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*Note on a Social System Composed of Hierarchies with Overlapping Personnel**

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Introductory remarks

Casual observation shows that many people are subject to supervision by different superiors for different activities (economic, political, scientific, recreational, social, educational, religious, ...) in which they engage. At the same time, one can view many societies as being composed of organizations that have an essentially hierarchical internal structure. The purpose of this note is to show that these two observations are logically entirely compatible. They are reconciled if we regard the supervision relation that generates hierarchical structure as being not total, but limited to one or a few specific activities in each instance. Since this is a matter of a formal, logical or mathematical, character we can deal with it without facing the more difficult sociological questions as to what "supervision" really consists of and on what terms it is accepted by the supervisee. [1]) Neither need we spell out in the present context what the activities in question are or might be, thus maintaining flexibility of interpretation.

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*) The ideas developed in this note originated from a joint investigation with J. Michael Montias, the results of which will be published as a paper, "On the Description and Comparison of Economic Systems," in a volume edited by Alexander Eckstein, reporting on a Conference on Comparison of Economic Systems held at the University of Michigan, Nov. 1968. I am indebted to Truus W. Koopmans, Gerald Kramer and J. M. Montias for valuable comments.

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[1]) For discussions of these questions, see, for instance, H. A. Simon, "A Formal Theory of the Employment Relation," Econometrica, Vol. 19, July, 1951, pp. 293-305, reprinted in H. A. Simon, Models of Man, Wiley, 1957. Also P .M. Blau and W. R. Scott, "Formal Organizations," Chandler, 1962.

Hierarchies for a Single Activity

Let **A** denote a finite set of *activities,* a, a^{I} ..., in which people engage, **B** a finite set of *persons,* b, b^{I}, ..., each of whom pursues one or more of these activities. (One may think of persons as individuals, but also as committees, boards, governing bodies with specific decision-making procedures). Denote by $c = (a, b)$ an *engagement,* that is, a pair composed of an activity a and a person b engaged in it.

Denote by **C** the set of all engagements $c = (a, b)$ assumed or observed to be in operation in a given state of society. Then **C** is a given subset

P1 $$\mathbf{C} \subset \mathbf{A} \times \mathbf{B}$$

of the set of all imaginable engagements (a, b) in any activity a of **A** by any person b of **B**, technically known as the "Cartesian product"

$$\mathbf{A} \times \mathbf{B} = \{ (a, b) \mid a \varepsilon \mathbf{A}, \ b \varepsilon \mathbf{B} \}$$

of **A** and **B**.

For each activity a of **A** we define the set

D1 $$\mathbf{B}_a = \{ b \mid (a, b) \varepsilon \mathbf{C} \}$$

of all persons engaged in that activity. On that set we define a supervision relation, denoted by

$$b^{\mathrm{I}} \overset{1}{>}_a b ,$$

and interpreted as " b^{I} supervises b in activity a ." We postulate that

P2 $$b^{\mathrm{I}} \overset{1}{>}_a b , \quad b^{\mathrm{II}} \overset{1}{>}_a b \quad \text{imply} \quad b^{\mathrm{I}} = b^{\mathrm{II}} ,$$

a roundabout way of saying that each person b engaged in an activity a has at most one supervisor in that activity.

By iterated application of the supervision relation $\overset{1}{>}_a$ we define an n-th superior b^n of b in activity a by:

D2 $b^n \overset{n}{>}_a b$ if there exists a sequence of n—1 persons $b^1, \ldots, b^{n-1}$, where $n \geqq 2$, such that

$$b^n \overset{1}{>}_a b^{n-1} \overset{1}{>}_a \ldots \overset{1}{>}_a \overset{1}{b} \overset{1}{>}_a b .$$

It follows from P2 that for each n there exists at most one n-th superior. If one exists for given n, the sequence $b^1, b^2, \ldots, b^{n-1}$ consists of the unique 1st, 2nd, $(n-1)$th superiors, provided we define the first superior as the supervisor of b. If we add the postulate

P3 for no a, b, n does $b \overset{n}{>}_a b$ hold,

then no one is superior to himself in any activity (a good organisational rule!), the persons $b^1, \ldots, b^n$ in D2 are all distinct, and the number n is also unique, given b^n. Finally we define the general superiority relation for the activity a by

D3 $b^{\mathrm{I}} >_a b$ means $b^{\mathrm{I}} \overset{n}{>}_a b$ for some $n \geqq 1$.

In order to explore the organizational structure implicit in these definitions and postulates, we define, for any $a \varepsilon \mathbf{A}$ and $b \varepsilon \mathbf{B}_a$,

D4 *the hierarchy* $H_a(b)$ *for the activity* a, *headed by* b, is the set of all persons to whom b is superior for a, with b himself added in.

