TOWARD A STUDY OF BIDDING PROCESSES
PART IV — GAMES WITH UNKNOWN COSTS*

James H. Griesmer and Richard E. Levitan

IBM Watson Research Center
Yorktown Heights, New York

and

Martin Shubik†

Yale University
New Haven, Connecticut

ABSTRACT

This paper represents a continuation of three previous papers [1-3] in the study of competitive bidding processes. It treats the case where a bidder's knowledge of his competitor's cost is given by a probability distribution over a certain interval. The results obtained extend the work of Vickrey [4] to the case where the cost intervals are not necessarily symmetric.

1. INTRODUCTION

Sealed bidding when the individuals have known fixed costs and known valuations presents both an institutionally and mathematically uninspiring problem. The noncooperative equilibrium point will award the contract to the individual with the lowest cost (or equivalently the highest valuation for an item) at a price just under that of the individual with the next higher cost. Even this fairly simple solution contains an unforeseen difficulty, as will be noted below; however, the "game of bidding" by sealed bids with known costs and values is basically simple and possibly the main lesson that we can learn from it is that the problems with bidding lie elsewhere.

In industry, auctions, sales, and all economic activities which use sealed bids or other auctioning devices the problems which call for analysis appear to arise from considerations such as capacity constraints, sequential choices in dynamic markets, and, above all, lack of knowledge of the costs and valuations of others and possibly a lack of knowledge of one's own costs and valuations. Indeed, much of the success of art dealers and other professional buyers must be attributed to their greater sensitivity to value.

In this paper following the work of Vickrey [4] we represent the competitors' lack of knowledge of each others' true costs by distributions which we assume represent their beliefs concerning the costs of others.

A minor point overlooked in the analysis of the simple bidding problem is the possibility of "dog-in-the-manager" tactics. Consider two individuals with costs of 1 and 2 bidding for a contract (say in units of 0.1). The immediate simple solution is that the first obtains the

*Research supported, in part, by the Office of Naval Research under Contract N00014-70-A-0655(01) with the Cowles Foundation, Yale University.
†Professor of the Economics of Organization.
contract for a price of 1.9 and makes a profit of 0.9; however, the pair of strategies \((1.1, 1.2)\) also defines an equilibrium point where the first player still obtains the contract, but makes only 0.1. In our analysis we distinguish between "serious bids" and "nonserious bids." The distinction is that serious bids are those which have a positive probability of winning. These are never below cost, while non-serious bids can be below cost and are explainable only in terms of spite, incompetence, or other motives than the obtaining of an immediately profitable contract. In our work, both serious and non-serious bids are considered.

The formal statement of the problem of bidding with unknown costs together with the proofs of our results is given in Section 2. In a future paper, numerical calculation of some examples will be included.

2. THE ASYMMETRIC BIDDING GAME WITH UNKNOWN COMPETITIVE COSTS

This game is defined as follows: each of two players draws a cost \(c_i\) from a rectangular distribution on an interval \([a_i, b_i]\), where \(a_i < b_i\), \(i = 1, 2\). Based on the drawn cost, a bid of \(y_i\) is made by each player. The contract is awarded to the low bidder if there is one. He makes a profit \(y_i - c_i\), i.e., a profit equal to his bid minus his cost. In the event of a tie the contract is awarded to either player with probability 1/2.

A strategy for player \(i\) consists of a family of distribution functions \(\theta_i(y; c)\) which gives his bidding distribution when his cost is \(c\). A pure strategy is a function \(\gamma_i(c)\) which gives a unique bid as a function of cost, i.e., a pure strategy is the special case in which \(\theta_i(y; c)\) is a family of step functions.

Our goal is to characterize the Nash equilibrium solutions of this game. An equilibrium solution is a pair of strategies \((\hat{\theta}_1, \hat{\theta}_2)\) such that if \(\pi_i(\theta_1, \theta_2)\) is the expected profit to player when strategies \((\theta_1, \theta_2)\) are employed by the respective players, then

\[
\pi_1(\hat{\theta}_1, \hat{\theta}_2) \geq \pi_1(\theta_1, \theta_2)
\]

and

\[
\pi_2(\hat{\theta}_1, \hat{\theta}_2) \geq \pi_2(\theta_1, \theta_2)
\]

for all alternative strategies \(\theta_1\) and \(\theta_2\).

This game is equivalent to a game proposed by Vickrey [4]. In his paper Vickrey assumes that each player drew a "value" from the same rectangular distribution. The award is made to the high bidder, who makes a profit equal to his value less his bid. Vickrey gave an equilibrium solution to the game for the symmetric case of equal distributions. Corollary 4.1 reproduces this result, using our formulation of the bidding model in terms of cost.

We introduce some definitions and conventions.

1) We assume, without loss of generality, that \(b_1 \geq b_2\), and define \(\bar{b} = 1/2 (b_1 + b_2)\).

2) We define \(\Phi_i(y)\) as the probability that player \(i\) makes a bid of \(y\) or more, and \(\Psi_i(y)\) as the probability that he bids exactly \(y\). If the indicated sets exist, we define

\[
\bar{\nu}_i = \text{least upper bound } \{y \mid \Phi_i(y) = 1\},
\]

\[
\bar{\nu}_i = \text{greatest lower bound } \{y \mid \Phi_i(y) = 0\},
\]

\[
\bar{\nu} = \text{least upper bound } \{y \mid \Phi_1(y) \cdot \Phi_2(y) = 1\},
\]

\[
\bar{\nu} = \text{greatest lower bound } \{y \mid \Phi_1(y) \cdot \Phi_2(y) = 0\}.
\]
Surely in an equilibrium, \( y_1 \) and hence \( y \) must be defined since a bid below cost with a positive probability of winning must be non-optimal. We shall show that \( y \) will always be defined in an equilibrium. We shall convene, however, that, if \( y \) is not defined by the above, we take it to be \(+\infty\).

3) A serious bid by a player is a bid which has a positive probability of winning. Thus, if \( y \) is a serious bid made by player \( i \), \( i = 1, 2 \), then \( \Phi_j(y) > 0 \) (where \( j \) denotes the other player).

4) We characterize an essential equilibrium as one in which both players make serious bids with positive probability. An inessential equilibrium is one which is not essential; i.e., one player is the winning bidder with probability 1.

We shall generalize Vickrey’s solution to the case of asymmetric cost intervals and show that, for all nondegenerate cost intervals \( (a_i < b_i, i = 1, 2) \), an equilibrium pair exists. In fact, we shall show that a pure strategy equilibrium pair exists. We shall find that all equilibria are essentially of the pure type in the sense that serious bids are, with probability 1, single valued functions of the bidder’s cost. In the essential case, the function is a strictly monotone one, except of course on a set of measure zero in which a bidder may randomize and not disturb the equilibrium. With the exception aside, we show that the serious bids in an equilibrium are characterized by a single parameter \( \bar{y} \), the least upper bound on the serious bids of both players.

