A RE-EXAMINATION OF THE PURE CONSUMPTION LOANS MODEL

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I. INTRODUCTION

Professor Samuelson's article (1958) on an exact consumption-loan model of interest led to an interesting controversy. At issue were the determination and properties of interest rates in a dynamic economic system with no capital. One might have thought that these exchanges, between Samuelson (1959) and Lerner (1959) on the one hand, and between Samuelson (1960) and Meckling (1960) on the other, would result, if not in a complete resolution of the disagreements, at least in the emergence of a clear picture of what the issues are and how they might be treated. Unfortunately, such was not the case. The Samuelson-Lerner and the Samuelson-Meckling dialogues leave the reader rather perplexed, as though he had just watched a New Wave film—executed with brilliance, enjoyable while in progress, but not quite clear as to what is happening and never giving one a sense of resolution.

The 1960's have brought an upsurge of interest in capital theory and, more generally, in questions of allocation over time. Now Samuelson's model, even though it has no capital, is of interest to capital theorists because it has many of the features of a model of capital accumulation with decentralized decision making. Since the 1958–60 discussions of this model by Samuelson, Lerner, and Meckling left some questions unanswered, we feel that a restudy of the model is in order. The following is an attempt at such a restudy.

We shall take the liberty of deviating somewhat from Samuelson's original notation.

II. THE MODEL

We shall concentrate throughout most of our discussion on the simpler of Samuelson's two models, namely, the one in which people live for two periods, earning a fixed income in the first and earning nothing in the second. In the first period of his lifetime, a person earns one unit of output, where "output" is something usable directly (and exclusively) in consumption, and we do not inquire wherefrom it comes.

The generation which is born at time $t$ will be called generation $t$, and we shall assume that there are $(1 + n)^t$ people in it. Thus, $n$ is the (relative) rate of growth of population, which is assumed constant. Members of generation $t$ are thought of as being alike in all respects, so one can speak of a member of generation $t$ without specifying the individual. Let the symbol $C^1_t$ stand for the first-period consumption of a member of generation $t$, and let $C^2_t$ stand for his second-period consumption. A member of generation $t$ is assumed to value any given consumption plan $(C^1_t, C^2_t)$ according to the value, $U(C^1_t, C^2_t)$, taken on by a "regularly shaped" utility function $U$ at $(C^1_t, C^2_t)$. The utility function $U$ is assumed to be the same for all generations.

Continuing in Samuelson's footsteps,
we now proceed to assume that output is non-durable and thus cannot be carried over from one period to the next. This assumption reduces the production possibilities in the model (that is, the possibilities of using output in one period in the production of output in another period) to naught. It is clear, furthermore, that the assumptions made so far are so restrictive as to rule out from the outset any possibility of trades, markets, or prices. A member of generation \( t \) who wishes to engage in a transaction cannot find anyone willing and able to participate in the transaction on the opposite side.

Given that production and trade have both been dispensed with, there remains only one other economic activity to be considered—distribution. This function is still open in our economy, for output can be taken from the young who earn it and given to the old who do not. Thus, our first task will be the examination of alternative distribution schemes. However, before proceeding with this examination, we must define the notion of “the rate of interest.” Writing \( r_t \) for the rate of interest in period \( t \), we define:

\[
\frac{C_{t-1} + C_{t-1}}{1 - C_{t-1}}
\]

or

\[
1 + r_t = \frac{C_{t-1}}{1 - C_{t-1}}.
\]

Two things should be noted: (a) \( 1 + r_t \) is not a price; that is, no transactions are ever held using it as a rate of exchange. It is not even an “implicit price,” in the sense of a price which emerges as a by-product of efficient allocation of resources. It is, rather, an ex post rate of exchange which is inferred from observation of the consumption pattern of a member of generation \( t - 1 \), and it has reference neither to trade nor to efficiency.1 (b) When \( C_{t-1} = 1, r_t \) is clearly not defined. If \( C_{t-1} = 1 \) and \( C_{t-1}^2 < 0 \), we shall say that \( r_t \) can be any real number, and if \( C_{t-1} = 1 \) and \( C_{t-1}^2 > 0 \), we shall say that \( r_t = +\infty \).

