Individual Saving, Aggregate Capital Accumulation, and Efficient Growth

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1. Introduction

In the present essay we shall try to explore the growth-theoretic implications of what is essentially the Modigliani-Brumberg [7] life-cycle theory of saving. More specifically, we shall attempt to study growth and capital accumulation in a model where production is neoclassical but aggregate saving is due entirely to the desire of individuals to achieve an optimal lifetime consumption pattern.

Upon casual examination, the Modigliani-Brumberg life-cycle hypothesis may seem especially well suited for incorporation in a neoclassical growth theory. For this hypothesis implies that after the economy has “settled down” aggregate saving will depend on such parameters as the rate of population growth and the rate of technological improvement. However, attempts so far to construct a dynamic theory on this basis suggest that the task is not as simple as it appears at first sight. The particular studies to which we have reference here are Samuelson’s 1958 article on an exact consumption-loan model of interest [10] and Diamond’s 1965 article on a neoclassical model of national debt [3]. This essay may very well be thought of as a continuation of the investigation begun by these authors.²

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² Elsewhere [2], we have presented an extensive analysis of Samuelson’s model.
On the production side of the economy we shall assume a single, competitive sector that hires labor and capital, produces a homogeneous output under neoclassical conditions, and sells this output (indifferently) on the capital and consumption goods markets. Our view of aggregate production and distribution is thus essentially that of Solow [11].

On the consumption side we assume that each individual is born with a labor endowment only and that he plans his lifetime consumption pattern given the relevant wage rates, interest rates, and output prices. Since, in general, a particular individual's desired level of consumption at a moment of time will not coincide with his earnings at that moment, he will be engaged in saving or dissaving. Net aggregate saving, and therefore aggregate capital accumulation, is thus simply the sum of all individuals' saving or dissaving. This is, essentially, the Modigliani-Brumberg view.

Given the analogue of perfect information in static equilibrium analysis, perfect foresight, the interaction of the production and consumption sectors determines a competitive equilibrium growth path. Our major concern will be the properties, for instance, efficiency and Pareto optimality, of this path.a

2. The Model

The population at time $t$ is composed of all individuals born at times $v$ for $t - 1 \leq v \leq t$; that is, all individuals live a lifetime of length one "year." The group of individuals born at time $v$ will be referred to as "generation $v."$

Generation $v$ consists of $n^v dv$ individuals, where $n$ is thus the constant rate of population growth. We assume that $n$ is nonnegative. It is further assumed that generation $v$ is born into the labor force and continues therein until death. Hence, the total labor force at time $t$, denoted $L(t)$, is given by

$$L(t) = \int_{t-1}^{t} e^{nv} dv = \frac{1 - e^{-n}}{n} e^{nt}. \quad (1)$$

Total output at time $t$, denoted $Y(t)$, is produced with the cooperation of this labor force and the capital stock at time $t$, denoted $K(t)$, in many competitive firms whose aggregate activity can be described by a neoclassical production function. More precisely, letting lower-case letters stand for per capita quantities, it is assumed that b

$$y(t) = f[k(t)],$$

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a After this essay was completed, we became aware of Meade's investigation of the bequest motive in a similar model [6].

b The implications of modifying this assumption will be explored in section 6.

c There is no technical progress in this economy. However, as is usual in neoclassical growth theory, labor augmenting or Harrod-neutral technical progress at a constant rate could easily be assumed. The only difference that this assumption would make is that it would require a reinterpretation of all intensive variables and some parameters of the model.
where $f$ is a real function that is twice continuously differentiable, strictly concave, and that satisfies the derivative conditions $f'(0) > n > f'(\infty)$. It is also assumed that $f$ is strictly increasing for $0 \leq k \leq k$, where $0 < k < \infty$ is the point of capital saturation. The distribution of total output is determined in competitive factor markets, where the competitive real wage and real interest rates are given by the marginal productivities of labor and capital

$$w(t) = f[k(t)] - k(t)f'[k(t)]$$

and

$$r(t) = f'[k(t)],$$

respectively.

In order to complete the picture, we must describe the behavior of an individual from generation $r$. For such a description we have at our disposal the classical analysis of consumer behavior over time, as introduced, basically, by Irving Fisher [4]. The difficulty is that we do not have much else at our disposal, and using the Fisher analysis involves one assumption that, in the present context, is very drastic indeed. We are referring here to the perfect foresight assumption. To plan his consumption today, the Fisher consumer must know with certainty what he will earn in wages during the balance of his lifetime and what returns can be earned in the future from the ownership of assets. In the model under discussion, however, the effects of consumer decisions on wage and interest rates play a central role, so that the perfect foresight assumption amounts to the assertion that consumers take into account wage and interest rates that accurately reflect their own present and future actions. Clearly, what lies in the background of such an assumption is the notion of a general equilibrium model for which the commodities are dated output, labor, and capital, prices (with output as numéraire) are the wage and interest rates at every moment of time, and the equilibrium conditions are summarized in Equation 8 from the sequel.

The existence of such an equilibrium is by no means obvious; in particular, the restrictions on trades, which are imposed by the time structure of the model and are not found in the standard static general equilibrium theory, may preclude existence. Only if this equilibrium exists is the perfect foresight assumption legitimate. Demonstrating the existence of equilibrium for initial conditions that give rise to a stationary path is a fairly easy matter, as will be seen in the following section. For arbitrary initial conditions, we shall present an existence argument only in the case of a specific example that is discussed in section 4. It is possible to generalize the methods of section 4 and thereby

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$^6$ For the time being we assume that all individuals of generation $r$ behave identically. This assumption will be relaxed slightly in section 6.
prove the existence of equilibrium in the general case; however, this requires a rather lengthy argument, and we have decided to omit it.

Notice, in passing, that the alternatives to assuming perfect foresight are either to adopt an arbitrary rule of expectation (and a concomitant arbitrary rule by which to reconcile ex ante with ex post) or to incorporate into the model a full-fledged theory of decisionmaking under uncertainty; the former alternative leads to rather unconvincing results, while the latter lies beyond our present investigative ability.

Given the perfect foresight assumption, one can write down the allocation problem confronting an individual of generation \( v \). Let \( C(t, v) \) be the rate of consumption and let \( A(t, v) \) be the asset holdings (in capital goods) of an individual of generation \( v \) at time \( t \) where \( v \leq t \leq v + 1 \). We assume that the individual's preferences depend only on his consumption of output\(^{7}\) and, furthermore, that they can be represented by a utility function of the form

\[
\int_v^{v+1} U[C(t, v)]e^{-\delta(t-v)} \, dt,
\]

where \( U \) is a nondecreasing, concave, and twice-differentiable real function, and \( \delta \) is the individual's constant and nonnegative subjective rate of time preference. The allocation problem may now be stated as follows:

Maximize \( \int_v^{v+1} U[C(t, v)]e^{-\delta(t-v)} \, dt \)

subject to

\[
C(t, v) = \omega(t) + r(t)A(t, v) - \frac{\partial A(t, v)}{\partial t},
\]

\[
A(v, v) = A(v + 1, v) = 0,
\]

and

\[
C(t, v) \geq 0.
\]

The first of the constraint equations is simply the individual's budget identity. The second equation states that an individual is born with no assets, that is, that bequests are absent in this economy and that an individual dies with no assets. (Actually, the wealth constraint should be written \( A(v + 1, v) \geq 0 \), but the monotonicity of the function \( U \) entails that it hold with equality.) Finally, the last of the constraints merely formalizes the fact that a negative rate of consumption of output is meaningless.

\(^{7}\) Leisure as a second consumption good can easily be incorporated into the model. In fact, we originally carried out a good deal of the subsequent analysis under this assumption. However, as it is cumbersome and yields little if any added insight into the central results, we now just mention it as a feasible extension.
By solving the budget equation for \( A(t, v) \), which yields
\[
A(t, v) = \exp \left[ \int_{v}^{t} r(x) \, dx \right] \\
\times \left\{ \int_{v}^{t} [w(s) - C(s, v)] \exp \left[ -\int_{v}^{s} r(x) \, dx \right] \, ds + A(v, v) \right\}, \tag{2}
\]
it is possible to rewrite the allocation problem as
\[
\text{Maximize } \int_{v}^{t+1} U[C(t, v)]e^{-\delta t - w} \, dv
\]
subject to
\[
\int_{v}^{t+1} [w(s) - C(s, v)] \exp \left[ -\int_{v}^{s} r(x) \, dx \right] \, ds = 0
\]
and
\[
C(t, v) \geq 0.
\]

The solution to this maximization problem is well known from the theory of consumer behavior over time.\(^8\) For each moment \( t \) at which the non-negativity constraint is not binding, the optimal consumption plan satisfies the following differential equation:
\[
\frac{\partial C(t, v)}{\partial t} - \eta[C(t, v)] = r(t) - \delta,
\]
where \( \eta(x) = -xU''(x)/U'(x) \) is the elasticity of marginal utility.

At this point we make the simplifying assumption that \( \eta \) is identically equal to one, which reduces the function \( U \) to the logarithm.\(^9\) Such an assumption is essentially equivalent to postulating a Friedman consumption function, with permanent income elasticities all equal to unity. The optimal consumption plan is now given by the differential equation
\[
\frac{\partial C(t, v)}{\partial t} = r(t) - \delta,
\]
which has the closed solution
\[
C(t, v) = C(v, v) \exp \left\{ \int_{v}^{t} [r(x) - \delta] \, dx \right\}. \tag{3}
\]

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\(^8\) See, for example, Yaari [13].