We shall also show that the side condition that no player bid below cost imposes the restriction that \( \bar{y} = \max (\bar{a}_2, \bar{b}) \) and thus fixes all serious bidding behavior.

In the case that \( b_1 < b_2 \), we show that in an equilibrium Player 2 must make nonserious bids; i.e., bids above \( \bar{y} \), with positive probability. No unique characterization for this bidding can be given except an upper bound for \( \Phi_2(y) \), when \( y > \bar{y} \). In this case, \( \Phi_2(y) \leq (\bar{y} - b_1)/y - b_1 \). In the inessential case, \( \Phi_2(\bar{y}) = 1 \), and in the essential case, \( \Phi_2(\bar{y}) = (b_2 - \bar{y})/(b_2 - a_2) \). If the equilibrium is essential, Player 2 may lump his nonserious bids at \( \bar{y} \); in the inessential case, however, he may not. In no case does Player 1 make nonserious bids with positive probability.

The characterization of the serious bidding behavior for the essential and inessential cases is as follows. In the inessential case, Player 1 always bids \( \bar{y} \), such that \( b_1 < \bar{y} \leq a_2 \), and Player 2 makes nonserious bids according to a distribution \( \Phi_2(y) \) satisfying the above inequality.

In the essential case, \( b_1 \leq \bar{y} \leq \bar{b}_2 \), and both players make serious bids according to differentiable monotonic functions of cost in the respective ranges, \( [a_1, b_1] \) and \( [a_2, \bar{b}] \). \( y \), the greatest lower bound for both players’ serious bids is equal to

\[
\frac{\bar{y}(2\bar{b} - \bar{y}) - a_1a_2}{2\bar{b} - (a_1 + a_2)}.
\]

The question of whether the possible equilibria are essential or inessential is determined from the values of the cost parameters. In some cases all equilibria are essential, in some cases all are inessential, and finally there are cases where both types of equilibria can occur. Theorems 1, 2, and 3 formalize this discussion.

We present the following diagram illustrating the different possibilities. In the plane, we plot points corresponding to such bidding games, where the abscissa is equal to \((b_2 - b_1)/(b_2 - a_2)\) and the ordinate is equal to \((b_1 - a_1)/(b_2 - a_1)\). Points in the first quadrant of the plane
correspond to games in which $b_2 \geq b_1$. Considering this region for the moment, we find that it is decomposed into three subregions, denoted I, II, and III, by the vertical lines $b_1 - a_2$ and $b_1 = 2a_2 - b_2$; i.e., $a_2 = \bar{b}$. In I, $a_2 \leq b_1$ and essential equilibria only are possible. A mixed situation obtains for region II, where both essential and inessential equilibria can occur. If $\bar{y} > a_2$, then the equilibrium is essential; in the contrary case, the equilibrium is inessential. In either case, $b_1 < \bar{y} \leq \bar{b}$. In region III, inessential equilibria only can occur. Here, $\bar{b} \leq a_2 < b_2$.

We now consider briefly the case $b_1 \geq b_2$; i.e., the case of points in the second quadrant of the plane. In a manner analogous to the first quadrant, the second quadrant is also divided into three subregions, labelled I', II', and III', which correspond in terms of their properties, precisely to regions I, II and III, respectively.

Since these regions of the second quadrant are cones with a common vertex, the origin, we can infer immediately that the (degenerate) game corresponding to the origin has no equilibrium solution. This would be a bidding game in which $a_1 = b_1 = b_2$. Such a game, however, can be shown to possess an $\epsilon$-equilibrium solution; i.e., a strategy pair such that a unilateral deviation by either player can achieve an increase of at most $\epsilon$ above his profit when both players are using the prescribed strategies.

We now proceed with the formal development.

THEOREM 1: An equilibrium is essential or inessential depending on whether, respectively, $\bar{y} > a_2$ or $\bar{y} \leq a_2$.

PROOF: Suppose $\bar{y} \leq a_2$. Then it cannot be the case that Player 2 bids below $\bar{y}$ with positive probability since such a bid will entail a positive expected loss. Also, any bid by Player 2 at the level of $\bar{y}$ must win with probability zero, since his cost is above $\bar{y}$ with probability 1. Thus only Player 1 can be making serious bids and the equilibrium is inessential.

Now suppose the equilibrium is inessential, and that $\bar{y} > a_2$. Since Player 2 can afford to make a serious bid when his cost is in the interval $(a_2, \bar{y})$ he must be the sole serious bidder and all his bids must be serious. Hence, $\bar{y} \leq b_2$. But then $\bar{y} \leq b_1$, and Player 1 must also make serious bids with probability 1, thereby contradicting the assumption that the equilibrium is inessential and proving the theorem.
THEOREM 2: In an inessential equilibrium, \( b_1 < \overline{y}, \) and hence \( b_1 < a_2. \) Player 1 bids \( \overline{y} \) with probability 1. Player 2 bids above \( \overline{y} \) with probability 1, and \( \phi_2(y) \leq (\overline{y} - b_1)/(y - b_1) \) for \( y \geq \overline{y}. \)

PROOF: Since Player 1 is the only serious bidder, it must be the case that \( b_1 \leq \overline{y}. \) We claim that Player 1 must bid \( \overline{y} \) with probability 1. For, if \( y_1 < \overline{y}, \) then \( y_2 = 1/2 (y_1 + \overline{y}) \) dominates \( y_1, \) since it wins with probability 1 and has a larger payoff. Hence, the only undominated bid is \( \overline{y}. \) Suppose now that Player 2 bids at the level of \( \overline{y} \) with probability \( p > 0. \) Let Player 1's cost be \( c. \) Player 1 has a payoff for a bid of \( \overline{y} \) equal to

\[
(\overline{y} - c) \left(\frac{p}{2} + (1 - p)\right) = (\overline{y} - c) \left(1 - \frac{p}{2}\right),
\]

while for a bid of \( y < \overline{y}, \) his payoff is \( (y - c). \) Thus any bid in the open interval \((\overline{y} - \frac{p}{2} (\overline{y} - c), \overline{y})\) dominates a bid of \( \overline{y}, \) and, in turn, is dominated itself. Hence, \( p \) must be 0. The optimality of bidding \( \overline{y} \) with probability 1 by Player 1 requires that for \( y > \overline{y} \) and all \( c \leq b_1, \)

\[
(y - c) \phi_2(y) \leq (\overline{y} - c)
\]

or

\[
\phi_2(y) \leq \frac{(\overline{y} - c)}{y - c}.
\]

The right hand side attains its minimum when \( c = b_1, \) so we have

\[
\phi_2(y) \leq \frac{\overline{y} - b_1}{y - b_1}.
\]

Thus, \( \overline{y} = b_1 \) implies that \( \phi_2(y) = 0 \) for \( y > \overline{y}, \) and hence implies that Player 2 always bids \( \overline{y}. \) This contradicts the result that he bids \( \overline{y} \) with probability zero. Hence, \( \overline{y} > b_1. \)

COROLLARY 2.1: There exists an inessential equilibrium if, and only if, \( b_1 < a_2. \)

PROOF: The necessity follows immediately from Theorem 2. Let \( b_1 < a_2 \) and choose \( \overline{y} \in (b_1, a_2]. \) Let Player 1 always bid \( \overline{y} \) and Player 2 bid so as to satisfy \( \phi_2(y) = (\overline{y} - b_1)/(y - b_1) \) for \( y \geq \overline{y}. \) The proof follows from the optimality of this pair of strategies with respect to each other.