### III. DISTRIBUTION OF OUTPUT

As has already been remarked, the only economic function remaining in the model of the foregoing section is that of distribution of output. A pair of sequences,

\[
\{C_t^1, t = 0, \pm 1, \pm 2, \ldots \},
\]

\[
\{C_t^2, t = 0, \pm 1, \pm 2, \ldots \},
\]

with non-negative elements will be referred to as a distribution scheme. It specifies how much each member of any given generation shall consume in each period of his lifetime. A distribution scheme will be called feasible if it does not use up more output than is available in any period. In other words, the scheme \((\{C_t^1\}, \{C_t^2\})\) is feasible if and only if

\[
(1 + n)^t C_t^1 + (1 + n)^{t-1} C_{t-1}^2 \leq (1 + n)^t
\]

or

\[
C_t^1 + \frac{C_{t-1}^2}{1 + n} \leq 1
\]

for all \( t \), where the inequality arises from an assumption of free disposal.

Suppose that in period \( t \) no disposal takes place. Then,

\[
C_t^1 + \frac{C_{t-1}^2}{1 + n} = 1.
\]

On the other hand, we have, by the definition of the rate of interest, that

\[
C_{t-1} + \frac{C_{t-1}}{1 + r_t} = 1.
\]

1 Note, however, that if a member of generation \( t - 1 \) were to maximize utility subject to a given \( r_t \), then \( 1 + r_t \) would, as usual, equal the marginal rate of substitution of second-period consumption for first-period consumption.
Subtracting the latter from the former, we get

\[ C_i - C_{i-1} + \frac{r_i - n}{(1 + r_i)(1 + n)} C_{i-1} = 0. \]

As an immediate consequence one now obtains that

if \( C_{i-1} = C_i \neq 1 \), then \( r_i = n \).

If \( C_{i-1} = C_i = 1 \), then \( r_i \) can be any real number, and we might as well agree once again that \( r_i = n \).

This equality of \( r_i \) and \( n \) in case \( C_{i-1} = C_i \) is the manifestation in the present framework of what Samuelson (1958) calls “the biological rate of interest,” which he finds “paradoxical,” even “astonishing” (pp. 471 and 473, respectively) and which Meckling finds very hard to accept. The main cause for suspicion seems to be the fact that a rate of interest has been determined without any reference to impatience and time preference or, more generally, to the utility function \( U \). Somehow, the fact that \( r_i \) is a completely mechanistic construct, having no reference to markets, seems to have become blurred.

**IV. STATIONARY SCHEMES**

In order to analyze the relation between the rate of interest and the rate of growth of population somewhat further, let us define a *stationary* distribution scheme by the requirement

\[
C_i = C_i^1 \\
C_i^2 = C_i^2
\]

for all \( i \),

which implies that

\[ r_i = \frac{C_i^1 + C_i^2 - 1}{1 - C_i^1} \quad \text{for all } i. \]

We are assuming for the moment that \( C_i^1 < 1 \), so \( r_i \) is well defined. Now let us check the feasibility of a stationary distribution scheme. By direct substitution, we find that

\[ C_i^1 + \frac{C_i^2}{1 + n} \leq 1 \quad \text{if and only if} \quad r_i \leq n. \]

Thus, the feasibility of a stationary distribution scheme is equivalent to the statement that the rate of interest is no greater than the rate of growth of population. Furthermore, the same algebraic operation which yielded the equivalence of the above inequalities also yields the equivalence of the strict inequality \( r < n \) and the strict inequality \( C_i^1 + C_i^2/(1 + n) < 1 \). With a stationary distribution scheme, a rate of interest which falls short of the rate of growth of population means that some output is being discarded in every period. In other words, the inequality \( r < n \) means that the distribution scheme under consideration is inefficient (unless consumers are satiated). This result has a familiar ring to it. In models where investment and capital accumulation are possible, we often find that, among all feasible stationary paths, the path which maximizes per capita consumption is the so-called golden-rule path, which is characterized, among other things, by the equality of the rate of interest and the rate of growth of population. Indeed, we know that a stationary path along which the (constant) rate of interest is lower than the rate of growth of population is in fact inefficient in the sense that everybody’s consumption can be increased (see, for example, Phelps, 1965).

Let us recapitulate: Every stationary distribution scheme is characterized by a pair of non-negative real numbers, \( C^1 \) and \( C^2 \). The set of all feasible schemes is represented by the shaded area in Figure 1, and it corresponds precisely to the set of all schemes with a rate of interest which is no greater than the rate of growth of population. Among the latter,
only the schemes that are represented by points on the northeastern boundary line of the shaded area are efficient. These efficient schemes are precisely those for which the rate of interest is, in fact, equal to the rate of growth of population.