\(^9\) Here \( \eta \) equal to any constant could just as well be assumed. Note that this assumption ensures that the nonnegativity constraint is never binding (except in the trivial case where the economy produces no output).
for all \( t \) satisfying \( v \leq t \leq v + 1 \). The initial rate of consumption is determined from the wealth constraint after substitution from Equation 3:

\[
C(v, v) = \frac{\int_v^{v+1} w(s) \exp \left[ - \int_v^s r(x) \, dx \right] \, ds}{1 - e^{-\delta}/\delta}.
\]  

(4)

Finally, the asset holdings corresponding to the optimal consumption plan are derived from Equation 2 after substitution from Equations 3 and 4:

\[
A(t, v) = \int_t^v w(s) \exp \left[ - \int_t^s r(x) \, dx \right] \, ds
- \frac{1 - e^{-\delta t - v}}{1 - e^{-\delta}} \int_v^{v+1} w(s) \exp \left[ - \int_t^s r(x) \, dx \right] \, ds
\]  

(5)

for all \( t \) satisfying \( v \leq t \leq v + 1 \).

Let \( c(t) \) be the rate of aggregate consumption per capita and \( a(t) \) be aggregate asset holdings per capita at time \( t \). From Equations 1, 3, 4, and 5 these two quantities are given explicitly by

\[
c(t) = \frac{n}{1 - e^{-\delta}} \frac{\delta}{1 - e^{-\delta}}
\times \int_{t-1}^t \left\{ \int_v^{v+1} w(s) \exp \left[ - \int_t^s r(x) \, dx \right] \, ds \right\} \exp \left[ - \int_t^s 0.5 \frac{1 - e^{-\delta t - v}}{1 - e^{-\delta}} \right] \, dv
\]  

(6)

and

\[
a(t) = \frac{n}{1 - e^{-\delta}} \int_{t-1}^t \left\{ \int_v^{v+1} w(s) \exp \left[ - \int_t^s r(x) \, dx \right] \, ds \right\} \exp \left[ - \int_t^s 0.5 \frac{1 - e^{-\delta t - v}}{1 - e^{-\delta}} \right] \, dv.
\]  

(7)

If we now identify the asset holdings of individuals with the capital stock,\(^\text{10}\) we get the basic equilibrium condition for the economy

\[a(t) = k(t).\]  

(8)

By differentiating Equation 8 and simplifying the result, or, more directly, by equating the rate of net saving per capita to the rate of gross investment per capita, we also have the equilibrium growth equation for the economy

\[
k(t) = f[k(t)] - nk(t) - c(t)
= [r(t) - n]k(t) + \left[ w(t) - c(t) \right].
\]  

(9)

\(^\text{10}\) Negative asset holdings (debt) for some consumers could be expressly accounted for by assuming competitive financial intermediaries which hold as assets capital goods and consumer-issued bonds and as liabilities intermediary-issued bonds. Equation 8 then represents a netting of the consumer and financial intermediary sectors. We shall have more to say about intermediaries of this sort in sections 6 and 8.
For some purposes the representation in Equation 9 will be more useful than
the representation in Equation 8; note, however, that Equation 8 implies
Equation 9, but not conversely.

3. Competitive Growth Equilibrium: Balanced Growth Paths

We initiate the analysis of the equilibrium condition of Equation 8 by
restricting attention to balanced growth paths, that is, to growth paths on
which the capital-labor ratio remains constant over time. The question to be
answered is thus: Are there constant capital-labor ratios which satisfy
Equation 8?

To begin with, observe that if \( k = \text{constant} \), then \( r = f'(k) = \text{constant} \)
and \( w = f(k) - f'(k)k = \text{constant} \). (In fact, because of these constancies,
balanced growth in our economy is easily shown to be equivalent to stationary
consumer behavior, that is, to the statement that \( C(t, \nu) \) and \( A(t, \nu) \)
depend only on the difference \( t - \nu \) for all \( t \) satisfying \( \nu \leq t \leq \nu + 1 \).) Hence,
utilizing Equation 9 with appropriate simplification, it follows that a necessary
condition for balanced growth equilibrium is

\[
0 = [f(k) - nk] - w\left[ n \frac{\delta}{1 - e^{-\delta}} \frac{1 - e^{-r}}{1 - e^{-e^{-\delta} - r}} \frac{1 - e^{-r(n + \delta - n)}}{n + \delta - r} \right].
\]  

(10)

For convenience, let us rewrite Equation 10 as an equation in the rate of
interest \( r \) (which is legitimate by virtue of the strict concavity of the function
\( f \)),

\[
\psi(r) = \phi(r),
\]

(11)

where we define the function \( \psi \) as

\[
\psi(r) = \frac{f(k) - nk}{f(k) - f'(k)k} \quad \text{with} \quad r = f'(k)
\]

and the function \( \phi \) as

\[
\phi(r) = \frac{n}{1 - e^{-\delta}} \frac{\delta}{1 - e^{-\delta}} \frac{1 - e^{-r}}{1 - e^{-e^{-\delta} - r}} \frac{1 - e^{-r(n + \delta - n)}}{n + \delta - r}.
\]

Properties of the functions \( \psi \) and \( \phi \), relevant now and later on, are for \( \psi \),

\[
\psi(r) \lesssim 1 \quad \text{as} \quad r \lesssim n
\]

(12a)

and

\[
\psi'(r) = \frac{1}{f'(k)} \frac{f''(k) - n}{f(k) - f'(k)k} + \psi(r) \frac{k}{f(k) - f'(k)k}.
\]

(12b)
while for $\phi$,

$$
\phi\left(\frac{n + \delta}{2} + x\right) = \phi\left(\frac{n + \delta}{2} - x\right) \quad \text{for all } x, \quad (12c)
$$

$$
\phi(\delta) = \phi(n) = 1, \quad (12d)
$$

$$
\phi'(r) = \left[ g(r) - g(n + \delta - r) \right] \phi(r) \leq 0 \quad \text{as } r \leq \frac{n + \delta}{2} \quad (12e)
$$

and

$$
\phi''(r) = \left[ g'(r) + g'(n + \delta - r) \right] + \left[ g(r) - g(n + \delta - r) \right]^2 \phi(r) > 0, \quad (12f)
$$

where it is useful to introduce the function $g$, defined by

$$
g(x) = \frac{(1 + x)e^{-x} - 1}{x(1 - e^{-x})} \quad \text{for all } x,
$$

and shown in Figure 1.

![Figure 1](image_url)

**Figure 1.** Properties of the function $g(x) = \frac{(1 + x)e^{-x} - 1}{x(1 - e^{-x})}$.

From the properties just listed it becomes clear that Equation 11 yields at least one candidate $r = n$, and in general more than one candidate $r = n$, $r = \bar{r}_1, \ldots, r = \bar{r}_m$ for balanced growth equilibrium. As an illustration, in Figure 2, $\psi, \phi$ and the two solutions to Equation 11 are depicted under the assumptions $0 < \delta < n$ and $f(k) = Ak^\alpha$, $0 < \alpha/(1 - \alpha) < n[g(n) - g(\delta)]$.

Now, by examining Equation 8 we can show that $r = n$ represents a balanced growth equilibrium if and only if $\psi'(n) = \phi'(n)$ (that is, if and only if the
functions $\psi$ and $\phi$ are actually tangent to each other at $r = n$) whereas $r = \tilde{r}, \neq n, j = 1, \ldots, m$, always represents a balanced growth equilibrium.

Let $k^*$ be the golden rule capital-labor ratio, that is, let $f'(k^*) = n$. Then the first part of the assertion follows simply by writing out the expression for assets per capita from Equation 7 when $r = n$,

$$a^* = \left[ f(k^*) - f'(k^*)k^* \right] \left[ \frac{1}{n} \left( \frac{n}{1 - e^{-n}} - 1 \right) - \frac{\delta}{\delta} \left( \frac{1}{1 - e^{-\delta}} - 1 \right) \right]$$

$$= \left[ f(k^*) - f'(k^*)k^* \right] \phi'(n),$$

and noting that therefore $k^* = a^*$ if and only if

$$\frac{k^*}{f(k^*) - f'(k^*)k^*} = \phi'(n),$$

or, from Equations 12a and 12b, $\psi'(n) = \phi'(n)$. On the other hand, let $k$ be any other capital-labor ratio. Then the second part of the assertion follows

![Figure 2](image-url)

Figure 2. Illustration of possible balanced growth equilibria.

by differentiating the expression for assets per capita from Equation 7 when $r \neq n$,

$$0 = (r - n)a + [f(k) - f'(k)k][1 - \phi(r)]$$

or

$$a = \frac{[f(k) - f'(k)k][1 - \phi(r)]}{n - r},$$

and noting that therefore $k = a$ if and only if

$$\frac{f(k) - nk}{f(k) - f'(k)k} = \phi(r)$$
or

$$\psi(r) = \phi(r).$$

It should be mentioned that if $\phi'(n) < \psi'(n)$, then a balanced growth equilibrium may not exist. For example, consider the production technology described by

$$f(k) = k(A + Bk), \quad A > 0, \quad B < 0,$$

so that

$$\psi(r) = \frac{A - 2n + r}{A - r} \quad \text{for } r \leq A.$$

If $n < A < \delta$, then $\phi'(n) < 0 < \psi'(n)$, while $r = n$ is the only solution to Equation 11. (Note that if $\phi'(n) > \psi'(n)$, then there must be at least one solution to Equation 11 in the interval $(0, n)$; that is, for the opposite case the situation depicted in Figure 2 is typical.) In the following discussion, however, we simply assume the existence of a balanced growth equilibrium, which is equivalent to imposing appropriate additional conditions on the production function. For example, the conditions

$$\lim_{k \to 0} r = \infty \quad \text{(15)}$$

and

$$\psi'(r) \leq 0 \quad \text{for } r \geq \hat{r}, \quad 0 \leq \hat{r} < \infty, \quad \text{(16)}$$

entail that if $\phi'(n) < \psi'(n)$, then there must be at least one solution to Equation 11 in the interval $(n, \infty)$, and thus these conditions would be sufficient (if not especially transparent).

