariant to the supply of bonds by any single firm.

II. Limitations on Individual Borrowing

One of the main objections raised to the M-M analysis is that individuals cannot borrow at the same rate of interest as firms. First, it should be noted (see [13])
that the analysis does not require that individuals actually borrow from the market, but only that they change their holdings of bonds. A problem can arise then only if an individual has no bonds in his portfolio.

Although the requirement that all individuals hold bonds does place restrictions on the possible debt-equity ratios of different firms, there still need not be an optimal debt-equity ratio for any single firm. Assume we have some general equilibrium situation where \( B_j \geq 0 \) for all \( j \). Then so long as \( B_i \) satisfy the inequalities

\[
(9) \quad \sum_i \alpha_i B_i \geq \sum_j \alpha_j V - \sum_i \alpha_i \nu_i \quad \text{for all } j
\]

all individuals will be lenders. If there were two firms, the constraints (9) would imply that \((B_1, B_2)\) lie in the shaded area shown in Figure 1. For any pair of \((B_1, B_2)\) in the region, there will exist a general equilibrium in which the values of both firms are identical to that in the original situation.

So far, none of our results have depended on the existence of risk classes.\(^6\) The following two results depend on more than one firm having the same pattern of returns across the states of nature.

We shall first show that if there are two (or more) firms with the same pattern of returns and individuals can sell short, then the two firms must have the same value, independent of the debt-equity ratio.

We follow M-M in assuming for simplicity that one of the two firms has no outstanding debt, so \( V_1 = E_2 \). The second firm issues \( B_2 \) bonds, so \( V_2 = B_2 + E_2 \).

Consider first an individual who owns \( \alpha_1 \) of the shares of the first firm, yielding an income pattern \( \alpha_1 X_1(\theta) \). If instead he purchases \( \alpha_2 \) of the shares of the second firm, at a cost of \( \alpha_2 E_2 \) and buys \( \alpha_2 B_2 \) bonds, his income in state \( \theta \) is \( \alpha_1 (X_1(\theta) - rB_2) + \alpha_2 rB_2 = \alpha_1 X_1(\theta) \) which is identical to his income in state \( \theta \) in the previous situation. But the cost of purchasing \( \alpha_1 \) of the shares of the first company is \( \alpha_1 V_1 \) which is greater than \( \alpha_1 (E_2 + B_2) = \alpha_1 V_2 \) if \( V_1 > V_2 \). Accordingly, if \( V_1 \) were greater than \( V_2 \), all holders of shares in the first company would sell their shares and purchase shares in the second firm, driving the value of the second firm up and that of the first down. Now consider an individual who wishes to lend money. If he sells short \( \alpha_2 \) of the shares of the second firm and buys \( \alpha_2 \) of the shares of the first firm, he receives a perfectly safe return of \( -\alpha_2 (X_2 - rB_2) + \alpha_2 X_1 = \alpha_2 rB_2 \) at a net cost of \( -\alpha_2 (V_2 - B_2) - \alpha_2 V_1 \) so the return per dollar is

\[
r^* = \frac{B_2}{V_1 - (V_2 - B_2)} = r^* \frac{1}{1 + \frac{V_1 - V_2}{B_2}}
\]

If \( V_1 < V_2 \), the individual can obtain a perfectly safe return in excess of \( r^* \). It follows

\(^{6}\) Two firms, \( i \) and \( j \) are in the same risk class if \( X_i(\theta) = \lambda X_j(\theta) \) for all \( \theta \). In the remainder of the discussion we shall assume, for convenience, that \( \lambda = 1 \).

\(^{7}\) As usual, we assume no transactions costs and that there is no cash margin requirement on short sales. (See fn. 9.)
immediately that equilibrium in the capital market requires \( V_1 = V_2 \).

Similar arguments can be used to show the following.

If there are three or more firms in the same risk class, and the firms with the highest and lowest debt equity ratios have the same value, then the value of all other firms must be the same.