Note that Figure 1 actually contains part of Samuelson’s (1959, p. 519) Figure 1 in his reply to Lerner. In that figure, Samuelson marks two specific schemes, represented by the points $S$ and $L$, which he labels the “Samuelson plan” and the interest can be taken to be any real number, including zero (and also including $n$).

Let us turn now to the question of choosing between alternative distribution schemes. If we take all the feasible distribution schemes and ask a member of generation $t$ which of these he prefers, he will no doubt say that the schemes which satisfy the requirement

$$C_t^* = 1, \quad C_{t+1}^* = 1 + n,$$

are highest on his preference scale. These are the schemes which give generation $t$ all the output in period $t$, as well as all the output in period $t + 1$ (under the assumption that within each generation all share alike). He will be indifferent as to which of the schemes satisfying this requirement is picked. It is obvious, therefore, that no single distribution scheme will be preferred to all other feasible schemes by all individuals. In a situation of this sort, the economist usually resorts to one of two things: Either he defines an over-all social welfare function and picks the feasible distribution scheme which maximizes it, or he restricts the choice of a scheme to a subclass of the original class of all feasible schemes, a subclass such that individual maximization over it will result in compatible choices. The subclass of all stationary distribution schemes clearly has this property, since stationarity means that everybody has the same lifetime consumption profile. Thus, if we agree to restrict the search for a distribution scheme to the class of all stationary schemes (and this agreement is extraneous to the analysis, just as the

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**Fig. 1**

"Lerner plan," respectively. Now both $S$ and $L$ are in fact on the efficient frontier of the shaded set and therefore both represent distribution schemes for which the rate of interest is equal to $n$. The discussions by both Samuelson and Lerner, in which the point $L$ is referred to as corresponding to a zero rate of interest, therefore seem to be in error. The only point on the efficient frontier which could possibly be taken to represent a distribution scheme with zero interest is the point $(1, 0)$, where the rate of

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$^2$Samuelson and Lerner both refer to the case where $n = 1$ and $L$ is the point $(3, 3)$. Clearly, at that point every person foregoes $\frac{1}{2}$ units of output in the first period and receives $\frac{1}{2}$ in the second period, which corresponds to a rate of interest of 100 percent.

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$^3$It seems that Lerner’s concern for the equality of income distribution ought to lead him to stationarity (everybody getting the same consumption profile) and not to equality of consumption for all within each time period (that is, $C_t^* = C_t$), which he seems to advocate in the above-cited references.
choice of a social welfare function would be) we can find one distribution which maximizes everybody’s utility. We write

$$\max U(C^t, C^s)$$

subject to the constraint that the stationary scheme be feasible, that is, subject to

$$C^t + \frac{C^s}{1+n} \leq 1.$$  

This is indeed Samuelson’s maximization problem (leading to the point S in his diagram as the solution) but stated in terms of choice among distribution schemes rather than in terms of the opportunities open to a “representative man.” It should be stressed again that all the efficient points in the set over which the maximization takes place are points which correspond to a rate of interest equal to $n$ (except the point $(1, 0)$, where the rate of interest is indeterminate).

Before leaving this part of the discussion, let us consider the problem of decentralization. It is clear that the distribution schemes discussed above (whether stationary or not) are not, in general, attainable by having each individual act on his own in a decentralized fashion. Indeed, the only distribution schemes which are attainable with each individual acting on his own through the (inactive) market are distribution schemes for which

$$\begin{align*}
C_t^t &\leq 1, \\
C_t^s & = 0
\end{align*}$$  

for all $t$.

Among these, the only efficient scheme is given by $C_t^t = 1$ and $C_t^s = 0$ for all $t$, which also happens to be a stationary scheme. However, this is obviously not (in general) the scheme that individuals would pick, among all stationary schemes, if they had the choice. Thus, as Samuelson points out, we have here an example in which decentralized (competitive) behavior fails to lead to an optimum. This conclusion can be sharpened considerably if one drops the assumption that output is non-durable (see Sec. VI below).

V. CONSTANT RATE OF INTEREST PATHS

In the foregoing section we have seen that the rate of interest is constant along every stationary path. We now ask whether every efficient path along which the rate of interest is constant is, in fact, stationary.