As a special case, $H_a(b)$ may consist of the single member b. If more than one member exist, one readily verifies that, for any two distinct members $b^{\mathrm{I}}, b^{\mathrm{II}}$ of $H_a(b)$, one of the following mutually exclusive statements holds,

- (i) $b^{\mathrm{I}} >_a b^{\mathrm{II}}$,
- (ii) $b^{\mathrm{II}} >_a b^{\mathrm{I}}$,
- (iii) neither (i) nor (ii) hold, and there exists a unique b^{III} with the properties
  - (α) $b^{\mathrm{III}} >_a b^{\mathrm{I}}$ *and* $b^{\mathrm{III}} >_a b^{\mathrm{II}}$,
  - (β) there exists no b^{IV} *for which* $b^{\mathrm{III}} >_a b^{\mathrm{IV}} >_a b^{\mathrm{I}}$ *and* $b^{\mathrm{IV}} >_a b^{\mathrm{II}}$.

The member b^{III} may be called the *lowest common superior* of b^{I} and b^{II}. The situation is clarified if one represents $H_a(b)$ as in Figure 1 by a directed linear graph or *digraph* [2] $D_a(b)$, in which the members of

[2]) See, for instance, F. Harary, R. Z. Norman and D. Cartwright, Structural Models: An Introduction to the Theory of Directed Graphs, Wiley, 1965.

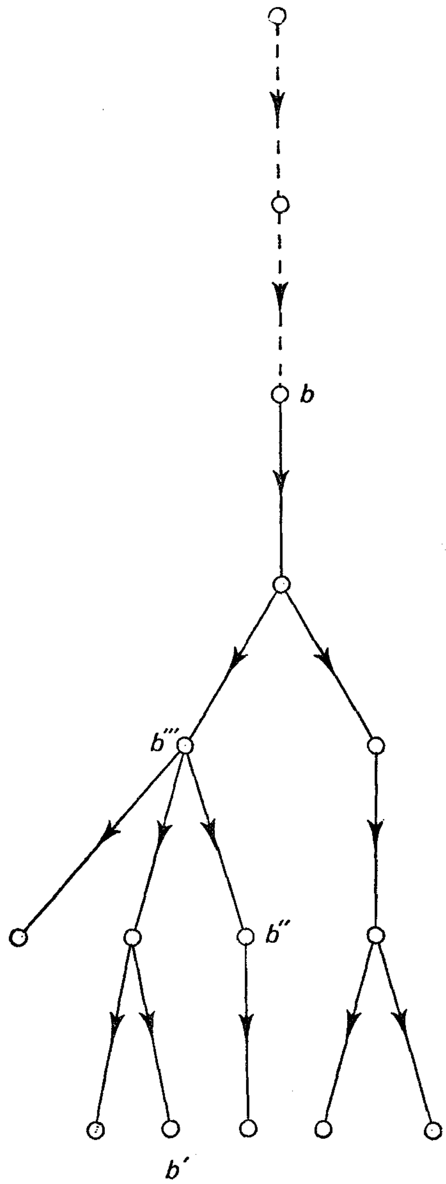

Figure 1

$H_a(b)$ are shown as *nodes,* all instances of supervision within $H_a(b)$ as *directed arcs.* For any given member b^{I}, there is a unique directed chain (sequence of nodes with successive directed connecting arcs) leading from b to b^{I} (a single node only if $b^{\mathrm{I}} = b$). Cases (i) and (ii) arise if one of the chains leading to b^{I} and b^{II}, respectively, is contained in the other. Case (iii) arises if each chain has at least one node with connecting arc outside the other chain. The member b^{III} corresponds to the node from which the two chains bifurcate.

A digraph with the properties required so to represent a hierarchy for a single activity is known as a *directed tree from* b *:* every node is connected with b , and no cycle can be formed from any set of arcs.

Figure 1 also shows, by dotted arcs drawn from nodes outside $D_a(b)$, the possible existence of superiors of b for a . We now define:

D5 *a complete hierarchy* $H_a(b^*)$ *for an activity* a is a hierarchy for a whose head b^* has no supervisor for a ,

and prove the following theorem:

T1 Each person b engaged in an activity a belongs to one and only one complete hierarchy $H_a(b^*)$ for a .

To construct one such hierarchy from any given such b , take $b^* = b$ if b has no supervisor for a . If b has a supervisor b^1 construct the unique sequence $b^1, b^2, \dots, b^n$ of successive supervisors (as in D2) until a superior b^n is met who has no supervisor. Take $b^* = b^n$. To show there is only one such hierarchy, assume that $b \varepsilon H_a(b^*)$, $b \varepsilon H_a(b^{**})$ with $b^* \neq b^{**}$. Then, in graph language, the supervision chains leading from b^* and b^{**}, respectively, to b would meet (see Figure 2) at a member b^{I} (which could be b) having two supervisors for a , a violation of P2.