COROLLARY 2.2: There exists an inessential equilibrium with no player bidding below cost with positive probability if, and only if, \( \overline{b} \leq a_2. \)

PROOF: Sufficiency. Let \( \overline{b} \leq a_2 \) and let \( \overline{y} = a_2 \) with Player 1 always bidding \( \overline{y}. \) Let Player 2 always bid his cost. For \( y \in [a_2, \overline{b}], \) \( \phi_2(y) = (b_2 - y)/(b_2 - a_2), \) and, for \( y > \overline{b}, \)

\[
\phi_2(y) = 0. \] The given strategies are obviously optimal for Player 2 and also for Player 1 if \( \phi_2(y) \leq (\overline{y} - b_1)/(y - b_1). \) By substitution, we have
\[
\frac{\bar{y} - b_1}{y - b_1} - \Phi_2(y) = \frac{\bar{y} - b_1}{y - b_1} - \frac{b_2 - y}{b_2 - a_2} = \frac{(y - a_2)(y + a_2 - 2\bar{b})}{(y - b_1)(b_2 - a_2)},
\]

for \(a_2 \leq y \leq b_2\). The denominator above is positive, and the first factor of the numerator above is non-negative. The second factor becomes \((y - \bar{b}) + (a_2 - \bar{b}) \geq 0\), since \(y \geq a_2 \geq \bar{b}\).

Necessity. Let such an inessential equilibrium be given by \(\bar{y}\) and \(\Phi_2(y)\) for \(y \geq \bar{y}\). Define the function \(c_2(y) = b_2 - (b_2 - a_2) \Phi_2(y)\) on the interval \([\bar{y}, \infty)\), and the strategy pair consisting of Player 1's bidding at the level \(\bar{y}\) and Player 2's bidding according to the function \(c_2(y)\); i.e., \(c_2\) is the cost at which Player 2 bids \(y\). Obviously, \(\Phi_2(y)\) is the probability of a bid above \(y\), when Player 2 is bidding according to \(c_2\). We claim that \(y - c_2(y) \geq 0\). Suppose \(y_1 - c_2(y_1) < 0\) for some \(y_1\). This implies that

\[
\Phi_2(y_1) < \frac{(b_2 - y_1)}{(b_2 - a_2)},
\]

which is the probability that cost is above \(y_1\). Hence, in the original strategy, Player 2 must bid above \(y_1\) with a smaller probability than that associated with his cost being above \(y_1\); i.e., he must bid below cost with positive probability. Thus, we have two inequalities:

\[
\frac{\bar{y} - b_1}{y - b_1} - \frac{b_2 - c_2(y)}{b_2 - a_2} \geq 0
\]

or

\[
c_2(y) \geq b_2 - \frac{(\bar{y} - b_1)(b_2 - a_2)}{(y - b_1)}.
\]

In combining, we obtain

\[
y \geq b_2 - \frac{b_2 - c_2(y)}{b_2 - a_2} \frac{(\bar{y} - b_1)(b_2 - a_2)}{(y - b_1)}
\]

or

\[
(\bar{y} - b_1)(b_2 - a_2) \geq (y - b_1)(b_2 - y)
\]

for \(y \geq \bar{y}\).

Letting \(y = \bar{y}\) reduces this to \(b_2 - a_2 \geq \bar{b} - \bar{y}\), or \(\bar{y} \geq a_2\). Because of Theorem 1, we must have \(\bar{y} = a_2\). Substituting \(a_2\) for \(\bar{y}\), we have

\[
a_2 b_1 + a_2 b_2 - a_2^2 \geq y b_1 + y b_2 - y^2, \quad \text{for } y \geq a_2
\]

or

\[
(y - a_2)(y + a_2 - 2\bar{b}) \geq 0.
\]
Thus

$$(y - \bar{b}) + (a_2 - \bar{b}) \geq 0$$

for all $y \geq a_2$. This can only happen if $a_2 \geq \bar{b}$, proving the result.

**THEOREM 3:** There exists an essential equilibrium if and only if $\bar{b} > a_2$. In such an equilibrium

i) $b_1 \leq \bar{y} \leq \bar{b}$,

ii) $\bar{y} > a_2$,

and

iii) $y = \frac{\bar{y}(2\bar{b} - \bar{y}) - a_1 a_2}{2\bar{b} - (a_1 + a_2)} > \max(a_1, a_2)$.

The serious bids of Player $i$, $i = 1, 2$, are defined in the non-degenerate half-open interval $[\bar{y}, \bar{y}]$ by the solution of the differential equations,

$$c_i'(y) = \frac{(b_i - c_i(y))(y - c_i(y))}{(y - \bar{y})(y + \bar{y} - 2\bar{b})},$$

passing through the initial point $(y, a_i)$. At $\bar{y}$, $c_i(\bar{y})$ is given by $\min(b_i, \bar{y})$. If $b_1 < b_2$, Player 2 makes nonserious bids when his cost exceeds $\bar{y}$. The only restriction on this nonserious bidding behavior is that

$$\phi_2(y) \geq \frac{\bar{y} - b_1}{y - b_1} \frac{b_2 - \bar{y}}{b_2 - a_2}.$$

**PROOF:** We shall first prove a sequence of lemmata.

**LEMMA 3.1:** Given a pair of equilibrium strategies, if one player bids in a closed interval $[e, f]$ with probability zero, his opponent makes serious bids in the half-open interval $[e, f]$ with probability zero.

**PROOF:** Let $i, j \in \{1, 2\}, i \neq j$. If $i$ bids in $[e, f]$ with probability zero, $\phi_i(e) = \phi_i(f)$. If $g \in [e, f]$, then $g$ is serious for $j$ if $\phi_j(g) > 0$. If $j$ bids in $[e, f]$ with positive probability, then he must realize a profit, and $a_j < f$. Let $c < g$ be a cost for which it is optimal for $j$ to bid $g$. Then

$$(g - c) \phi_i(g) \geq (f - c) \phi_i(f)$$

or

$$\phi_i(g) \geq \frac{(f - c)}{(g - c)} \phi_i(f) > \phi_i(f),$$

a contradiction.