What conclusions can be drawn from the foregoing? First, that there may be no, one, or several balanced growth equilibria. And second, that any feasible, positive rate of interest may represent a balanced growth equilibrium, depending, in particular, on the specific production technology, rate of population growth, and rate of time preference prevailing in the economy. Citing a result conjectured by Phelps [8], proved by Koopmans, and then elaborated by Phelps [9]—that if $r(t) \leq n - \epsilon$ for some $\epsilon > 0$ and all $t \geq \hat{t}$, then the growth path represented by $r(t)$ is inefficient—we can therefore deduce a further important result: the balanced growth equilibrium in our stylized competitive economy need not be an efficient growth path.

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11 As we have not forestalled the possibility of capital saturation, for a sufficiently "perverse" rate of time preference Equation 11 might have a nonpositive solution. This exceptional case is at present excluded by assumption $(\delta \geq 0)$ but will receive further attention in section 6.

12 In this context, inefficiency means simply that, given the same initial capital-labor ratio, there is another feasible growth path that provides at least as much total consumption all of the time, and in fact provides more total consumption some of the time. Additional results on inefficiency in the neoclassical growth model, which will be utilized later on in the essay, are contained in the appendix.
4. Competitive Growth Equilibrium: An Example

In this section we shall present a complete characterization of the competitive growth equilibria for a specific production function. The example is both interesting in its own right and suggestive of the general case, which we discuss briefly in the following section.

Suppose that the production technology in our economy is described by

\[ f(k) = k(A + B \log k), \quad B < 0, \quad (17) \]

which is the general representation of the class of production functions for which

\[ w = -Bk, \quad (18a) \]

that is, the wage rate is proportional to the capital-labor ratio. Notice also that the production function of Equation 17 exhibits capital saturation at \( k = e^{-(1 + A/B)} \). Some further properties we shall use are

\[ r = A + B + B \log k \quad (18b) \]

and

\[ \psi(r) = 1 + \frac{n}{B} \frac{r}{B}, \quad (18c) \]

The significance of the production function of Equation 17 is that it permits a substantial simplification of the equilibrium condition of Equation 8. More specifically, if we substitute from Equation 18a into the right-hand side of Equation 8, invert the order of integration in the right-hand side of Equation 8, and then multiply both sides of the equation by the factor

\[ \exp \left[ -\int_0^t r(x) \, dx \right], \]

we get the following standard linear integral equation for Equation 8:

\[ x(t) = \int_{-\infty}^t G(t - s) x(s) \, ds, \quad (19) \]

where

\[ x(t) = k(t) \exp \left[ -\int_0^t r(x) \, dx \right], \quad (20) \]

and

\[ G(t - s) = B \frac{n}{1 - e^{-n}} \int_{t-1}^s \left( 1 - \frac{1 - e^{-B(x - \psi)}}{1 - e^{-s}} \right) e^{-\eta(t - \psi)} \, dv, \]

\[ t - 1 \leq s \leq t, \quad (21) \]

\[ = -B \frac{n}{1 - e^{-n}} \int_{s-1}^t \left( 1 - \frac{1 - e^{-B(x - \psi)}}{1 - e^{-s}} \right) e^{-\eta(t - \psi)} \, dv, \]

\[ t \leq s \leq t + 1, \]

\[ = 0 \quad \text{otherwise}. \]
In order to characterize the competitive growth equilibria for an economy having such a production function, we need only characterize the real, positive solutions of Equation 19. But the solutions to this integral equation are well known; they are essentially exponentials in which the exponents are the roots of the equation

$$\int_{-1}^{1} G(s)e^{us} \, ds = 1. \tag{22}$$

More precisely, all solutions of Equation 19 are of the general form,

$$x(t) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} t^{j-1} e^{-u_j t}, \tag{23}$$

where $u_1, u_2, \cdots, u_n$ are the $n$ distinct roots of Equation 22 and $m_i$ is the multiplicity of the root $u_i$. The $a_{ij}$ are arbitrary constants. Thus, we need only characterize the roots of Equation 22.\footnote{See, for example, Titchmarsh [12], pp. 305-307.}

Rather than attempt this task directly, it is much easier to utilize the fact that, for the production function of Equation 17, the equilibrium growth equation may be rewritten in terms of the present value of capital per head as

$$\frac{\dot{x}(t)}{B} + \left(1 + \frac{n}{B}\right)x(t) = \int_{-\infty}^{\infty} H(t-s)x(s) \, ds, \tag{24}$$

where

$$H(t-s) = \frac{n}{1 - e^{-n}} \frac{\delta}{1 - e^{-\delta}} \int_{s-1}^{t} e^{-(n + \delta)(t-s)} \, dv, \quad t-1 \leq s \leq t,$$

$$= \frac{n}{1 - e^{-n}} \frac{\delta}{1 - e^{-\delta}} \int_{s-1}^{t} e^{-(n + \delta)(t-s)} \, dv, \quad t \leq s \leq t+1, \tag{25}$$

$$= 0 \quad \text{otherwise.}$$

What this means is that any root of Equation 22 must also be a root of the equation obtained by substituting

$$x(t) = e^{-u t}$$

into Equation 24, that is, of the equation

$$1 + \frac{n}{B} - \frac{u}{B} = \int_{-1}^{1} H(s)e^{us} \, ds. \tag{26}$$

Consider first the real roots that Equations 22 and 26 have in common.
If \( u \) is real, for instance \( u = r \), then it is easily verified that Equations 22 and 26 are in fact the balanced growth counterparts of Equations 8 and 9, respectively. In particular, Equation 26 reduces to Equation 11 with \( \psi \) given by Equation 18c. But then from the analysis of the preceding section we know immediately that \( \bar{r} \), the rate of interest corresponding to balanced growth equilibrium, is the unique real root that Equations 22 and 26 have in common, as is illustrated in Figure 3 under the assumptions \( 0 < \delta < n \) and \( \psi(n) = -1/B > g(n) - g(\delta) = \phi' \).

\[
\phi(r) = \int_{-1}^{1} H(s)e^{rs} \, ds
\]

\[
\psi(r) = (1 + \frac{n}{B}) - \frac{r}{B}
\]

**Figure 3.** Balanced growth equilibrium for the example \( f(k) = k(A + B \log k) \).

Consider now the complex roots that Equations 22 and 26 have in common. If \( u \) is complex, for instance \( u = \alpha + i\beta \), then by equating real and imaginary parts Equation 26 may be rewritten as the pair of equations

\[
1 + \frac{n}{B} - \frac{\alpha}{B} = \int_{-1}^{1} H(s)e^{\alpha s} \cos \beta s \, ds \tag{28}
\]

and

\[
-\frac{\beta}{B} = \int_{-1}^{1} H(s)e^{\alpha s} \sin \beta s \, ds. \tag{29}
\]

It follows directly from Equation 28 that if \( \beta \neq 0 \), then \( \alpha \neq \bar{r} \) (as \( H \) is a positive function while the cosine is an even function less than or equal to
one). Thus, without any further analysis we may conclude that if a solution to Equation 23 is to be real and positive, then it can only be of the form

$$x(t) = Ce^{-\bar{a}t}, \quad C > 0. \quad (30)$$

Converting from present to current values of capital per head, the foregoing can be summarized in the following proposition:

For an economy with the production function of Equation 17, every competitive growth equilibrium is represented by the asymptotically balanced growth path

$$k(t) = [r(t) - \bar{r}]k(t), \quad k(0) > 0, \quad \lim_{t \to \infty} k(t) = \bar{k}, \quad (31)$$

along which total consumption is a constant proportion of labor's competitive income

$$c(t) = \left(1 + \frac{n - \bar{r}}{\bar{B}}\right)w(t). \quad (32)$$

Equation 32 is easily checked by substituting from Equations 17, 18a, and 18b into Equation 31 in order to obtain Equation 9.

We mention for emphasis that the equilibria for this example are not necessarily efficient, because (again appealing to the Phelps-Koopmans result)

$$\lim_{t \to \infty} r(t) = \bar{r} < n \quad \text{whenever} \quad -1 < \frac{1}{\bar{B}} < g(n) - g(\bar{\delta}),$$

the rate of interest will eventually lie below its natural value for certain parameter configurations. This leads naturally to certain sorts of questions: Under what conditions is the competitive growth equilibrium efficient? Given that it is efficient, is it therefore Pareto optimal? For this example, the following proposition and attached footnote contain a complete answer: \(^{14}\)

If \(-1 < \bar{B} > g(n) - g(\bar{\delta}), \) or \(\bar{r} > n,\) then the competitive growth equilibrium represented by the growth path of Equation 31 maximizes

$$\int_{-1}^{\infty} \left[\int_{\max(0,v)}^{n+1} \log C(t,v)e^{-\bar{a}(s-v)} dt e^{-\bar{a}t}\right]e^{n\bar{a}} dv$$

$$= \int_{0}^{\infty} \left[\int_{-1}^{\rho} \log C(t,v)e^{\rho(s-v)} dv\right]e^{-\rho t} dt, \quad \text{with} \quad \rho = \bar{r} - n, \quad (33)$$