As a corollary of T1 we have

CT1 The set $\mathbf{B}_a$ of persons engaged in an activity a can be partitioned into (regarded as the union of non-overlapping) complete hierarchies for that activity.

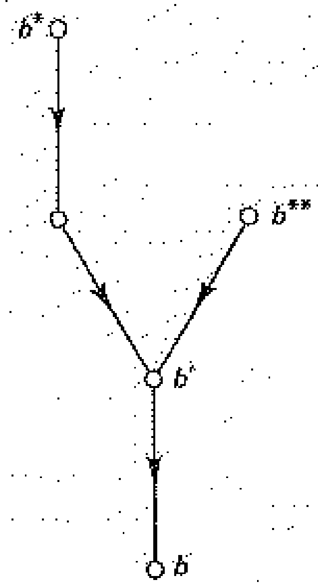

Figure 2

A graph consisting of a union of directed trees no two of which have a common node is called a *directed forest.*

Hierarchies for a Set of Activities

Everything said so far would apply equally if only a single activity were pursued in the society considered. If more than one activity is present, connections between hierarchies for different activities also need to be explored. A useful image[3]) is to think of a sheet of paper (an underlay, say) on which one node has been drawn in for each person of **B**. Fore each activity a of **A**, one then imagines a transparent overlay on which, in a color specific to a, nodes for all persons engaged in a have been dotted, and arcs for all instances of supervision in a have been drawn in. A picture of the entire hierarchical structure of society is obtained if all overlays are placed in proper position over the underlay.

The discussion of hierarchies for a set of activities is helped by a change in terminology and notation. We now define a supervision relation $>^1$ on the set **C** of all engagements as follows:

D6 $(a', b') >^1 (a, b)$ implies $a' = a$,

$(a, b') >^1 (a, b)$ if and only if $b' \overset{1}{>}_a b$.

The first condition in D6 limits supervision, as before, to persons engaged in the same activity. The second changes notation in such a way that $>^1$ now combines the information previously contained separately in all the $\overset{1}{>}_a$, $a \varepsilon \mathbf{A}$. It is represented by the graph obtained when all overlays are in position, provided each node b originally on the underlay is regarded as generating a different new node (a, b) in every instance in which it is found dotted in a specific color.

As before, we can define a superiority relation $>$ on **C** by iterating $>^1$. It satisfies conditions obtained from D6 by omitting all superscripts 1 to the relation symbols.

We further change both the meaning of, and the notation for, the term "hierarchy for a single activity" by applying

3) Suggested to me by Lloyd Shapley.

it now to a set of engagements rather than to a set of persons. The new definition is

D7 $$\mathbf{H}_a(b) = \{ (a, b^{\mathrm{I}}) \mid b^{\mathrm{I}} \varepsilon H_a(b) \},$$

in which every person b^{I} originally in the hierarchy is replaced by his "engagement" (a, b^{I}) in the activity a in question. The set of persons involved is now renamed the *personnel* of the hierarchy, and continues to be denoted by $H_a(b)$. We now define

D8 a *hierarchy* $\mathbf{H}_A(b)$ *for a set* A *of activities* is the union $\mathbf{H}_A(b) = \bigcup_{a \varepsilon A} \mathbf{H}_a(b)$ of a set of hierarchies $\mathbf{H}_a(b)$, one for each $a \varepsilon A$, all having the same head b;

D9 *a complete hierarchy* $\mathbf{H}_A(b^*)$ *for a set* A *of activities* is a hierarchy for that set of which each component hierarchy $\mathbf{H}_a(b^*)$ is complete.

As an example of a hierarchy for a set A of activities let A consist of all the activities pursued in the blast furnace department of an integrated iron and steel plant, and let b be the head of that department. Then $\mathbf{H}_A(b)$ consists of all engagements (a, b^{I}), $a \varepsilon A$ to which (a, b) is superior, that is, all engagements (a, b^{I}) in blast furnace operation activities by persons b^{I} linked to b by a chain of successive supervisors in one and the same activity $a \varepsilon A$. As one goes up a step in any such chain of supervisors, the set of all activities engaged in by the supervisor remains the same or expands. The head b is engaged in all activities of A. Figure 3 gives a simplified example with A consisting of just three activities, a, a^{I}, a^{II}.