**LEMMA 3.2:** If $c' < c''$ are costs at which Player $i$ makes serious bids $y'$ and $y''$ respectively, then $y' \leq y''$. 

PROOF: Let \( j \neq i \). Let \( z'_j(y) = \Phi'_j(y) - 1/2 \Psi'_j(y) \) denote the probability that a bid of \( y \) is winning for Player \( i \). Since \( y' \) and \( y'' \) are serious bids, \( z'_j(y') > 0 \) and \( z'_j(y'') > 0 \). Since \( y' \) and \( y'' \) are optimal for \( c' \) and \( c'' \), respectively, we have

\[(y' - c') z'_j(y') \geq (y'' - c') z'_j(y'')\]

and

\[(y'' - c'') z'_j(y'') \geq (y' - c'') z'_j(y').\]

Multiplying these inequalities by \((y'' - c'')\) and \((y'' - c')\), respectively, we obtain

\[(y'' - c'') (y' - c') z'_j(y') \geq (y'' - c'') (y'' - c') z'_j(y'')\]

\[\geq (y'' - c') (y' - c'') z'_j(y').\]

Thus

\[c' y' + c'' y'' \geq c' y'' + c'' y',\]

or

\[y''(c'' - c') \geq y'(c'' - c'),\]

or

\[y'' \geq y'.\]

LEMMA 3.3: If \( g \) is a serious bid for Player \( i \), and \( \Psi'_i(g) > 0 \), then \( i = 1 \), \( g = \bar{y}_1 \), and \( b_1 < b_2 \).

PROOF: Let \( \Psi'_i(g) > 0 \) and \( j \neq i \). The optimality of \( g \) for Player \( i \) requires that for each \( \Delta > 0 \), \( \Phi'_i(g) > \Phi'_i(g + \Delta) \) by Lemma 3.1. Suppose that \( j \)'s bids in \([g, g + \Delta]\) are serious. We shall show that all costs corresponding to such bids are at least \( g \).

Let \( c < g \). When \( j \)'s cost is \( c \), his expected profit for a bid of \( y \) is

\[v'_j(y) = (y - c) \left( \Phi'_i(y) - 1/2 \Psi'_i(y) \right).\]

If \( \delta > 0 \), then

\[v'_j(g) - v'_j(g + \delta) = (g - c) \left[ \Phi'_i(g) - \Phi'_i(g + \delta) - 1/2 \left( \Psi'_i(g) - \Psi'_i(g + \delta) \right) \right] - \delta \left[ \Phi'_i(g + \delta) - 1/2 \Psi'_i(g + \delta) \right] \]

\[\geq 1/2 (g - c) \left[ \Psi'_i(g) + \Psi'_i(g + \delta) \right] - \delta \]

\[\geq 1/2 (g - c) \Psi'_i(g) - \delta,\]

since \( \Psi'_i(g) \leq \Phi'_i(g) - \Phi'_i(g + \delta) \) and \( \Phi'_i(g) - 1/2 \Psi'_i(g) \leq 1 \). Also for \( \eta > 0 \), since \( \Phi'_i(g - \eta) - 1/2 \Psi'_i(g - \eta) - \Phi'_i(g) \leq 0 \), we have

\[v'_j(g - \eta) - v'_j(g) \geq 1/2 (g - c) \Psi'_i(g) - \eta.\]
If we choose \( \delta = \eta = \frac{1}{4} (g - c) \Psi_i(g) \), then it must follow that a bid \( y \) which is appropriate to \( c \) cannot lie in the open interval \( (g - \frac{1}{4} (g - c) \Psi_i(g), g + \frac{1}{4} (g - c) \Psi_i(g)) \). Hence, such a \( y \) cannot lie in the half-open interval \( [g, g + \frac{1}{4} (g - c) \Psi_i(g)) \).

Now suppose for each \( \Delta > 0 \), there is a bid \( y_\Delta \in [g, g + \Delta] \) such that \( c(y_\Delta) < g \). Selecting a monotone decreasing sequence \( \{\Delta_k\}, \Delta_k > 0 \), we have a sequence \( \{y_k\} \) which is strictly decreasing, which converges to \( g \), and whose cost \( c(y_k) \) appropriate to the bid \( y_k \) is below \( g \). Hence \( \{c(y_k)\} \) has a limit point \( c^0 \geq g \). If \( c^0 < g \), then there is a subsequence

\[
c^k < \frac{c^0 + g}{2} < g
\]

such that \( y(c^k) \rightarrow g \). But this is impossible by the argument in the preceding paragraph. Hence, \( c^0 = g \), and we can find a monotone increasing subsequence \( \{c^k\} \) such that \( c^k < c^{k+1} \); but \( \{y(c^k)\} \) satisfies \( y(c^k) > y(c^{k+1}) \) which contradicts Lemma 3.2. Thus it must be the case that such a cost \( c \geq g \).

Now we claim that \( i \) must bid above \( g \) with probability zero, and hence, \( j \)'s bids above \( g \) are not serious. Let \( [g, g + \Delta] \) be an interval in which \( j \) bids only with cost also in the interval. Consider \( j \)'s behavior when his cost is in the half open interval \( [g - \eta, g) \). Either he bids in \( [g - \eta, g) \) or else he bids above \( g + \Delta \). By Lemma 3.2, his optimal bid must be below \( g \).

Thus we have, at a cost \( g - \eta \), where \( \eta \geq \epsilon > 0 \),

\[
v_j(g - \epsilon) - v_j(g + \Delta) = (g - \epsilon - g + \eta) \left[ \phi_1(g - \epsilon) - 1/2 \Psi_1(g - \epsilon) \right] - (g + \Delta - g + \eta) \left[ \phi_1(g + \Delta) - 1/2 \Psi_1(g + \Delta) \right]
\]

\[
\leq (\eta - \epsilon) - \Delta \left[ \phi_1(g + \Delta) - 1/2 \Psi_1(g + \Delta) \right].
\]

If \( \phi_1(g + \Delta) > 0 \), we can set \( \eta = 1/2 \Delta \left[ \phi_1(g + \Delta) - 1/2 \Psi_1(g + \Delta) \right] \), and get \( v_j(g - \epsilon) - v_j(g + \Delta) < 0 \). Thus, \( \phi_1(g + \Delta) = 0 \), and \( j \)'s bids above \( g \) are not serious.

Suppose \( i = 2 \). We have Player 2 bidding above \( g \) with zero probability and Player 1 with positive probability. It must be the case that \( g \geq b_2 \), for, if not, when Player 2's cost is in \( (g, b_2) \), he bids below cost and wins with positive probability. But we have shown that \( j \)'s bids are at or above \( g \) when he bids at or above \( g \). Player 1 (i.e., \( j \)), however, has no such costs. We obtain a similar contradiction when \( i = 1 \) and \( b_1 = b_2 \).