\(^{14}\) It may puzzle the reader to find that Pareto optimality is defined only for welfare enjoyed after time zero, because up to now we have been implicitly assuming that the competitive growth equilibrium exists forever. The answer to this seeming contradiction is that we are also implicitly assuming that the behavior of the economy before time zero is past history, that is, that observation of the economy only begins at time zero. Perhaps a better (though certainly more difficult) assumption to accord with the latter would be that, given an arbitrary distribution of the capital stock among generations existing at time zero, just then does perfect foresight become a pervasive phenomenon.
the social welfare function which gives the total welfare of an individual of
generation \( v \) (measured from time zero) the weight \( e^{-rv} \). This proposition is
based on the facts, whose proof we do not spell out here, first, that the
individual consumption rate

\[
C(t, v) = \frac{1 - e^{-h}}{n} \frac{n + \delta - \tilde{r}}{1 - e^{-(n + \delta - \tilde{r})}} c(t)e^{\tilde{r}t - \delta v(t - v)}
\]  

(34)

is the solution to the variational problem

\[
\text{Maximize} \int_{t-1}^{t} \log C(t, v)e^{\psi - \delta v(t - v)} dv
\]  

(35)

subject to

\[
\frac{n}{1 - e^{-n}} \int_{t-1}^{t} C(t, v)e^{-n(t - v)} dv = c(t)
\]

and

\[
C(t, v) \geq 0
\]

of selecting the "best" intergenerational distribution of consumption at
each point along an arbitrary feasible growth path, and second, that the
growth path of Equation 31 is the solution to the variational problem

\[
\text{Maximize} \int_{0}^{\infty} \log c(t)e^{-\psi t} dt
\]  

(36)

subject to

\[
k(t) = [A - n + B \log k(t)]k(t) - c(t), \quad \text{given } k(0) > 0
\]

and

\[
c(t) \geq 0
\]

of selecting the "best" feasible growth path. From inspection of the social
welfare function of Equation 33 it is easily seen that, by solving the first
problem, substituting the value of its solution into the social welfare function,
and then solving the second problem, the asserted conclusion is obtained.

One further point is worth remarking in connection with the relationship
between the efficiency and Pareto optimality of the growth path of Equation
31. Namely, by virtue of the property of Equation 18a, the weight given to

---

\footnote{Furthermore, if \(-1/B = g(n) - g(\tilde{\delta})\), or \(\tilde{r} = n\), then the growth path of Equation 31 maximizes the social welfare function that gives the amount by which each individual's total welfare deviates from his hypothetical golden rule welfare (measured from time zero) the weight \(e^{-rv}\).}

\footnote{The distribution problem is, given constant wage and interest rates, precisely that
analyzed in section 2, while the growth problem is, given specific production and utility
functions, precisely that analyzed in Cass \([1]\) and Koopmans \([5]\). We should also note
that Koopmans' analysis provides a justification for the claim made in the preceding
footnote.}
the welfare of an individual of generation \( v \) in the social welfare function of Equation 33 is proportional to the present value of his labor endowment in the competitive growth equilibrium itself

\[
e^{-r_x} = \left[ w(0) \frac{1 - e^{-\frac{x}{f}}}{r} \right]^{-1} \int_{v}^{u+1} w(s) \exp \left[ - \int_{0}^{s} r(x) \, dx \right] \, ds. \tag{37}
\]

This feature, which will be the basis for generalization in the following section, is peculiar to the specific utility function assumed, as is well known from the standard static general equilibrium theory.

5. Competitive Growth Equilibrium: The General Case

Unlike the example just discussed, for a general neoclassical production function it is not possible to derive an explicit expression representing the competitive growth equilibria. However, taking our cue from that example, it is possible to construct a conventional fixed-point argument for the existence of a competitive growth equilibrium that starts close to and goes asymptotically to a balanced growth path. We omit this argument, partly because it is somewhat tedious but mostly because it is somewhat beside the point.

By the last we mean simply that much more pertinent to our understanding of the nature of competitive growth equilibrium would be, for example, the demonstration of the asymptotic balance of all competitive growth equilibria. And while, primarily on the basis of the example of the preceding section, we conjecture this property, we have as yet been unable to demonstrate it.

What we shall do in this section is to present a general theorem on the relationship between the efficiency and Pareto optimality of the equilibrium for our model. This theorem is closely related, though not perfectly analogous to the proposition outlined in the last part of the preceding section. It is intended to answer, at least in part, the same questions posed there.

For this purpose we require two things, first, a more sophisticated criterion for recognizing inefficiency than the Phelps-Koopmans result, and second, a weaker definition of Pareto optimality than the usual notion (which was employed earlier).

With regard to the first requirement, in the appendix the following necessary and sufficient condition for inefficiency in the neoclassical growth model is provided: a feasible growth path \( k^0(t) \) for \( t \geq 0 \) is inefficient if and only if there exists another feasible growth path \( k^1(t) \) for \( t \geq 0 \) such that\(^{17}\)

---

\(^{17}\) The notion of feasible growth from a given initial capital-labor ratio is precisely defined in the appendix to this essay.
\[ 0 < \lim_{T \to +\infty} \inf \int_0^T \left[ c'(t) - c''(t) \right] \exp \left\{ - \int_0^t [r(x) - n] dx \right\} dt. \] (38)

This condition simply says that a growth path is inefficient when there are feasible deviations in aggregate consumption whose present value, calculated at the rates of interest generated by the growth path, is unambiguously positive.

With regard to the second requirement, an increase in an individual's welfare from society's viewpoint is defined thus: the individual consumption \( \mathcal{C}(t, v) \) for \( 0 \leq t \leq v + 1 \) provides more welfare than the individual consumption \( \mathcal{C}(t, v) \) for \( 0 \leq t \leq v + 1 \) if and only if

\[ 0 \leq \int_{\max(0,v)}^t \left[ \log \mathcal{C}(t, v) - \log \mathcal{C}(t, v) \right] e^{-\theta(t-v)} dt \]

for \( 0 \leq v \leq v + 1 \). That is, an increase in an individual's welfare must occur throughout his lifetime to be recognized as such from society's viewpoint.\(^{10}\)

We proceed to show that if a competitive growth equilibrium is not Pareto optimal in the latter sense, then it is inefficient.

We shall let variables with a bar represent quantities along the particular competitive growth equilibrium under consideration and variables with a tilde represent quantities along a dominating growth path, that is, another feasible growth path that, after time zero, provides at least as much and sometimes more individual welfare than the competitive growth equilibrium. Without loss of generality we can assume that along the dominating growth path all individuals of generation \( v \) are treated identically. Also let

\[ \mathcal{P}(0, v) = \int_0^{v+1} \bar{u}(s) \exp \left[ - \int_0^s \bar{r}(x) dx \right] ds \]

\[ = \bar{P}(t, v) \exp \left[ - \int_0^t \bar{r}(x) dx \right] > 0 \] (40)

represent the present value of the labor endowment of an individual of generation \( v \) along the competitive growth equilibrium. From Equations 39

\(^{10}\) The appeal of our weaker definition of Pareto optimality in the present dynamic context arises from the following observation. Namely, if uncertainty about a particular individual's (or perhaps generation's) presence in the future is introduced explicitly (or even implicitly) into the model, then it more accurately reflects the fundamental idea that reallocation is obviously called for only when in fact somebody's welfare will be increased but nobody's welfare will be decreased. However, it is arguable and is adopted basically because it permits a definite conclusion.
and it follows directly that
\[ 0 < \lim_{T \to \infty} \int_{-1}^{T} \left\{ \int_{\max(0,v)}^{\min(T,v+1)} \left[ \log \bar{C}(t, v) - \log \bar{C}(t, v) \right] e^{-\delta(t-v)} dt \bar{P}(0, v) \right\} e^{x} dv \]
\[ = \lim_{T \to \infty} \int_{0}^{T} \left\{ \int_{-1}^{t} \left[ \log \bar{C}(t, v) - \log \bar{C}(t, v) \right] \bar{P}(t, v) e^{-(n+\delta(t-v)} dv \right\} \times \exp \left\{ -\int_{0}^{t} [\bar{r}(x) - n] dx \right\} dt, \]

some positive number is assigned to the dominating growth path by the social welfare function that gives the amount by which each individual's welfare deviates from his competitive equilibrium welfare (measured from time zero) the weight \( \bar{P}(0, v) \).

Now consider the intergenerational distribution of consumption along the dominating growth path. From Equation 41 it should be clear that if, say, \( \bar{C}(t, v) \) represents the solution to the variational problem

\[ \text{Maximize } \int_{-1}^{t} \log C(t, v) \bar{P}(t, v) e^{-(n+\delta(t-v)} dv \] (42)

subject to

\[ \frac{n}{1 - e^{-n}} \int_{t-1}^{t} C(t, v) e^{-(n+\delta(t-v)} dv = \bar{c}(t) \]

and

\[ C(t, v) \geq 0 \]

of picking the "best" intergenerational distribution of consumption at each point along the dominating growth path, then

\[ 0 < \lim_{T \to \infty} \int_{0}^{T} \left\{ \int_{-1}^{t} \left[ \log \bar{C}(t, v) - \log \bar{C}(t, v) \right] \bar{P}(t, v) e^{-(n+\delta(t-v)} dv \right\} \times \exp \left\{ -\int_{0}^{t} [\bar{r}(x) - n] dx \right\} dt. \] (43)

But each problem of this type is essentially the consumer lifetime allocation problem discussed in section 2 (now with a variable subjective discount rate). Again employing the result quoted there, we can easily derive its closed solution

\[ \bar{C}(t, v) = \frac{\bar{C}(t, v)}{\bar{c}(t)} \bar{c}(t). \] (44)