Constancy of the rate of interest means that there exists a real number $r$ such that

$$C_t^t + \frac{C_t^s}{1+r} = 1 \text{ for all } t.$$  

Now, feasibility and efficiency of the distribution scheme mean that

$$C_t^t + \frac{C_{t-1}^s}{1+n} = 1 \text{ for all } t.$$  

Subtracting the former from the latter and rearranging leads to

$$\frac{C_t^s}{C_{t-1}^s} = \frac{1+r}{1+n}.$$  

Since $C_t^s$ is bounded by

$$C_t^s \leq 1+n \text{ for all } t,$$

we find that the only rate of interest which can be constant for all $t = 0,1,2,\ldots$, is the rate $r = n$, which, indeed, corresponds to a stationary scheme.

However, if we only require the rate of interest to be constant from some point on, say $t = 0,1,2,\ldots$, then we find that $r \leq n$ is possible, while $r > n$
is not. If \( r < n \) then as \( t \to \infty \), \( C_t^2 \) tends to 0, and therefore \( C_t^1 \) tends to 1. In fact, by a straightforward calculation one can show that if there exists an \( \epsilon > 0 \) such that \( r \leq n - \epsilon \) for \( t = 0, 1, 2, \ldots \), then
\[
\lim_{t \to \infty} C_t^2 = 0.
\]

As a particular instance, this discussion applies to paths along which the rate of interest is always zero, which is what Lerner seems to advocate.

Suppose that with this new regime we now require our economy to proceed in an entirely decentralized fashion, each individual acting on his own. The opportunity set available to each individual is now given by the triangle \( OAB \) in Figure 2. (Recall that, when output was non-durable, the individual’s opportunity set consisted only of the line segment \( OA \).)

An individual of generation \( t \) will therefore pick his consumption plan \( (C_t^1, C_t^2) \) so as to maximize \( U(C_t^1, C_t^2) \) subject to \( C_t^1 + C_t^2 = 1 \), which will lead him to a point, say \( D \), on the boundary of his opportunity set. Clearly, \( D \) will not in general coincide with \( A \); that is, the individual will in general choose to consume positive amounts in both periods. Note also that the point \( D \) will be optimal for individuals of all generations (since we are assuming a common utility function for all) so that decentralized behavior leads, once again, to a stationary path.

Now let us look at distribution schemes which are not necessarily decentralized. Obviously, any scheme which was feasible under the assumption that output is non-durable is also feasible under the assumption that output is completely durable. Hence, the set of feasible distribution schemes in our new regime certainly contains the feasible set of the old regime. In particular, the set of all feasible stationary schemes must contain the shaded set in Figure 1, which appears in Figure 2 as the triangle \( OAC \).

A brief look at Figure 2 tells the story: While in the old regime decentralized behavior led to an efficient but (in general) non-optimal distribution scheme, in the new regime decentralization leads (in general) to an inefficient (to say nothing of optimality) distribution scheme.

This phenomenon, the inefficiency of

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4 Similarly, if the rate of interest is required to be constant up to some point, then \( r \geq n \) is possible, but \( r < n \) is not.

4 Actually, the two sets coincide.
decentralized behavior, does not disappear if we drop yet another assumption and permit capital (which in this model is in the form of inert inventories) to become productive. This has been shown recently by Diamond (1965). In Diamond’s model, the inefficiency appears only if people “want to save too much” (in some well-defined sense). In the present model, people in the decentralized scheme always want to save too much, in the sense that the efficient distribution schemes involve zero inventories at every moment of time, while the decentralized scheme (in general) involves positive inventories at every moment of time. (In Diamond’s model this is not always the case because, roughly speaking, as long as inventories reproduce faster than people, it is efficient to hold them.)

VII. FINANCIAL INTERMEDIATION

Thus far, we have avoided the question of how an efficient distribution scheme might be brought about. Is there an economic agent that could be introduced into the model and whose activity would ensure efficiency? Before attempting to answer this question we must probe a little further into the nature of the inefficiency of decentralization in the model of the foregoing section.

At the heart of the inefficiency in our new model (with durable output) lies the fact that decentralization forces people to hoard output in their first period so as to be able to consume what they had hoarded in the second period. The result is that in every period a fraction of total output is put aside in the form of a savings fund, to be carried over to the next period. But when the next period comes around and the older generation consumes its savings, the younger generation establishes its own savings fund and the economy ends up carrying a load of dead weight, in the form of output which is never consumed. Indeed, under the assumption of the same utility function for all, this load of dead weight keeps growing like a geometric progression, because the savings fund of generation $t$ is $1 + n$ the size of the savings fund of generation $t - 1$.