We shall show below that we can rule out even the remaining case of lumped probability for all serious bids in an essential equilibrium. But we have shown that \( \phi_1 \) and \( \phi_2 \) are continuous in the half-open interval \( [y, \infty) \).

**Lemma 3.4:** In an essential equilibrium, \( \phi_i(y) \) is a strictly decreasing function of \( y \) in \( [y, \infty) \) for each player; i.e., each player has a positive probability of bidding into each non-degenerate subinterval of \( [y, \infty) \).

**Proof:** Let \( (e, f) \) be an open interval such that both players bid into it with zero probability (because of Lemma 3.1) and in all closed subintervals \( [e - \Delta, e] \) and \( [f, f + \Delta] \), both players bid with positive probability. Let \( c_\Delta \) be a cost for which it is appropriate for player 1 to bid in the range \( [e - \Delta, e] \). For a bid \( y \) in the range \( [e - \Delta, e] \), the expected profit for player 1 is (since there are no lumped probabilities)
\[ v_1(y, c_\Delta) = (y - c_\Delta) \phi_2(y) = (y - c_\Delta) \left( \phi_2(y) - \phi_2(e) + \phi_2(f) \right), \]

and for a bid at level \( f \)

\[ v_1(f, c_\Delta) = (f - c_\Delta) \phi_2(f). \]

Since \( y \) is optimal, we have

\[ 0 \leq v_1(y, c_\Delta) - v_1(f, c_\Delta) = (y - c_\Delta) \left( \phi_2(y) - \phi_2(e) + (y - f) \phi_2(f) \right). \]

Since \( (y - f) \phi_2(f) \leq -(f - e) \phi_2(f) \) we have

\[ (y - c_\Delta) \left( \phi_2(y) - \phi_2(e) \right) \leq (f - e) \phi_2(f). \]

But this contradicts the continuity of \( \phi_2(y) \).

**Lemma 3.5**: In an essential equilibrium, except possibly on a set of measure zero, each player's serious bids are given by a continuous, increasing function of his cost. Indeed, this function is strictly increasing, except possibly for Player 1 in some interval \([e, b_1]\), where \( e > a_1 \).

**Proof**: Let \( c' \) and \( c'' \) be costs for which it is appropriate for Player 1 to bid \( y' \) and \( y'' \), respectively, with \( c' < c'' \). By Lemma 3.2, \( y' \leq y'' \). Suppose \( y' = y'' \). Let \( c''' \) satisfy \( c' < c''' < c'' \), and \( y''' \) be a bid appropriate to \( c''' \). Again, \( y' \leq y''' \leq y'' \), and, hence, \( y' = y''' = y'' \). Hence, for each cost in \([c', c'']\), the only optimal bid is \( y' \). Hence, Player 1 bids \( y' \) with probability at least \((c'' - c')/(b_1 - a_1)\), which contradicts Lemma 3.3, unless \( y' = \bar{y} \) with \( 1 = 1 \) and \( b_1 < b_2 \).

Finally, we show that for each cost, there is a unique bid. Suppose that \( y^* < y^{**} \), and both are optimal and serious at a cost of \( c^* \). By Lemma 3.2, any bid in \([y^*, y^{**}]\) is optimal only at cost \( c^* \). Hence, bids are made in this interval with probability zero, contradicting Lemma 3.4.

We can now define \( c_1(y) \) and \( c_2(y) \) as the respective costs appropriate to a bid of \( y \). By the previous lemmata, \( c_2(y) \) is a continuous strictly increasing function in \([y, \bar{y}]\), and \( c_1(y) \) is also in \([y, \bar{y}]\), but it may possibly be given as an interval at \( \bar{y} \).

**Lemma 3.6**: \( c_1(y) \) and \( c_2(y) \) are differentiable functions of \( y \) in the half-open interval \([y, \bar{y}]\).

**Proof**: Let \( c = c_1(y) \) and \( c + \epsilon = c_1(y + \delta) \), where \( \delta > 0 \), and \( y \) and \( y + \delta \) are serious bids. We have

\[ (y - c) \phi_2(y) \leq (y + \delta - c) \phi_2(y + \delta) \]

and

\[ (y - c - \epsilon) \phi_2(y) \leq (y + \delta - c - \epsilon) \phi_2(y + \delta). \]
By rearranging, we obtain

\[(y - c) \left[ \Phi_2(y + \delta) - \Phi_2(y) \right] \leq -\delta \Phi_2(y + \delta)\]

and

\[(y - c - c) \left[ \Phi_2(y + \delta) - \Phi_2(y) \right] \leq -\delta \Phi_2(y + \delta)\]

so that

\[
\frac{\Phi_2(y + \delta)}{y - c - \epsilon} \leq \frac{\Phi_2(y + \delta) - \Phi_2(y)}{\delta} \leq \frac{\Phi_2(y + \delta)}{y - c}.
\]

A similar calculation for \(\delta < 0\) would reverse the signs of the previous inequalities. In either case, since \(\Phi_2\) is continuous, both bounds on the differential quotient have the limit \(-\Phi_2(y)/(y - c)\) as \(\delta \to 0\). Hence \(\Phi_2\) is differentiable, and since

\[\Phi_2(y) = \left( b_2 - c_2(y) \right)/(b_2 - a_2),\]

so is \(c_2\). A similar argument for \(c_1\) completes the proof.

**LEMMA 3.7:** \(c_1\) and \(c_2\) satisfy the following pair of differential equations over the range of serious bids:

\[
c'_1(y) = \frac{b_1 - c_1(y)}{y - c_2(y)}
\]

\[
c'_2(y) = \frac{b_2 - c_2(y)}{y - c_1(y)}.
\]

**PROOF:** Using the proof in the preceding lemma, on the one hand, \(\Phi'_2(y) = -c'_2(y)/(b_2 - a_2)\), while, on the other, \(\Phi'_2(y) = -\Phi_2(y)/(y - c_1(y))\). These relations combine to produce one of the differential equations. The other one is obtained in a similar manner.

**LEMMA 3.8:** \(\bar{y}_1 = \bar{y}_2 = \bar{y}\).

**PROOF:** Suppose \(\bar{y}_1 < \bar{y}_2\). Player \(i\) bids in \([\bar{y}_1, \bar{y}_2]\) with positive probability, and always wins with such bids. But for any bid \(y_1\) in \([\bar{y}_1, \bar{y}_2]\), \(y_1\) is strictly dominated by a bid of \(1/2 (y_1 + \bar{y}_1)\).