In this example $\mathbf{H}_A(b)$ is not complete if b has one or more superiors in at least one activity of A, such as the head of the integrated plant, and possibly a board of directors above him. In that case, however, $\mathbf{H}_A(b)$ can be supplemented to form a complete hierarchy $\mathbf{H}_A(b^*)$ for the *same* set A of activities *only* if any superior b^{I} of b in any activity of A is a superior of b in all activities of A. If so, then the highest superior b^* to b in any $a \varepsilon A$ is also

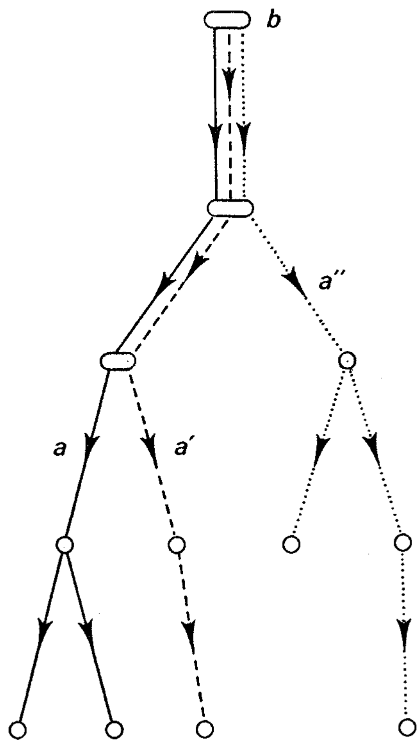

Figure 3

highest superior to b in all $a \varepsilon A$, and $\mathbf{H}_A(b^*)$ is a complete hierarchy for A.
Note that the personnel of $\mathbf{H}_A(b^*)$ may contain additional persons, other than b, who are neither superior to nor supervised by b in any $a \varepsilon A$.

Complete Hierarchies

It is natural to ask whether the set of activities A for which $\mathbf{H}_A(b^*)$ is a complete hierarchy can be further expanded. In the example just given, $\mathbf{C}$ may be such that A can be expanded, hence also the personnel $H_A(b^*)$ of $\mathbf{H}_A(b^*)$ enlarged, by adding in all engagements of the steelmaking and possibly other departments, supervised by the same head b^*. The following definition expresses this idea.

D10 A complete hierarchy $\mathbf{H}(b^*)$ is the union of all complete hierarchies $\mathbf{H}_A(b^*)$ for single activities a, of which b^* is the head.

This definition leads directly to our second theorem:

T2 each engagement $(a, b) \varepsilon \mathrm{C}$ belongs to one and only one complete hierarchy $\mathbf{H}(b^*)$.

To construct this hierarchy uniquely from any given $(a, b) \varepsilon \mathrm{C}$, first find the complete hierarchy $\mathbf{H}_a(b^*)$ for a containing (a, b), which by T1 is unique. Next find the set A of all activities pursued by the head b^* of $\mathbf{H}_a(b^*)$ in which he has no supervisor. Then the unique complete hierarchy containing (a, b) is $\mathbf{H}(b^*) = \mathbf{H}_A(b^*)$.

It will be seen from this construction that the complete hierarchy containing any arbitrary engagement in a blast furnace operation by an employee of the integrated iron and steel plant considered excludes the private practice that the medical director of the company may pursue on the side — provided that medical care for private patients is interpreted as an activity different from those he engages in as a medical director. It includes those activities by members of the family of the head of the integrated plant that he supervises — unless he is in turn supervised by a board of directors in the running of the integrated plant, but *not* in the engagements of his family.

Putting T2 in a different way,

CT2 The set C of all engagements partitions into complete hierarchies.

Concluding Remarks

Whether the definitions and postulates here proposed lead to a meaningful concept of complete hierarchies depends on a trait of real-life hierarchic organizations not recognized in these postulates. Often, and for good reasons, the set of activities engaged in by the personnel of a hierarchy, for which ultimate responsibility is placed in the same individual or supervisory body, is determined on the basis of interdependence [4] of the activities concerned. By introducing one or more relations of interdependence on the set of activities, one may be able to refine the definition of a complete hierarchy in such a way as to avoid the inconsistency that the family activities supervised by, or the boy scout group led by, a member of the firm is included with a complete hierarchy containing that firm if the member is its head but not if he is second in command.

Some activities of governments, especially in countries with central planning and direction of the economy, can be naturally fitted into the concepts introduced. A fuller coverage can be obtained if besides a relation of supervision of engagements one introduces a relation of suppression of engagements. In that way the law-making activities of governments can be looked upon as defining, and the law enforcement activities as aimed at suppressing, illegal engagements.

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[4] For further discussion of the concept of interdependent activities, see Koopmans and Montias, l.c.