Finally, substituting from Equation 44 into Equation 43 and manipulating the resulting expression we find that
\[ 0 < \lim_{\tau \to \infty} \int_0^\tau \left\{ \int_{t-1}^t \left[ \log \frac{\bar{C}(t, v)}{\bar{c}(t)} - \log \frac{\bar{C}(t, v)}{\bar{c}(t)} \right] \bar{P}(t, v) e^{-\gamma(t)\bar{v}} \, dv \right\} \times \exp \left\{ -\int_0^t [\bar{r}(x) - n] \, dx \right\} \, dt \]

\[ = \lim_{\tau \to \infty} \int_0^\tau \left\{ \int_{t-1}^t \bar{P}(t, v) e^{-\gamma(t)\bar{v}} \, dv \right\} \left[ \log \bar{c}(t) - \log \bar{c}(t) \right] \times \exp \left\{ -\int_0^t [\bar{r}(x) - n] \, dx \right\} \, dt \]

\[ = \frac{1 - e^{-\gamma}}{n} \frac{1 - e^{-\delta}}{\delta} \lim_{\tau \to \infty} \int_0^\tau \bar{c}(t) \left[ \log \bar{c}(t) - \log \bar{c}(t) \right] \times \exp \left\{ -\int_0^t [\bar{r}(x) - n] \, dx \right\} \, dt \]

\[ \leq \frac{1 - e^{-\gamma}}{n} \frac{1 - e^{-\delta}}{\delta} \lim_{\tau \to \infty} \int_0^\tau \bar{c}(t) \left[ \bar{c}(t) - \bar{c}(t) \right] \exp \left\{ -\int_0^t [\bar{r}(x) - n] \, dx \right\} \]

By virtue of the condition of Equation 38, this completes the proof of our assertion.

6. An Interpretation

In this section, for convenience of exposition, we concentrate on balanced growth. Hereafter (for the most part) we also adopt the simplifying assumption that the production technology satisfies the conditions of Equations 15 and 16 with \( \bar{r} = 0 \), which guarantees a unique balanced growth equilibrium. The central question we are addressing is: Why may competitive capital accumulation go astray?

The answer to this question is, in fact, straightforward. At efficient rates of interest consumers may want to hold more real assets than are available in the existing capital stock. (Or, to put the matter another way, at the rate of interest that equilibrates desired real assets and the actual capital stock, the private rate of return may differ from the social rate of return—which is surely minus infinity on an inefficient balanced growth path.) More precisely, given the conditions of Equations 15 and 16 with \( \bar{r} = 0 \), from Equations 13 and 14 we know that

\[ a \geq k \quad \text{as} \quad r \geq \bar{r} \]

and

\[ \bar{r} \geq n \quad \text{as} \quad \psi(n) \geq \phi(n), \]

which, taken together, entail the relationship

\[ a > k \quad \text{for} \quad r \geq n \quad \text{whenever} \quad \psi(n) < \phi(n). \]
An extreme example points up this difficulty especially well. Suppose, in a more classical tradition, that population is stationary \((n = 0)\) and that capital saturation is possible \((f'(\bar{k}) = 0)\) for some \(0 < \bar{k} < \infty\). Also suppose that consumers exhibit “perverse” time preference \((\delta < 0)\). Finally, suppose that production technology entails that \(\psi'(0) < \phi'(0)\). Then, there is a unique stationary solution to Equation 8 at some \(\bar{r} < 0\) (depicted in Figure 1 by shifting the vertical axis to the point \(n\) on the horizontal axis) that obviously represents a grossly inefficient situation; merely by utilizing less than the whole capital stock, total output could be increased. However, for this example, because consumers can carry inventories without cost or hold capital as assets, the rate of interest would never fall below zero, and Equation 8 is an incomplete statement of the equilibrium condition, which instead should be

\[
a(t) = k(t) + z(t),
\]

with

\[
k(t) \leq \bar{k}
\]

and

\[
z(t) \geq 0, \quad \text{equality for } k(t) < \bar{k},
\]

where \(z(t)\) stands for inventories of consumption goods per head at time \(t\). Therefore, it is easily seen that the true stationary state for this stylized, classical competitive economy must occur at the rate of interest \(r = n = 0\), which, on the production side, is clearly efficient. On the other hand, on the consumption side, this stationary state is just as clearly inefficient: from the relationship of Equation 46 it follows that \(z > 0\) when \(r = 0\), or that, in the stationary state, consumers desire to carry an inventory of consumption goods over and above their holdings of capital, an inventory that will in actuality never be consumed. We recall for emphasis that from each consumer’s viewpoint, carrying such a dead weight is quite sensible; he is doing the best he can given the wage rate \(w = f(\bar{k})\) and interest rate \(r = 0\) prevailing.

Given our interpretation, the question naturally arises: Have we overlooked something, some aspect of consumer behavior or perhaps some institution, that will in fact ensure the “right” relationship between real asset preferences and opportunities? Let us consider these possibilities in turn.

One point, purposely left in the background until now, is that a sufficiently high rate of time preference would guarantee (at least) efficient balanced growth equilibrium under all conceivable neoclassical production technologies. That is, if \(\delta \geq n\), then

\[
\phi'(n) = g(n) - g(\delta) \leq 0,
\]

which rules out the possibility of a solution to Equation 11 for \(r < n\). Intuitively this is a rather plausible result. If a consumer strongly prefers consumption today to consumption tomorrow, then, even at relatively high rates of interest, his total savings will be small algebraically at every instant.
during his lifetime (cf. Equation 5). However, the requisite balancing of parameters cannot be depended on in theory and, indeed, loses a degree of support if we admit some variation among consumers' rates of time preference.

Suppose now that each generation has a proportion of individuals $\lambda_j > 0$ with rate of time preference $\delta_j \geq 0, j = 1, \ldots, m$ and $\sum_{j=1}^{m} \lambda_j = 1$. Then we have

$$\phi(r) = \sum_{j=1}^{m} \lambda_j \phi_j(r),$$

where

$$\phi_j(r) = \frac{n}{1 - e^{-r}} \frac{e^{-\delta_j}}{1 - e^{-\delta_j}} \frac{1 - e^{-r} - e^{-(n+\delta_j-r)}}{n + \delta_j - r},$$

and the stylized competitive economy will exhibit efficient balanced growth equilibrium for every neoclassical production technology if and only if

$$\phi'(n) = \sum_{j=1}^{m} \lambda_j \phi'_j(n) = g(n) - \sum_{j=1}^{m} \lambda_j g(\delta_j) \leq 0. \quad (48)$$

If we now associate the rate of time preference $\delta$ introduced in section 2, with the average rate of time preference here $\sum_{j=1}^{m} \lambda_j \delta_j$, then by the strict concavity of $g(x)$ for $x > 0$, it follows that

$$-g(\delta) < -\sum_{j=1}^{m} \lambda_j g(\delta_j),$$

which means that the inequality in Equation 48 might not be satisfied even if that in Equation 47 were satisfied. While not placing too much weight on this example, it illustrates that diversity in consumer behavior cannot be expected, a priori, to reduce the likelihood that consumers may want to hold more real assets than are available in the capital stock at efficient rates of interest.

Another aspect of consumer behavior whose effects we have investigated is biological or sociological restrictions on the length of and efficiency during the work life. As a gross approximation to these restrictions, suppose that the individuals of each generation enter the labor force at age $m_1$ (because they are maturing and undergoing education from age 0 to age $m_1$) and leave

---

19 Some support is regained in the concluding section, where we analyze an economy in which uncertainty about the end of the world contributes to positive time preference.

20 Both Diamond [3] and Samuelson [10] adopt one such restriction, forced retirement, as an integral part of their models. Unlike them, we prefer to treat these restrictions as less fundamental, partly because their particular formulation seems somewhat more arbitrary than that of the other elements in the basic model but mostly because we feel that their introduction at the outset tends to obscure the basic issue presently being discussed.
it at age $m_2$ (because they are no longer able to work) where $0 \leq m_1 \leq m_2 \leq 1$. In the spirit of the two preceding paragraphs, it is easily shown that under these additional assumptions the function $\phi$ remains strictly convex, while $\phi'(n) \leq 0$ if and only if

$$-g(\delta) \leq -(m_2 - m_1)g[n(m_2 - m_1)] + m_1.$$  \hspace{2cm} (49)

The right-hand side of this inequality is depicted in Figure 4 under the further assumption that the growing-up and retirement periods are of equal length, $m_1 = \epsilon$ and $m_2 = 1 - \epsilon$ for $0 \leq \epsilon < 1/2$. Again, though one can certainly derive conditions from such restrictions that would ensure the efficiency of competitive behavior (for instance, $\epsilon \geq \epsilon^*(n, \delta)$ in Figure 4), a presumption that they would be satisfied is not justified.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Illustration of the sufficient condition for efficient balanced growth equilibrium when the growing-up and retirement periods are equal.}
\end{figure}

It is clear to us, after analyzing these and other aspects of consumer behavior, that further complications along similar lines do not basically alter the general conclusion stated at the beginning of this section. What, then, about an overlooked institution, one that might be expected to exist in an inefficient economy\footnote{Hereafter this term signifies an economy in which (1) the production technology satisfies the conditions of Equations 15 and 16 with $\dot{r} = 0$ (balanced growth equilibrium is unique) and (2) the production technology, rate of population growth, and rate of time preference together entail $\psi(n) < \phi'(n)$ (the balanced growth equilibrium is inefficient).} and whose existence might be expected to forestall the possibility of inefficiency?