The only way to restore efficiency to our system is to find an arrangement whereby the savings of generation $t$ (when it is young) are used to provide for the consumption of generation $t - 1$ (when it is old). In this way, the situation in which positive amounts of real output are always being carried over, from period to period, will not arise. But this cannot be done in a decentralized manner within the present framework of assumptions. The first thing which comes to mind at this point is that maybe the trouble lies in the lack of sufficient overlap between generations, which prevents the right trades from being made.

In order to investigate this possibility, we introduce into the picture a financial intermediary of some sort (say, a banking system or a system of pension funds) which is assumed to exist concurrently with all generations. People are now able to save by holding the liabilities of this financial intermediary, which they will in fact prefer to do if by holding such liabilities for one period they can earn a positive rate of interest. Now, the intermediary can, in fact, offer holders of its liabilities a rate of interest of $n$ per period simply by using the output deposited with it by generation $t$ to redeem the liabilities which are held by generation $t - 1$. The result is that all output available in period $t$ is in fact consumed.

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6 Note that the present model, with durable output, may be looked upon as a special case of Diamond’s model, with the intensive production function (that is, the function relating output per man to capital per man) identically equal to unity.
in period $t$, partly by generation $t$ and partly by generation $t - 1$. Generation $t - 1$ receives whatever generation $t$ decides to save as payment in full (principal as well as interest) of the debt incurred to it one period earlier by the financial intermediary. There will no longer be a carryover of output from period to period and, as a result, efficiency will be restored. Indeed, optimality will be restored as well, because in response to a rate of interest of $n$ per period people will choose to save exactly the amount which leads to the optimal distribution scheme.

The outlook seems rosy until one takes a brief look at the balance sheet of our financial intermediary, where things are rather unfortunate: The balance sheet as of the end of period $t$ shows zero assets and liabilities of $s(1 + n)^t$, where $s$ is the (stationary) saving ratio of people in their first period of life. This means that at the end of period $t$ the net worth of the intermediary is given by $-s(1 + n)^t$. Now by not doing anything (that is, by shutting down) the intermediary can guarantee itself a net worth of zero, and so one might argue that it will never choose to engage in the aforementioned transactions. On the other hand, it might be argued that the intermediary should be looked upon as a social security system which is not privately owned, so that its net worth position is of no concern (or should not be of any concern). This is what Lerner seems to be saying when he decries (1959, p. 517) “those of the accountants who insist on the ‘solvency’ of the Social Security Administration.” Later, Lerner seems to be taking a 180° turn by insisting on the very accounting practice which he had previously decried: “Business is fine,” said the optimistic contractor. “It is true that I lose money on every contract, but I always start a bigger one and get an advance that more than covers the loss on the old one.” (1959, p. 523).

Be that as it may, it is certainly the case that a privately owned financial intermediary will not rescue the economy from inefficiency. In other words, decentralization and competitive behavior still fall to result in efficient behavior.

VIII. LACK OF BORROWING AS A SOURCE OF INEFFICIENCY

Let $x_t$ be the amount of output carried over by society from period $t$ into period $t + 1$. If we restrict our attention to stationary states, then it is clear that a necessary and sufficient condition for efficiency is $x_t = 0$ for all $t$. However, if we wish to consider the non-stationary cases as well, the condition for efficiency becomes somewhat more complicated. For it is possible, for instance, for generation $t$ to underconsume and for generation $t + 1$ to overconsume, with the result that $x_t$ will be positive and yet the economy will remain efficient. More generally, the building up of inventories does not destroy efficiency so long as these inventories are eventually consumed. The exact statement in this respect is as follows: A necessary and sufficient condition for the economy to be efficient is that there exist a subsequence \( \{x_{t_k}\} \) of the sequence \( \{x_t\} \) such that

$$\lim_{k \to \infty} x_{t_k} = 0.$$  

The proof of this assertion will be given in the Appendix. Roughly speaking, efficiency requires that inventories return periodically to a level which is “practically zero.”