This is true whether individuals can borrow or can sell securities short. This result rules out the possibility of a U-shaped curve relating the value of the firm to the debt-equity ratio.

III. Bankruptcy

Bankruptcy presents a problem for the usual proofs of the M-M theorem on two accounts: first, it means that the nominal rate of interest which the firm must pay on its bonds will increase as the number of bonds increases. (M-M have treated the case where it increases at exactly the same rate for all firms and individuals.) Second, if a firm goes bankrupt, it is no longer possible for an individual to replicate the exact patterns of returns, except if he can buy on margin, using the security as collateral; and if he defaults, he only forfeits the security and none of his other assets. To see this, consider the two alternative policies considered in Section I: in the one case, the firm issues no bonds (hence no chance of default) and in the other it issues \( \beta \) bonds. We have shown how the individual by buying stock on margin in the latter case can exactly replicate the returns in the former situation in those states where the firm does not go bankrupt if the value of the firm is the same in the two situations. But if the firm goes bankrupt in some state, \( \theta' \), in the one case his return is zero, while in the other his return per dollar invested is

\[
\frac{X(\theta')}{V} \left(1 + \frac{\beta}{E} \right) - \frac{\beta}{E} < 0.
\]

If, however, he can forfeit the security then his return will again be zero.

Of course, if the firm has a positive probability of going bankrupt, it will have to pay a higher nominal rate of interest. But if the individual is to use the security as collateral, he, too, will have to pay a higher nominal rate of interest. And indeed, it is clear that the two will be exactly the same, since the pattern of returns on the bonds in bankruptcy will be the same. Thus, we have shown that

- if a firm has a positive probability of going bankrupt, and an individual can borrow using those securities as collateral (so that if his return from the securities is less than his borrowings, he can forfeit the securities) the value of the firm is invariant to the debt-equity ratio.

It should be noted that the validity of this proposition does not require 100 percent margins. The required margin is only \( \beta/V \).

Individuals may, of course, not be able to make the limited liability arrangements or to obtain the level of margin required
by the above analysis. Then, a firm by pursing alternative debt-equity policies may be able to offer patterns of returns which the individual cannot obtain in any other manner (i.e. by purchasing shares in one or more other firms), and the value of the firm may consequently vary as the firm changes its debt-equity ratio. In the following subsections, we consider some special situations in which M-M results may still be valid, even though there is a finite probability of bankruptcy.

Risk Classes

If there are a large number of firms in the same risk class, then potentially they can all supply the same pattern of returns. If all firms maximize their value, then in market equilibrium all firms will have the same value.\(^{10}\) Firms may have different debt-equity ratios and the same value for a number of reasons. For instance, assume that some individuals, for some reason or other, prefer a low debt equity-ratio, and some prefer a high debt-equity ratio. Then, some firms may have a high debt-equity ratio, some a low one. If one firm observes another firm in the same risk class with a different debt-equity ratio but a higher value, it will change its debt-equity ratio. Thus the observation that all firms in a given risk class have the same value but possibly different debt-equity ratios can be taken as evidence that firms are value maximizers and are in market equilibrium. It is not necessarily evidence that the arbitrage activities described by Modigliani and Miller have occurred, or that the value of the firm would be the same at some debt-equity ratio other than those actually observed.

Assume the market is in equilibrium, with \(V = \rho EX\) for all members of the risk class. The securities sold by a firm are completely described by the risk class and the debt-equity ratio. A new, small firm is created, belonging to the same risk class, with mean return \(\bar{X}\). If it chooses a debt-equity ratio used by other firms in the same risk class, the price of its shares must be the same as those of the other firms (since they are identical) so its value will be \(X_0\). But, if it chooses some other debt-equity ratio, its value may be lower (if, for instance, there is a positive probability of bankruptcy).\(^{11}\)