**LEMMA 3.9:** \(\bar{y}_1 = \bar{y}\) and \(\bar{y} \leq b_2\).

**PROOF:** If both \(\bar{y}_1 > b_2\) and \(\bar{y}_2 > b_2\), then \(\bar{y}_1 = \bar{y}_2\). For, if not, either player could improve his expected profit by lowering his maximum bid. Let \(\bar{c}_1\) be a cost for which it is appropriate for Player 1 to bid \(\bar{y}_1\). His expected profit for a bid of \(\bar{y}_1\) at this cost is

\[v_1(\bar{y}_1) = 1/2 (\bar{y}_1 - \bar{c}_1) \psi_2(\bar{y}_1);\]
but, by Lemma 3.3, $\psi_2(\bar{y}_1) = 0$, so $v_1(\bar{y}_1) = 0$. At a bid of $y_1 - \epsilon$ and with a cost of $\bar{c}_1$, however, he can expect

$$v_1(\bar{y}_1 - \epsilon) = (\bar{y}_1 - \bar{c}_1 - \epsilon) \Phi_2(\bar{y}_1 - \epsilon) > 0,$$

which contradicts the optimality of $\bar{y}_1$.

Now suppose $\bar{y}_1 > b_2$ and $\bar{y}_2 \preceq b_2$. Indeed, because Player 2 is bidding optimally, $\bar{y}_2 = b_2$; however, the optimality requirement on Player 1's bidding forces him to bring his maximum bid below $b_2$ to improve his expected profit. Thus $\bar{y}_1 \preceq b_2$.

Suppose finally that $\bar{y}_2 < \bar{y}_1 \preceq b_2$. For costs in $[\bar{y}_2, b_2]$, Player 2 must bid with a positive expected loss. Hence, $\bar{y}_2 \preceq \bar{y}_1$ and $\bar{y}_1 = \bar{y}$, thereby completing the proof.

**Lemma 3.10:** $b_1 \preceq \bar{y} \preceq \bar{b}$.

**Proof:** For bids $y$ in the range $[\bar{y}, \bar{y}]$, $c_2(y) \preceq y$. At the upper bound of the range $\bar{y}$, we assert $c_2(\bar{y}) = \bar{y}$. For, if $c_2(\bar{y}) < \bar{y}$, $\bar{y}$ would not be the optimum bid for Player 2 at $c_2(\bar{y})$. Also, $c_1(\bar{y}) = b_1$, by definition. At a cost of $b_1$ and a bid of $\bar{y}$, Player 1's expected profit is

$$v_1(\bar{y}) = (\bar{y} - b_1) \Phi_2(\bar{y})$$

$$= (\bar{y} - b_1) \left( \frac{b_2 - c_2(\bar{y})}{(b_2 - a_2)} \right)$$

$$= (\bar{y} - b_1) \frac{(b_2 - \bar{y})}{(b_2 - a_2)}.$$

Since $\bar{y}$ is optimal for $b_1$, for any $y \preceq \bar{y}$, we have

$$(\bar{y} - b_1) \Phi_2(y) \preceq (y - b_1) \Phi_2(y)$$

$$= (y - b_1) \left( \frac{b_2 - c_2(y)}{(b_2 - a_2)} \right)$$

$$\preceq (y - b_1) \left( \frac{b_2 - y}{(b_2 - a_2)} \right).$$

Thus

$$(\bar{y} - b_1) (b_2 - \bar{y}) \preceq (y - b_1) (b_2 - y),$$

or

$$2\bar{b} - 2\bar{y} \preceq \bar{y}^2 - y^2$$

or

$$2\bar{b} \preceq \bar{y} + y.$$

Since this relation holds for all $y$ in the range $[\bar{y}, \bar{y}]$, it must be the case that $\bar{b} \preceq \bar{y}$.

Also, $\bar{y} \preceq b_1$, since, otherwise, if Player 1 had a cost in $[\bar{y}, b_1]$, he would be accepting a positive probability of a loss, contradicting optimality.

**Lemma 3.11:** If both players are restricted to bids not less than cost, then $\bar{y} = \bar{b}$. 
PROOF: If $b_1 = b_2$, then the proof is immediate by Lemma 3.10. For the case $b_1 < b_2$, if $c_2(y) \leq y$ for all $y$ in $[y, b_2]$, then by optimality,

$$
(\bar{y} - b_1)(b_2 - \bar{y})/(b_2 - a_2) = (\bar{y} - b_1) \Phi_2(\bar{y}) \\
\geq (y - b_1) \Phi_2(y) \\
= (y - b_1)(b_2 - c_2(y))/(b_2 - a_2) \\
\geq (y - b_1)(b_2 - y)/(b_2 - a_2),
$$

so that

$$
0 \geq (\bar{y} - y) ([\bar{y} + y] - 2\bar{b})
$$

for all $y$ in $[y, b_2]$. In particular, for $\bar{y} < y \leq b_2$,

$$
\bar{y} + y \geq 2\bar{b}.
$$

Since this relation holds for all $y$ in $(\bar{y}, b_2)$, we must have $\bar{y} \geq \bar{b}$. Combining this inequality with the preceding lemma yields the desired result.

LEMMA 3.12: The differential equations of Lemma 3.7, together with the condition $c_2(\bar{y}) = \bar{y}$ are equivalent to

$$
c_1'(y) = \frac{(b_1 - c_1(y))(y - c_1(y))}{(y - \bar{y})(y + \bar{y} - 2\bar{b})}, \quad i = 1, 2.
$$

PROOF: In the differential equations of Lemma 3.7 make the substitution $d_i = y - c_i'$, $i = 1, 2$. The equations become $d_j(1 - d_i') = b_i - y + d_i$, $i = 1, 2$ and $j \neq i$. Adding these two equations, and cancelling, we have

$$
\frac{d}{dy}(d_1d_2) = d_1d_2' + d_1'd_2 = 2(y - \bar{b}).
$$

By integrating, this becomes

$$
d_1d_2 = (y - \bar{b})^2 + k.
$$

The condition $c_2(\bar{y}) = \bar{y}$ implies $d_2(\bar{y}) = 0$, and $k = -(\bar{y} - \bar{b})^2$. Thus $d_1d_2 = (y - \bar{y})(y + \bar{y} - 2\bar{b})$. Substituting with this equation, we have,

$$
c_1'(y) = \frac{b_1 - c_1}{y - c_j} = \frac{b_1 - c_1}{d_j} \\
= \frac{(b_1 - c_1)d_1}{(y - \bar{y})(y + \bar{y} - 2\bar{b})} = \frac{(b_1 - c_1)(y - c_1)}{(y - \bar{y})(y + \bar{y} - 2\bar{b})}.
$$
COROLLARY 3.12.1: The lower bound, $y$, in an essential equilibrium is given by

$$y = \frac{\bar{y}(2\bar{b} - \bar{y}) - a_1 a_2}{2\bar{b} - (a_1 + a_2)}.$$ 

PROOF: At $\bar{y}$ we have, by Lemma 3.5, $c_1(\bar{y}) = a_1$, or $d_1(\bar{y}) = \bar{y} - a_1$. Thus,

$$(\bar{y} - a_1) (\bar{y} - a_2) = (\bar{y} - \bar{b})^2 - (\bar{y} - \bar{b})^2.$$ 

This can be solved for $y$ to give the indicated result.