From the point of view of balance sheets, $x_t$ is clearly net assets (total assets minus total liabilities) in the economy at
the end of period \( t \). If we concentrate our attention at the time periods \( t_n \), we find that efficiency requires net assets at the end of these periods to be (practically) zero. In other words, in order to have efficiency, it must be the case that for each outstanding asset in the economy at the end of period \( t_n \) there exists a corresponding liability outstanding at the end of period \( t_n \). But now let us recall the time structure of our model. People live for two periods, they consume in both periods but earn income only in the first. This forces individuals to become net lenders (and never net borrowers) no matter what rate of interest prevails. In particular, members of generation \( t_n \) will want to be net lenders, that is, to have an end-of-period balance sheet which shows just assets and net worth and no liabilities. But efficiency requires that aggregate net worth in the economy be zero, so there must exist someone in the economy holding liabilities in excess of assets, that is, having a negative net worth at the end of period \( t_n \). This puts the unfavorable net worth position of the financial intermediary of Section VII in perspective. By the same token, the decentralized economy of Section VI is inefficient precisely because no one can be a net borrower while everyone wishes to be a net lender.

It is of some interest to investigate Samuelson's discussion concerning "the contrivance of money" in the light of the foregoing remarks. Clearly, if efficiency is to be attained, people must be dissuaded from holding output as a store of value and persuaded to hold another asset instead. If this other asset bears a positive rate of interest, then people will in fact make the desired shift. As Samuelson points out, the role of this other asset can very well be fulfilled by money. People would save by buying the existing money supply and disavow by selling it to next period's savers. On a stationary path, the price of money will rise by a factor of \( 1 + n \) in each period, which corresponds to a rate of interest of \( n \) per period, so individuals will in fact prefer holding money to holding output. Thus, at least with respect to stationary paths, the introduction of money leads to efficiency. (More precisely: Every stationary path in the money economy is efficient.) Samuelson's interpretation of this phenomenon is a philosophical one: An economy is inherently more than a mere mechanical system of particles in motion; it is, in fact, such a system plus something called a "Hobbes-Rousseau social contract" (Samuelson, 1958, p. 479). A physical system can operate efficiently without this added aspect, but a social system cannot. Now, it seems to us that the social contract is no more involved in Samuelson's money economy than it is in any other general equilibrium model. For this reason we feel that Samuelson's discussion in this area is liable to be misleading. In general equilibrium analysis one thinks of a single market convention in which prices are announced and economic agents determine the trades and the productive activities in which they wish to engage at these prices. If the totality of all trades clears all markets, then the announced prices are said to be equilibrium prices. The question which general equilibrium theory asks is the following: Under the

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7 The word "practically" is intended to convey the notion that, strictly speaking, net assets at the end of period \( t_n \) might be given by some \( \epsilon > 0 \), but by taking \( \epsilon \) large enough, this \( \epsilon \) may be taken as small as we wish. From here on, we shall neglect to remind the reader of this qualification.

8 As long as this rate of interest exceeds \(-1\).

9 It is not clear whether Samuelson intends his model to constitute a mathematical proof of the Locke-Hobbes-Rousseau thesis.
assumption that everybody at the market convention takes prices as given, is there a schedule of prices which leads to the clearing of all markets? In Samuelson's money economy, commodities are time-dated output and time-dated money, and all that one asks is whether or not a given price schedule is an equilibrium price schedule. It turns out that the schedule which sets the price of output in all periods equal to 1 and the price of money in period $t$ equal to $(1 + n)^t$ is, in fact, an equilibrium price schedule.

The element of public trust in the monetary unit is reflected by the fact that a person who buys money in period $t$, at a price of $(1 + n)^t$, assumes that he will be able to sell it in period $t + 1$, at a price of $(1 + n)^{t+1}$. But this is precisely what is meant in general equilibrium theory by the phrase "taking prices as given."

At the beginning of this section, we argued that, if efficiency is to be attained, someone will have to have a balance sheet showing liabilities in excess of assets. For this reason it seems appropriate to look upon money as a liability of a monetary authority that is committed to paying one dollar to whoever presents it with one dollar. The balance sheet of this monetary authority shows only liabilities and no assets, and the value of the authority's liabilities (quantity of money multiplied by the price of money) is precisely equal to the excess of assets over liabilities in the private sector. From this point of view, Samuelson's "contrivance of money" is, in essence, no different from the financial intermediary of Section VII.