Mean-Variance Analysis and the Separation Theorem

In this subsection we consider the special case where all individuals evaluate alternative income patterns in terms of their mean and variance. For simplicity, let us assume that only the first firm issues enough bonds to go bankrupt. If all individuals agree on the probability distribution of returns for each firm, it can be shown that the

\[\ldots \text{total market value of any stock in equilibrium is equal to the capitalization at the risk-free interest rate } r^*, \text{ of the certainty equivalent} \ldots \text{ of its uncertain aggregate dollar return;} \ldots \text{ the difference} \ldots \text{ between the expected value of these returns and their certainty equivalent is proportional for each company to its aggregate risk represented by the sum of the variance of these returns and their total covariance with those of all other stocks, and the factor of proportionality is the same for all companies in the market.}[11, \text{pp. 26-27}].\]

This implies that

\begin{equation}
E_i + Bi = \frac{\left\{\bar{X} - k \sum_{j=1}^{n} \delta(X_i - \bar{X}_i)(X_j - \bar{X}_j)\right\}}{r^*},
\end{equation}

\[i = 2, \ldots, n\]

\(^{10}\) Recall that we have assumed for expositional convenience that \(X_i(\theta) = X_j(\theta)\) for all firms in the risk class.

\(^{11}\) This also may occur with taxes if interest payments are tax deductible and if capital gains are treated preferentially. See [8].
(11) \[ E_t = \left\{ Z - k \sum_{j=1}^n \mathbb{E}(Z - \bar{Z}) (X_j - \bar{X}_j) \right\} / r^* \]

(12) \[ B_1 = \left\{ \bar{r}B_1 - k \sum_{j=1}^n \mathbb{E}(\bar{r} - \bar{r}) B_1 (X_j - \bar{X}_j) \right\} / r^* \]

where \( \mathbb{E} \) is the expectations operator,

\[ Z = \max(X_1 - rB_1, 0), \quad \mathbb{E}Z = \bar{Z}, \]

\[ \mathbb{E}X_i = \bar{X}_j, \quad \mathbb{E}r = \bar{r}. \]

and

\[ k = r^* \left( \frac{\sum_i (X_i - \bar{X}_i)}{\sum_i \sum_j \mathbb{E}(X_i - \bar{X}_j)(X_j - \bar{X}_j)} \right). \]

Then adding \( B_1 \) and \( E_t \), ((11) and (12)), we obtain

\[ V_1 = E_1 + B_1 = \left\{ \bar{X}_1 - k \sum_{j=1}^n \mathbb{E}(X_1 - \bar{X}_j) (X_j - \bar{X}_j) \right\} / r^* \]

independent of the debt-equity ratio.

The intuitive reason for this should be clear: it is well-known that if all individuals agree on the probability distribution of the risky assets, if there exists a safe asset, and if individuals evaluate income patterns in terms of mean and variance, the ratio in which different risky assets are purchased will be the same for all individuals, i.e. all the relevant market opportunities can be provided by the safe asset and a single mutual fund which (in market equilibrium) will contain all the risky assets, including the risky bonds. More generally, whenever the ratio in which different risky assets are purchased is the same for all individuals, then the M-M theorem will be true even with bankruptcy. For a complete discussion of the conditions under which the separation theorem obtains, see Cass and Stiglitz [4].

If, however, (a) all individuals do not agree on the probability distribution of \( X_i (\theta) \) or, (b) the conditions under which the separation theorem is valid do not obtain, then the value of the firm will in general depend on the debt-equity ratio.\[12\]

**Arrow-Debreu Securities**

Arrow [1] and Debreu [5] have formulated a model of general equilibrium under uncertainty in which individuals can buy and sell promises to pay if a given state of the world occurs. See also Hirshleifer [10]. A stock market security and a bond can be viewed as a bundle of these Arrow-Debreu securities. If there is a sufficient number of different firms, equal to or greater than the number of states of na-
ture, then the market opportunities available to the individual (by purchasing or selling short different amounts of the market securities) are identical to those of a corresponding Arrow-Debreu market. If a promise to pay one dollar in state \( \theta \) has a price \( p^*(\theta) \),
then the value of the firm's equity is
\[
E = \sum_{\theta} (X(\theta) - \bar{\rho} B) p^*(\theta).
\]
If
\[
\bar{\rho} = \left[ 1 - \sum_{\theta} \frac{X(\theta)}{B} p^*(\theta) \right] / \sum_{\theta} p^*(\theta)
\]
where \( \theta \{ X(\theta) \geq \bar{\rho} B \} \), i.e. the states of nature in which the firm does not go bankrupt, and \( \theta' \{ X(\theta) < \bar{\rho} B \} \), then
\[
E = \sum_{\theta} X(\theta) p^*(\theta) - B
\]
i.e.
\[
V = E + B = \sum_{\theta} X(\theta) p^*(\theta)
\]
independent of the debt-equity ratio.