That any such institution would have as its essential function the provision of assets to be held by consumers in lieu of capital goods should be apparent after a moment’s reflection on the relationship of Equation 46. The logical candidate is thus an intermediary sector, holding as assets capital goods
while issuing as liabilities various types of financial instruments. We deduce immediately, however, that in an inefficient economy private ownership of the intermediary sector and an efficient balanced growth equilibrium are mutually exclusive. This conclusion follows from the fact that the only efficient balanced growth path consistent with a competitive consumer sector is the golden rule path (that is, the rate of interest \( r = n \) is the only efficient solution to Equation 11 for an inefficient economy). Hence, assuming for simplicity that consumers hold only financial assets, if an intermediary sector were to provide the real assets consumers desire on this balanced growth path \( L(t)a^* \), then it would have negative and, indeed, continually decreasing net worth \( L(t)(k^* - a^*) < 0 \).

But, public ownership of the intermediary sector—for instance in the form of a social saving system or, like in Diamond’s model [3], a fiscal authority, or, as suggested by Samuelson [10], a monetary authority—is certainly possible. And, in contrast to the conclusion of the preceding paragraph, given proper policies a public intermediary sector could ensure an efficient balanced growth equilibrium precisely because it need not back its liabilities with specific assets—but rather with the general fiscal and monetary authority vested in government—and can therefore supply assets to the consumer sector somewhat independently of the existing capital stock.

In part to exemplify the last statement, we take up Samuelson’s suggestion and present a rigorous treatment of the effects stemming from the existence of a monetary system.

7. A Monetary Authority

Suppose that we introduce into our economy a monetary authority that has nominal liabilities equal to the nominal money supply \( M(t) > 0 \) and that pays a nominal rate of interest \(-\infty < \rho(t) < \infty\). Then the basic model of section 2 must be modified accordingly. First, there is an additional growth equation for the money supply per capita \( m(t) \),

\[
\frac{m(t)}{m(t)} = \rho(t) - n.
\]  \hspace{1cm} (50)

\footnote{And also somewhat reluctantly. Notice, especially, that this conclusion is independent of the type or behavior of private intermediaries postulated. In an earlier stage of this research we thought that a sector of private intermediaries with existence in their own right (that is, owning capital goods purchased from internal funds) would set matters right. Though this conjecture proved wrong, such a private intermediary sector does generate effects of some interest, and therefore will be detailed in section 8.}

\footnote{This is the simplest monetary policy to analyze in the framework of our model. However, essentially the same results would carry over for any monetary policy that is perfectly foreseen by all individuals, as it is the mere existence of a monetary system that is critical here.}
Second, the money price level \( p(t) \), that is, the price of output in terms of money, must be accounted for. Third, individual holdings of real assets may now be composed of both real money balances \( M(t, v)/p(t) \), and capital goods \( K(t, v) \),

\[
A(t, v) = \frac{M(t, v)}{p(t)} + K(t, v).
\]

(51)

Fourth, the equilibrium condition of Equation 8 becomes one requiring equality of real assets per capita to the sum of the real money supply per capita and the capital stock per head

\[
a(t) = \frac{m(t)}{p(t)} + k(t).
\]

(8')

And finally, observing that individuals will hold capital goods if and only if the rate of interest is at least as large as the real rate of return on money, while they will hold real money balances—say, money will matter—if and only if the real rate of return on money is at least as large as the rate of interest, we deduce a second equilibrium condition requiring equality of these two rates:

\[
r(t) = \rho(t) - \frac{\dot{p}(t)}{p(t)}.
\]

(52)

Clearly Equation 9 is still a necessary condition for equilibrium. It should also be obvious that the interpretation accompanying the model just outlined only makes sense\textsuperscript{24} when

\[
0 < \frac{m(t)}{p(t)} = a(t) - k(t) < \infty.
\]

(53)

This enables a simple characterization of balanced growth equilibrium in which \( 0 < m/p < \infty \) or money matters. Recall from the analysis in section 3 that \( r = n \) is the only solution to Equation 11 for which possibly \( a - k \neq 0 \), while \( a^* - k^* > 0 \) if and only if \( \psi'(n) < \phi'(n) \). We can conclude immediately that (1) for the inefficient economy with money there is a unique balanced growth equilibrium at the golden rule while (2) for the efficient economy with money (that is, an economy with money in which \( \psi'(n) \geq \phi'(n) \)) there is no balanced growth equilibrium.

The latter results are somewhat misleading, as, in addition to the possibility that there is no equilibrium in which money matters, there is also the possibility that there are equilibria in which money matters in the short run but not in the long run. This observation is borne out by further analysis based on the example of section 4: For the production function of Equation 17 with

\textsuperscript{24} But see section 8 for an alternative interpretation that only makes sense when, in effect, \(-\infty < m(t)/p(t) < 0 \).
INDIVIDUAL SAVING

\[ -1/B \neq g(n) - g(\delta), \] the equilibrium condition of Equation 8 can be reduced to

\[ x(t) = \int_{-\infty}^{\infty} I(t - s)x(s) \, ds, \quad (54) \]

where now

\[ x(t) = k(t) \exp \left\{ -\int_0^t [r(x) - n] \, dx \right\} + \frac{m(0)/p(0)}{1 - \int_{-1}^{t} I(s) \, ds} \quad (55) \]

and

\[ I(t - s) = G(t - s)e^{\alpha(t - s)}. \quad (56) \]

Detailed analysis of Equation 54, omitted here because it is basically similar to the earlier analysis of Equation 19, reveals that the nature of its real, positive solutions depends on whether the economy is efficient or inefficient. On the one hand, if \(-1/B > g(n) - g(\delta)\) (the efficient economy), then in fact there are no solutions with \(m(0)/p(0) > 0\). On the other hand, if \(-1/B < g(n) - g(\delta)\) (the inefficient economy), then all solutions with \(m(0)/p(0) > 0\), after converting from present values of capital to current values of capital per head and introducing explicitly the real value of the money supply per capita, are of the form

\[ k(t) = [r(t) - \bar{r}]k(t) + \left( n - \bar{r} \right) \left[ 1 - \int_{-1}^{t} I(s) \, ds \right] \frac{m(t)}{p(t)} \quad k(0) > 0. \quad (57) \]

and

\[ \left[ \frac{m(t)}{p(t)} \right] = [r(t) - n] \left[ \frac{m(t)}{p(t)} \right], \quad \frac{m(0)}{p(0)} > 0. \quad (58) \]

The permissible solutions to this pair of differential equations are depicted in the phase diagram in Figure 5, which shows up clearly that equilibria abound in which money does not matter in the long run (for example, the paths marked I and II are such equilibria).

Even though money may not matter, perhaps the most interesting property of the economy with money is that if money does matter in the long run, then the equilibrium is efficient. More formally, if

\[ \lim_{t \to \infty} \sup \frac{m(t)}{p(t)} > 0, \quad (59) \]

then the equilibrium is efficient and, therefore, by virtue of the theorem proved in section 5 (which, it is easy to see, also pertains here), Pareto optimal. We emphasize, in particular, that this property is essentially independent of the monetary policy pursued—in the present discussion it is the nominal rate of interest promised, but in general it could be any scheme
that systematically puts the monetary authority in debt to the public—provided that it is perfectly foreseen by all individuals.

To verify this property, we demonstrate in the appendix the following sufficient condition for efficiency in the neoclassical growth model: the feasible growth path $k(t)$ for $t \geq 0$ is efficient if there exists a finite number $M$ such that

$$\liminf_{t \to \infty} \exp \left\{ - \int_0^t [r(x) - n] \, dx \right\} < M. \quad (60)$$

As from Equations 50 and 52

$$\frac{m(t)}{p(t)} = \frac{m(0)}{p(0)} \exp \left\{ \int_0^t [r(x) - n] \, dx \right\}, \quad (61)$$

this condition is simply the asserted property.

To summarize the effects stemming from the existence of a monetary system: On the one hand, there may be no equilibrium in which individuals will desire to hold money balances. On the other hand, even when there is such an equilibrium, money may or may not matter in the long run. However, if money does matter in the long run, then the economy is assuredly efficient and Pareto optimal.
8. A Mixed Neoclassical-Marxian Model

One of the characteristics of neoclassical economics (in the broad sense of the term) is the notion that consumers are the sole source of independent decision making in the decentralized economy. Firms and financial institutions act as agents of their shareholders, who are consumers. At another extreme we find the Marxian view, which holds that in a private economy all the relevant decision making is concentrated in the hands of capitalists whose sole objective is accumulation, and consumption enters the picture only to the extent that the labor force must be kept at a subsistence level.

In the foregoing discussion we have seen that a purely consumer-oriented economy may be inefficient. Consumer decisions were not sufficient to guarantee an efficiently operating system. Indeed, sometimes in order to achieve efficiency we had to introduce into the model an institution that could not be thought of as a privately owned (that is, consumer-owned) firm whose actions represent the interest of its owners. In this section, we shall take a brief look at the role that a similar institution might play in the efficient economy. Our model will turn out to be a blend of the neoclassical (consumer-oriented) and the Marxian (accumulation-oriented) points of view.

Let us assume that individuals cannot hold capital goods as an instrument of saving but must, rather, save by holding corporate bonds. These bonds are issued by a multitude of competitive firms which, in turn, hold capital goods as assets. We now depart from the neoclassical tradition by assuming that the ownership of these firms is not located in the consumer sector. In other words, firms behave according to certain independent objectives that are not reducible to consumer decisions. In particular, we shall assume that firms act so as to maximize the rate of increase in net worth at every moment of time. This is indeed a Marxian postulate. Firms must of course repay their debt to consumers (at competitive interest rates), but with this repayment their commitment to the consumer sector ends.