COROLLARY 3.12.2: In an essential equilibrium, $y > a_i$, $i = 1, 2$.

PROOF: We consider the expression

$$\bar{y} - a_1 = \frac{\bar{y}(2\bar{b} - \bar{y}) - a_1 a_2 - a_1 \left(2\bar{b} - (a_1 + a_2)\right)}{2\bar{b} - (a_1 + a_2)}.$$ 

The numerator on the right hand side is a monotone increasing function of $\bar{y}$, and the denominator is a positive constant. We have, by Lemma 3.10 and Theorem 1, respectively, $\bar{y} \geq b_1$ and $\bar{y} > a_2$. For the case $i = 1$, we replace $\bar{y}$ by $b_1$, so that the numerator becomes

$$b_1 (2\bar{b} - b_1) - a_1 a_2 - a_1 (b_1 - a_1 + b_2 - a_2) = b_1 b_2 - a_1 (b_1 - a_1) - a_1 b_2$$

$$= (b_1 - a_1) (b_2 - a_1) > 0.$$ 

For the case, $i = 2$, upon substitution of $a_2$ for $\bar{y}$, the numerator becomes

$$a_2 (2\bar{b} - a_2) - a_1 a_2 - a_2 \left(2\bar{b} - (a_1 + a_2)\right) = 0.$$ 

Since $\bar{y} > a_2$, we must have $y > a_2$.

COROLLARY 3.12.3: If there is an essential equilibrium, then $\bar{b} > a_2$.

PROOF: Since $y < \bar{y}$, we have, from Corollary 3.12.1,

$$\bar{y}(2\bar{b} - \bar{y}) - a_1 a_2 < \bar{y} \left(2\bar{b} - (a_1 + a_2)\right)$$ 

which reduces to

$$(\bar{b} - a_1) (\bar{b} - a_2) > 0.$$ 

Since $\bar{b} \geq b_1 > a_1$, we must have $\bar{b} > a_2$.

LEMMA 3.13: If $\bar{b} > a_2$, then the differential equation of Lemma 3.12 has a unique solution in the half-open interval $[y, \bar{y})$, passing through the point $(y, a_1)$, where $y$ is defined in Corollary 3.12.1. The solution is a strictly increasing function of $y$ and has the property that
\[
\lim_{y \to \bar{y}} c_i(y) = \min (\bar{y}, b_i).
\]

Hence, neither player can have a lumped serious bid in an essential equilibrium.

PROOF: The function \( c_1(y) \) satisfies a Lipschitz condition with respect to \( c_1 \) for \( y \leq y \leq k < \bar{y} \). Hence, by the Picard Theorem (cf. Coddington and Levinson, Ordinary Differential Equations, p. 10), for all closed intervals \([y, k] \), for \( k < \bar{y} \), there is a unique solution passing through \((y, a_i)\), and, hence, also in the half-open interval \([y, \bar{y}]\).

We assert that \( y - c_i(y) > 0 \) and \( b_i - c_i(y) > 0 \), and, hence, \( c_i(y) > 0 \) on \([y, \bar{y}]\). Suppose that one of the inequalities does not hold. Then, by continuity and using Corollary 3.12.3, there is a smallest value \( \hat{y} \) such that either \( \hat{y} = c_i(\hat{y}) \) or \( b_i = c_i(\hat{y}) \), for \( i \) equal to either 1 or 2. At \( \hat{y} \), \( c_i'(\hat{y}) = 0 \). In case \( c_i(\hat{y}) = b_i \), then \( c_i(y) = b_i \) is the unique solution to the differential equation passing through \((\hat{y}, b_i)\). But this contradicts \( c_i(\bar{y}) = a_i < b_i \).

On the other hand, if \( c_i(\bar{y}) = \hat{y} \) with \( c_i(y) < y \) for \( y \leq y < \hat{y} \), we have

\[
c'_i(\bar{y}) = \lim_{y \to \bar{y}} \frac{c_i(y) - c_i(y)}{\hat{y} - y} = \lim_{y \to \hat{y}} \frac{\hat{y} - c_i(y)}{\hat{y} - y} \geq 1,
\]

which contradicts \( c'_i(\bar{y}) = 0 \).

Thus, we have shown that \( c_i(y) < \min (b_i, \bar{y}) \) for \( y \leq y < \bar{y} \). Let us extend \( c_i(y) \) to \( \bar{y} \) by defining

\[
c_i(\bar{y}) = \lim_{y \to \bar{y}} c_i(y) = g \leq \min (b_i, \bar{y}).
\]

Suppose \( g < \min (b_i, \bar{y}) \). Then, since \( y - c_i(y) > 0 \) on \([y, \bar{y}]\), \( y - c_i(y) \) has a minimum, \( h \), on \([y, \bar{y}]\). We have

\[
c_i(y) \geq \frac{(b_i - g)h}{(\bar{y} - y) (2\bar{b} - \bar{y} - y)} - G \cdot \frac{1}{(\bar{y} - y) (2\bar{b} - \bar{y} - y)}
\]

where \( G > 0 \). Thus

\[
c_i(y) \geq a_i + G \int_{\bar{y}}^{y} \frac{dn}{(\bar{y} - \eta) (2\bar{b} - \bar{y} - \eta)}
\]

\[
= a_i + G \int_{y}^{\bar{y}} \frac{dn}{(\bar{b} - \eta)^2 - (\bar{b} - \bar{y})^2}
\]

\[
= a_i + \frac{-G}{2(\bar{b} - \bar{y})} \ln \left[ \frac{(\bar{b} - y) - (\bar{b} - \bar{y})}{(\bar{b} - y) + (\bar{b} - \bar{y})} \cdot \frac{(\bar{b} - y) + (\bar{b} - \bar{y})}{(\bar{b} - y) - (\bar{b} - \bar{y})} \right].
\]
This last term is not bounded from above on \([y, \overline{y}]\), which contradicts \(g < \min(b_1, \overline{y})\) and we have proved

\[
\lim_{y \to \overline{y}} c_i(y) = \min(b_1, \overline{y}).
\]

\(y < \overline{y}\)

**Lemma 3.14:** In an essential equilibrium where \(\overline{y} < b_2\), \(\Phi_2(y)\) satisfies

\[
\Phi_2(y) \leq \frac{(\overline{y} - b_1) (b_2 - \overline{y})}{(b_2 - a_2) (y - b_1)} \quad \text{for} \quad y \geq \overline{y}.
\]

**Proof:** It is required that \(\overline{y}\) be optimal for Player 1 at cost \(b_1\). Hence, for \(y \geq \overline{y}\),

\[
(y - b_1) \Phi_2(y) \leq (\overline{y} - b_1) \Phi_2(\overline{y})
\]

\[
= \frac{(\overline{y} - b_1) (b_2 - \overline{y})}{(b_2 - a_2)} ,
\]

and the result follows.