Three observations are in order: First, individuals do not need to agree on the probability of different states of nature occurring, i.e. they may disagree on the probability distribution of the returns to any firm.\(^{14}\) Second, if there are fewer firms than states of nature, whether there are as many securities as states of nature is a function of the debt-equity ratio. If there are four states of nature and two firms, and if neither firm issues enough securities to go bankrupt, then there will only be three securities, but if one of the firms goes bankrupt, there will be four. Although the latter situation will be Pareto optimal (the marginal rate of substitution between consumption in any two states identical for all individuals), the value of the firm which goes bankrupt may be larger or smaller in the former situation than in the latter.\(^{15}\)

Third, if we take literally the Arrow-Debreu definition of a state of nature, there undoubtedly will be more states of nature than firms. Yet, in some sense, most of these states are not very different from one another. For example, much of the variation in the return on stocks can be explained by the business cycle. If in any given business cycle state, the variance of the return were very small, and there were a small number of identifiable business cycle states, then the economy might look very much as if it were described by an Arrow-Debreu securities market.\(^{16}\)

**Bankruptcy and Perfect Capital Markets**

The usual criterion for a perfectly competitive market is that the price of a commodity or factor an individual (or firm) buys or sells be independent of the amount bought or sold and be the same for all individuals in the economy. On this basis, it has been argued that the capital market is imperfectly competitive: (a) as a firm issues more bonds the rate of interest it pays may go up; (b) individuals may have to pay a higher interest rate than firms, and some firms higher than others; (c) lending rates may differ from borrowing rates. In this section, we have, however, considered perfectly competitive

\(^{12}\) If there are no Arrow-Debreu securities on the market, \( p^*(\theta) \) is the net cost to the individual of increasing his income in state \( \theta \) by one dollar, i.e. by buying and selling short different securities. If there are more securities than states of nature, market equilibrium requires that the set of market prices generated by considering any subset of market securities which span the states of nature be independent of the particular subset chosen. For a more thorough discussion of these problems, see [4].

\(^{14}\) They must, however, not assign zero probabilities to different states of nature occurring.

\(^{15}\) In this situation we cannot assume that firms will necessarily maximize market value. (See fn. 12.)

\(^{16}\) The point is that under these conditions the individual, by diversification of his portfolio, can essentially eliminate the variations in returns within a given business cycle state.
capital markets (with bankruptcy) in which all three of these would be true. See also [22]. Thus the possibility of bankruptcy makes somewhat questionable the interpretation of much of this evidence of an imperfect capital market. The crucial fallacy lies in the implicit assumption that one firm’s bond is identical to another firm’s bond, and that bonds a firm issues when it has a low debt-equity ratio and those which it issues when it has a high debt-equity ratio are the same. But they are not. They give different patterns of returns. If there is any chance of default, a bond gives a variable return (i.e. is a risky asset). Just as there is no reason to expect butter and cheese, even though they are related commodities, to have the same price, so there is no reason to expect the nominal rate of interest where there is a low debt-equity ratio to be the same as when there is a high debt-equity ratio. Even the discrepancy between borrowing and lending rates does not imply imperfect capital markets, for when a person lends to the bank and the account is insured by FDIC, he can assume there is a zero probability of bankruptcy, but when the bank lends back to the same individual, it cannot make the same assumption.

References

Transactions costs may also partly explain (b) and (c).