Let \( B(t) \) be the total number of corporate bonds\( ^{25} \) outstanding at time \( t \), and let \( S(t) \) be the aggregate net worth of firms at time \( t \). Then, the consolidated balance sheet equation for all firms is

\[
K(t) = B(t) + S(t),
\]

which is equivalent to

\[
k(t) = b(t) + s(t),
\]

where \( b(t) = B(t)/L(t) \) and \( s(t) = S(t)/L(t) \). At time \( t \) firms hire labor and issue bonds so as to maximize the rate of increase of net worth. To find this

\( ^{25} \) By a "bond" we mean a debt instrument that is traded by whoever issues it for one unit of output and that is recontracted at every instant at the current rate of interest.
rate of increase, we must write down the profit-and-loss statement for the firms at time $t$ under the assumption that profits are never distributed:

$$\dot{s}(t) = L(t)f[k(t)] - L(t)w(t) - r(t)b(t),$$

which, we note for later reference, reduces immediately to

$$\dot{s}(t) = f[k(t)] - ns(t) - w(t) - r(t)b(t).$$

It is the quantity $\dot{s}(t)$ that the firms are assumed to maximize, given the wage and interest rates. This maximization leads, once again, to the conditions

$$f[k(t)] - k(t)f'[k(t)] = w(t)$$

and

$$f''[k(t)] = r(t).$$

The first of these equations may be regarded as a demand-for-labor equation and the second as a supply-of-bonds equation. These two equations may now be used in our equation for $\dot{s}(t)$ to obtain the following result:

$$\frac{\dot{s}(t)}{s(t)} = r(t) - n. \quad (63)$$

The dynamic behavior of this system is described by Equations 62 and 63, together with the equilibrium condition for the bond market, namely,

$$a(t) = b(t). \quad (64)$$

Again, Equations 62, 63, and 64 entail the equilibrium growth equation

$$\dot{k}(t) = f[k(t)] - nk(t) - c(t). \quad (9)$$

Let us now consider balanced growth equilibrium. Along a balanced growth path $k$ is constant by definition and $a$ is constant by virtue of the stationarity of consumer decisions. Thus, from Equation 64, we find $b$ must also be constant and, since $s = k - b$, we conclude that along a balanced growth path $s$ must be constant. It now follows from Equation 63 that $r = n$; that is, the only possible balanced growth path in our new model is the golden rule path.

Let $k^*$, $a^*$, $b^*$, and $s^*$ denote the values of $k$, $a$, $b$, and $s$, respectively, along the golden rule path. We have already seen (Equation 13) that

$$a^* = [f(k^*) - nk^*]\phi'(n),$$

and since $a^* = b^*$ and $s^* = k^* - b^*$, we get

$$s^* = k^* - [f(k^*) - nk^*]\phi'(n).$$
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Thus, using Equation 12b, we obtain

\[ s^* \geq 0 \quad \text{if and only if} \quad \psi(n) \geq \phi(n). \]

In other words, the statement \( s^* \geq 0 \) is equivalent to the statement that the economy is efficient. By the same token, \( s^* < 0 \) means that the economy is inefficient, so that in the inefficient case our firms reduce to the negative net worth intermediaries that were discussed in sections 6 and 7. Indeed, by identifying the quantity \( s(t) \) with the quantity \(-m(t)/p(t)\) of section 7, we see immediately that our present model and the model of section 7 are really the same, except that now we are concentrating on the efficient case whereas in section 7 we concentrated on the inefficient case.

We turn now to a brief comment about the actual equilibrium path along which the economy will travel, a path that will not, in general, be the balanced growth path. We shall restrict our attention to the efficient case, that is, to the case \( s^* \geq 0 \). The first point to notice is that if \( s(0) = 0 \), then it follows from Equation 63 that \( s(t) = 0 \) for all \( t \geq 0 \). In other words, if our firms start out with zero net worth, there is no way for them to get to a state of positive net worth (since payments to laborers and creditors always exhaust the firms' receipts). Thus, when \( s(0) = 0 \), we find ourselves back in the original model of section 2. However, if \( s(0) > 0 \), then it follows from Equation 60 that \( s(t) > 0 \) for all \( t \geq 0 \). In other words, if our firms start out with positive net worth, then they will continue to have positive net worth. Even so, it may happen that the firms' net worth will become relatively insignificant in the long run, that is, that \( \lim_{t \to \infty} s(t) = 0 \). By the concluding argument of the preceding section, if this is not the case, then equilibrium will be efficient and (again taking account of only consumers' welfare) Pareto optimal.

9. The End of the World

Many of the properties of the economic system that we have been discussing in this study depend upon the assumption that civilization will survive forever. This fact is duly emphasized, indeed sometimes overemphasized, by both Diamond and Samuelson.\(^{27}\) Before bringing our discussion to a close,

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\(^{26}\) Of course, in the inefficient case our firms can no longer be viewed as maximizing the rate of growth of net worth because by shutting down they can guarantee themselves a net worth of zero.

\(^{27}\) A viewpoint pursued more thoroughly in (2) There it is argued that a more pertinent difference between the present dynamic general equilibrium model and the standard static general equilibrium model is that in the former we are explicitly confronted with the fact that a central role of markets is to intermediate indirect trade between individuals when direct trade is not feasible and/or desirable (for example, by means of the use of money).
let us, therefore, indicate briefly how the grim prospect of the end of the world might be incorporated in the foregoing discussion. We confess that this remark will be made somewhat tongue in cheek.

The end of the world only matters in the foregoing discussion if it enters into the expectations of decision makers. Now, it seems to us somewhat extreme to assume that all decision makers expect with certainty that the world will come to an end at some definite time, say \( T \). It is more plausible that if the end of the world enters decision makers’ expectations at all, it enters in a probabilistic fashion. In other words, the date \( T \) on which the world will end is not a fixed number but a random variable with a subjective probability distribution. To make things simple, suppose that each individual’s subjective density function for this random variable (measured from his birth date) is the same and given by the exponential density with parameter \( \lambda \). The subjective probability for each individual of the event that his world will end when he is \( \tau \) years old is thus given by

\[
\pi(\tau) = \lambda e^{-\lambda \tau}, \quad 0 \leq \tau < 1, \\
= e^{-\lambda}, \quad \tau = 1.
\]  
(65)

From Equation 65 it follows that each consumer believes he will be alive at age \( \tau \) with probability

\[
1 - \Pi(\tau) = e^{-\lambda \tau}, \quad 0 \leq \tau < 1, \\
= 0, \quad \tau = 1.
\]  
(66)

Finally, suppose that individuals behave according to the expected utility hypothesis, which simply means that the lifetime allocation problem of section 2 now becomes, from Equation 66,

\[
\text{Maximize } \int_{v}^{u+1} [1 - \Pi(t - v)] \log C(t, v) e^{-\delta'(t-v)} dt \\
= \int_{v}^{u+1} \log C(t,v)e^{-\delta'(t-v)} dt \\
\text{subject to } \int_{v}^{u+1} [w(t) - C(t,v)] \exp \left[ - \int_{v}^{t} r(x) \, dx \right] dt = 0 \\
\text{and } C(t,v) \geq 0,
\]

where \( \delta' = \delta + \lambda \), that is, the consumer’s rate of time preference is the sum of his “pure” rate of time preference and the reciprocal of his expectation of the world’s end (measured from his birth date). It is now obvious that the stylized competitive economy in which people have probabilistic expectations
with regard to the world's end behaves just like the stylized competitive economy that is infinite with certainty, except that in the former consumers have a higher rate of time preference. Indeed, this may serve as one rationalization for the existence of a positive rate of time preference.

APPENDIX

Our purpose here is to develop the two conditions for recognizing (in)efficiency in the neoclassical growth model that were utilized in the text. As a preliminary we require two definitions.

**Definition 1.** Given a positive initial capital-labor ratio \( k^0 \), the growth path \( k(t) \) for \( t \geq 0 \) is feasible if and only if

\[
k(t) \geq 0
\]

and

\[
k(t) = f[k(t)] - nk(t) - c(t) \quad \text{with } k(0) = k^0
\]

for some nonnegative, piecewise continuous rate of consumption per capita \( c(t) \) for \( t \geq 0 \).

An immediate implication of this definition is that any feasible growth path \( k(t) \) for \( t \geq 0 \) satisfies the condition

\[
k(t) \leq \max (k^0, \hat{k}) = \bar{k},
\]

where the maximum sustainable long-run capital-labor ratio \( \bar{k} \) is given by the unique solution to the equation

\[
f(\bar{k}) - n\bar{k} = 0.
\]

**Definition 2.** The feasible growth path \( k^0(t) \) for \( t \geq 0 \) is inefficient if and only if there exists another feasible growth path \( k^2(t) \) for \( t \geq 0 \) such that

\[
c^2(t) \geq c^0(t) \quad \text{with strict inequality for some } t \geq 0.
\]

Likewise, the feasible growth path \( k^0(t) \) for \( t \geq 0 \) is efficient if and only if there does not exist such a feasible growth path \( k^2(t) \) for \( t \geq 0 \).

An immediate implication of the first definition for the second is that the strict inequality in Equation A.5 must hold over an open interval.

Armed with these definitions we proceed to restate and demonstrate the conditions in the order of their appearance in the text.
THEOREM 1. A feasible growth path \( k^o(t) \) for \( t \geq 0 \) is inefficient if and only if there exists another feasible growth path \( k^1(t) \) for \( t \geq 0 \) such that

\[
\liminf_{t \to \infty} \int_0^t \left[ c^i(s) - c^o(s) \right] \exp \left( - \int_0^s \left[ r^o(x) - n \right] dx \right) ds > 0. \quad (A.6)
\]

Proof (necessity): This is obvious from Definition 2 and the remark following it.