This is the strongest restriction we can place on \(\Phi_2(y)\) above \(\overline{y}\), since by Lemma 3.2, if a bid \(y > \overline{y}\) is non-optimal at cost \(b_1\), it is non-optimal for any cost below \(b_1\).

We have at this point proved every proposition of Theorem 3 except the existence of an essential equilibrium when \(\overline{b} > a_2\). This we shall do in the following lemma, which completes the proof of the theorem.

**Lemma 3.15:** If \(\overline{b} > a_2\), there exists an essential equilibrium for all \(\overline{y}\) such that \(b_1 \leq \overline{y} \leq \overline{b}\) and \(\overline{y} > a_2\).

**Proof:** Let \(\overline{b} > a_2\), and \(\overline{y}\) satisfy \(b_1 \leq \overline{y} \leq \overline{b}\) and \(\overline{y} > a_2\). Let \(\overline{y}\) be defined as in the statement of Corollary 3.12.1.

We now define \(c_i(y)\), \(i = 1, 2\), as the unique solution of the differential equation of Lemma 3.12, extended to \([y, \overline{y}]\) as in Lemma 3.13. Since \(c_i(y)\) is strictly monotone increasing and continuous it is invertible on \([a_i, \min(b_i, \overline{y})]\). Let this inverse function be \(Y_i(c)\), the optimal bid at cost \(c\). For \(c\) in the range \([\min(b_i, \overline{y}), b_i]\), we extend \(Y_i(c)\) by defining it equal to \(\overline{y}\). We claim that \(Y_i(c)\), \(i = 1, 2\), thus defined form an equilibrium pair.

Let \(i, j = 1, 2\) and \(j \neq i\). Player 1's expected profit \(\pi_1(y, c)\), when he has a bid of \(y\) and a cost of \(c\), is given by

\[
\pi_1(y, c) = \begin{cases} 
  y - c & y < \overline{y} \\
  \frac{b_j - c_j(y)}{b_j - a_j} & y \leq \overline{y} < \overline{y} \\
  \frac{b_j - c_j(\overline{y})}{b_j - a_j} & y = \overline{y} \\
  0 & y > \overline{y}.
\end{cases}
\]
We note that $\pi_1$ is differentiable with respect to $y$ on $(y, \bar{y})$ and
\[
\lim_{y \to \bar{y}} \pi_1(y, c) \geq \pi_1(\bar{y}, c).
\]

First, for $c < \bar{y}$, $y$ in $[y, \bar{y})$,
\[
\frac{\partial}{\partial y} \pi_1(y, c) = \frac{1}{b_j - a_j} \left[ b_j - c_j(y) - (y - c) \right] c_j(y) \]
\[
= \frac{b_j - c_j(y)}{b_j - a_j} \left[ 1 - \frac{y - c}{y - c_j(y)} \right]
\]
by Lemma 3.12. Rewriting we get
\[
\frac{\partial}{\partial y} \pi_1(y, c) = \frac{b_j - c_j(y)}{(b_j - a_j) [y - c_j(y)]} \left[ c - c_j(y) \right].
\]

Since $\text{sgn} \ [c - c_j(y)] = \text{sgn} \ [Y_1(c) - y]$, by construction of $Y_1(c)$, $\pi_1(y, c)$ has its maximum on $[y, \bar{y})$, and, hence, on $[y, \bar{y}]$, precisely at $Y_1(c)$.

Finally, for $c \geq \bar{y}$, $\pi_1(y, c) \leq 0$ for all $y$ and $\pi_1(\bar{y}, c) = 0$. This completes the proof of the optimality of $Y_1(c)$, the lemma, and the theorem.

For the case where $b_1 = b_2 = \bar{b}$, a closed form solution for $(c_1(y), c_2(y))$ is given by the following theorem.

**THEOREM 4:** If $b_1 = b_2 = \bar{b}$, the equilibrium solution $(c_1(y), c_2(y))$ is given by
\[
c_i(y) = y - (\bar{b} - y) \frac{(y - a_i + e) e^2 + (y - a_i - e) (\bar{b} - y)^2}{(y - a_i + e) e^2 - (y - a_i - e) (\bar{b} - y)^2} \quad i = 1, 2
\]
where $e = \bar{b} - y$.

**PROOF:** The differential equation of Lemma 3.12 reduces to
\[
c_i'(y) = \frac{(\bar{b} - c_i(y)) (y - c_i(y))}{(y - \bar{b})^2} \quad i = 1, 2
\]
since $b_i = \bar{b} = \bar{y}$. By letting $d_i = y - c_i$ and $x = y - \bar{b}$, this becomes
\[
d_i' = 1 - \frac{d_i (d_i - x)}{x^2}.
\]

Since this is a homogeneous differential equation, we substitute $d_i = ux$ and $d_i' = u + xu$ to obtain, after rearrangement
\[
\frac{u'}{1 - u^2} = \frac{1}{x}.
\]

This equation can be integrated and solved for \( u \) to yield

\[
u = \frac{k_1 x^2 - 1}{k_1 x^2 + 1}.
\]

By substituting in \( d_1 = u x = u (y - \bar{b}) \),

\[
d_1 = (y - \bar{b}) \frac{k_1 (y - \bar{b})^2 - 1}{k_1 (y - \bar{b})^2 + 1}.
\]

At \( y \), \( d_1(y) = y - a_1 \), so we can solve for \( k_1 \):

\[
k_1 = \frac{(a_1 + \bar{b} - 2y)}{(y - \bar{b})^2 (\bar{b} - a_1)} \]

\[
= \frac{-(y - a_1 - e)}{e^2 (y - a_1 + e)},
\]

where \( e = \bar{b} - y \). Substituting this expression for \( k_1 \), and simplifying, yields the desired expression for \( c_1(y) \).

**COROLLARY 4.1:** For the case \( a_1 = a_2 \) and \( b_1 = b_2 = \bar{b} \), the solution becomes

\[
c_1(y) = 2y - \bar{b}.
\]

**PROOF:** By using the notation of the theorem, \( y = (a_1 + \bar{b})/2 \) and \( y - a - e = 0 \), from which the proof is immediate.

This corollary is equivalent to the result obtained by Vickrey [4].

**REFERENCES**


* * *

* * *