Proof (sufficiency): Assume that there is a feasible growth path \( k^i(t) \) for \( t \geq 0 \) satisfying this condition. Then there must be some finite time \( T \) such that

\[
\int_0^t \left[ c^i(s) - c^o(s) \right] \exp \left( - \int_0^s \left[ r^o(x) - n \right] dx \right) ds > 0 \quad (A.7)
\]

for all \( t \geq T \).

Substituting from Equation A.2 into Equation A.7 we derive

\[
\int_0^t \left( \{ f[k^i(s)] - nk^i(s) - k^1(s) \} - \{ f[k^o(s)] - nk^o(s) - k^o(s) \} \right)
\times \exp \left( - \int_0^s \left[ r^o(x) - n \right] dx \right) ds
\]

\[= - \int_0^t \{ f[k^o(s)] - f[k^i(s)] - f'(k^o(s))[k^o(s) - k^1(s)] \}
\times \exp \left( - \int_0^s \left[ r^o(x) - n \right] dx \right) ds \quad (A.8)
\]

\[+ \left[ k^o(t) - k^1(t) \right] \exp \left( - \int_0^t \left[ r^o(x) - n \right] dx \right) > 0
\]

for all \( t \geq T \).

Rearranging the latter inequality in Equation A.8 we find

\[
k^o(t) - k^1(t) > \int_0^t \left( \{ f[k^o(s)] - f[k^i(s)] - f'(k^o(s))[k^o(s) - k^1(s)] \}
\times \exp \left( - \int_t^s \left[ r^o(x) - n \right] dx \right) ds > 0 \quad \text{for all } t \geq T,
\]

where the second inequality holds because both \( k^o(t) \) and \( k^1(t) \) are continuous, while \( f \) is strictly concave. We shall utilize Equation A.9 to construct a growth path that is feasible and satisfies the condition of Equation A.5.
First, let
\[
\varepsilon(t) = \int_0^t \left\{ f[k'(s)] - f[k^3(s)] - f'(k^0(s))[k^0(s) - k^3(s)] \right\}
\times \exp \left\{ - \int_t^s \left[ r^0(x) - n \right] dx \right\} ds \quad \text{for all } t \geq T,
\] (A.10)
which by virtue of Equation A.9 satisfies the inequalities
\[
0 < \varepsilon(t) < k^0(t) - k^3(t) < k^0(t).
\] (A.11)
Second, let
\[
z(t) = f[z(t)] - nz(t) - [c^0(t) + c] \quad \text{with } z(T) = k(T)
\] for \( T \leq t \leq T' \) (A.12)
where \( c \) is an arbitrary positive number and \( T' \) is the first point \( t > T \) for which \( z(t) = k^0(t) - \varepsilon(t) \), or, if there is no such point, \( \infty \). Finally, let
\[
k^2(t) = k^0(t), \quad 0 \leq t \leq T,
\]
\[
= z(t), \quad T \leq t \leq T',
\] (A.13)
\[
= k^2(t) - \varepsilon(t), \quad T' \leq t.
\]
It is easily seen from Equations A.10 through A.12 that the growth path of Equation A.13 is feasible and satisfies the condition of Equation A.5 provided
\[
c^3(t) \geq c^0(t) \quad \text{for } t \geq T'.
\] (A.14)
To verify this inequality we simply write out \( c^3(t) \) for \( t \geq T' \):
\[
c^3(t) = f[k^3(t)] - nk^3(t) - k^3(t),
\]
\[
= \left\{ f[k^0(t)] - nk^0(t) - k^3(t) \right\}
\]
\[
+ \left\{ f[k^2(t)] - f[k^3(t)] - f'[k^0(t)][k^2(t) - k^3(t)] \right\},
\] (A.15)
\[
> c^0(t) + \left\{ f[k^0(t)] - f[k^3(t)] - f'[k^0(t)][k^2(t) - k^3(t)] \right\},
\]
\[
> c^0(t).
\]

**Theorem 2.** The feasible growth path \( k^0(t) \) for \( t \geq 0 \) is efficient if there exists a finite number \( M \) such that
\[
\lim \inf_{t \to \infty} \exp \left\{ - \int_0^t \left[ r^0(x) - n \right] dx \right\} < M.
\]
Proof: We shall show that for any feasible growth path $k^1(t)$ for $t \geq 0$

$$\liminf_{t \to \infty} \int_0^t [c^2(s) - c^0(s)] \exp \left\{ - \int_0^s [r^0(x) - n] \, dx \right\} \, ds \leq 0, \quad (A.16)$$

which by Theorem 1 implies that the growth path $k^0(t)$ for $t \geq 0$ is efficient.

To begin with, we know from the derivation of Equation A.8 that

$$\int_0^t [c^2(s) - c^0(s)] \exp \left\{ - \int_0^s [r^0(x) - n] \, dx \right\} \, ds$$

$$= - \int_0^t \left\{ f[k^0(s)] - f[k^1(s)] - f'(k^0(s))[k^0(s) - k^1(s)] \right\} \, ds$$

$$\times \exp \left\{ - \int_0^s [r^0(x) - n] \, dx \right\} \, ds + [k^0(t) - k^1(t)]$$

$$\times \exp \left\{ - \int_0^s [r^0(x) - n] \, dx \right\} \quad \text{for all } t \geq 0. \quad (A.17)$$

In order to evaluate the limit inferior of this expression, consider the sequence of points

$$t_0 = \min \left\{ t : t \geq 0, \exp \left\{ - \int_0^t [r^0(x) - n] \, dx \right\} \leq M \right\}$$

$$t_1 = \min \left\{ t : t \geq t_0 + 1, \exp \left\{ - \int_0^t [r^0(x) - n] \, dx \right\} \leq M \right\} \quad (A.18)$$

$$\vdots$$

$$t_i = \min \left\{ t : t \geq t_{i-1} + 1, \exp \left\{ - \int_0^t [r^0(x) - n] \, dx \right\} \leq M \right\}$$

$$\vdots$$

whose existence is guaranteed by hypothesis. For each point $t_i$ and some positive number $1 \geq \delta > 0$, define the closed interval

$$T_i = [t_i, t_i + \delta],$$

and for each interval $T_i$ define the lower bound

$$e_i = \min_{t \in T_i} \left\{ [k^0(t) - k^1(t)] \exp \left\{ - \int_0^t [r^0(x) - n] \, dx \right\} \right\} \quad (A.19)$$

Now because the function $f$ is concave, the first term in Equation A.17 must be nonpositive. Thus, on the one hand, if $\limsup_{t \to \infty} e_i \leq 0$, then the limit inferior of Equation A.17 is clearly nonpositive. On the other hand, suppose that $\limsup_{t \to \infty} e_i > 0$, so that there exists a subsequence $\{t_n\}$ and some positive number $\varepsilon > 0$ such that

$$0 < \varepsilon \leq [k^0(t) - k^1(t)] \exp \left\{ - \int_0^t [r^0(x) - n] \, dx \right\} \quad \text{for } t \in T_i. \quad (A.20)$$
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Without any loss of generality we can assume that the subsequence \( \{t_{n_i}\} \) is in fact the original sequence \( \{t_i\} \). On the basis of Equation A.21 we shall construct an upper bound to Equation A.17 for \( t = t_i \) that is strictly negative for \( i \) sufficiently large.

We know (by hypothesis and the definition in Equation A.19) that

\[
\exp \left( -\int_0^t [r^0(x) - n] \, dx \right) < M e^{n_0} \quad \text{for } t \in T_i \tag{A.22}
\]

and (by the feasibility condition of Equation A.1 and remark in Equation A.3) that

\[
[k^0(t) - k^1(t)] < \bar{K} \quad \text{for all } t \geq 0. \tag{A.23}
\]

By combining Equations A.21 through A.23 we can deduce further that

\[
\exp \left( -\int_0^t [r^0(x) - n] \, dx \right) > \frac{\varepsilon}{\bar{K}} \quad \text{for } t \in T_i \tag{A.24}
\]

and

\[
[k^0(t) - k^1(t)] > \frac{\varepsilon}{M e^{n_0}} \quad \text{for } t \in T_i. \tag{A.25}
\]

Hence, substituting from Equations A.22 through A.25 into Equation A.17, we can derive the following inequalities:

\[
\int_0^t [c^1(s) - c^0(s)] \exp \left( -\int_0^s [r^0(x) - n] \, dx \right) \, ds
\]

\[
< -\varepsilon \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left\{ f[k^0(s)] - f[k^1(s)] - f'[k^0(s)][k^0(s) - k^1(s)] \right\}
\]

\[
\times \exp \left( -\int_0^s [r^0(x) - n] \, dx \right) \, ds + \bar{K} M e^{n_0}
\]

\[
< -\frac{\varepsilon}{\bar{K}} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left\{ f[k^0(s)] - f \left[ k^0(s) - \frac{\varepsilon}{M e^{n_0}} \right] \right. \right.
\]

\[
- f'[k^0(s)] \frac{\varepsilon}{M e^{n_0}} \left. \right] \, ds + \bar{K} M e^{n_0}
\]

\[
\leq \left( \frac{\varepsilon \bar{K}}{\bar{K}} \left[ f(k) - f \left( k - \frac{\varepsilon}{M e^{n_0}} \right) - f'(k) \frac{\varepsilon}{M e^{n_0}} \right] \right) \tag{A.26}
\]

where \( k \) is the solution to the minimization problem

\[
\text{Minimize } \left[ f(k) - f \left( k - \frac{\varepsilon}{M e^{n_0}} \right) - f'(k) \frac{\varepsilon}{M e^{n_0}} \right].
\]

But because the function \( f \) is strictly concave, the value of the solution to this last problem is positive and the limit of the last expression in Equation A.26 is minus infinity.
References