We turn now to a brief investigation of a model in which people live for three periods, and in which efficiency can be achieved (under some circumstances) without introducing into the economy a sector with negative net worth.

IX. A THREE-PERIOD MODEL

We have argued that in order to have efficiency there must be someone in the economy who is, at least periodically, a net borrower. Along stationary paths, this must be the case in every period. However, there was nothing in that argument to suggest that the net borrower must be a net borrower throughout his (or its) lifetime. Indeed, this role may well change hands over time, which was not the case with the financial intermediary of Section VII.

The prime candidate for this state of temporary net borrowership is the consumer himself. To check on this possibility, let us consider the following modification of our model: Assume that people live for three periods rather than two. In the first period of life a person grows up and is educated and therefore earns nothing; in the second period he works and earns one unit of output; in the third he is retired. Every person will now be a net borrower at the end of his first period, a net lender at the end of his second period, and he will be neither a borrower nor a lender at the end of his last period. (We are not using the terms "negative net worth" and "positive net worth" because it is not clear that they are applicable in the present context.) We shall restrict our attention to stationary paths. Along a stationary path, everybody receives the same lifetime consumption profile, say $(C^1, C^2, C^3)$. Feasibility and efficiency now mean that the following equation holds:

$$(1 + n)C^1 + C^2 + \frac{C^3}{1 + n} = 1,$$

which is similar to the feasibility and efficiency equation in the two-period case and is derived in the same way. If we let the (constant) rate of interest along the
stationary path be denoted \( r \), then each person's budget constraint is given by

\[
(1 + r)C^1 + C^2 + \frac{C^3}{1 + r} = 1,
\]

which leads, once again, to \( r = \frac{n}{2} \). In other words, the only rate of interest which can be established along an efficient stationary path is \( n \). Each person is now viewed as maximizing the utility function \( U(C^1, C^2, C^3) \) subject to the budget constraint (with \( n \) replacing \( r \)). The maximization will result in an optimal consumption plan, say \((\bar{C}^1, \bar{C}^2, \bar{C}^3)\). This optimal triple must be examined to see if it can be brought about by purely competitive (decentralized) trades.

The optimal consumption plan \((\bar{C}^1, \bar{C}^2, \bar{C}^3)\) will be said to be competitively attainable if the following equation holds:

\[
\bar{C}^3 = (1 + n)^{\frac{1}{2}} \bar{C}^1.
\]

To see how this condition is obtained, consider a person of generation \( t \). When he is young, he borrows an amount \( \bar{C}^3 \) from members of generation \( t - 1 \), who are in their middle years. Next period, when he is middle-aged and generation \( t - 1 \) is retired, he returns the loan, plus interest. In other words, he returns the amount \( (1 + n)\bar{C}^1 \). Now aggregate borrowing by generation \( t \) (when young) is given by the amount \( (1 + n)^t \bar{C}^1 \), and therefore total payment by generation \( t \) (when middle-aged) to generation \( t - 1 \) (when retired) is given by \( (1 + n)^{t+1} \bar{C}^3 \). This quantity, divided by the size of generation \( t - 1 \), yields the per capita consumption of the retired, that is, \( \bar{C}^3 \). This is precisely what the foregoing equation says.

As an example, consider the utility function

\[
U(C^1, C^2, C^3) = \sum_{i=1}^{3} \log C^i.
\]

Maximizing it subject to the budget constraint leads to

\[
(1 + n)\bar{C}^1 = \bar{C}^3 = \frac{\bar{C}^3}{1 + n} = \frac{1}{2},
\]

which is clearly competitively attainable.

Why is the decentralized economy in this example efficient? Surely, it is possible to attribute this result to “the social contrivance of binding contracts.” It is clearly in the interest of the middle-aged to default and ignore the debt which they have incurred when young. Even if the rules are such that a person guilty of default is denied access to the capital market as a lender (and so must lose interest on his savings), it is still true in many cases that default will result in increased consumption in all periods. Here, once again, is an opportunity to appeal to the social contract and here, once again, it would seem to be beside the point, and for very much the same reasons as before: The assumption that contracts are not defaulted upon usually goes without saying in the theory of the competitive mechanism; it does not explain our result, it merely permits it.

But our example is a very lucky one. For it is not, in general, to be expected that the optimal solution of a consumer’s lifetime allocation problem will satisfy as stringent a condition as \( \bar{C}^3 = (1 + n)^{\frac{1}{2}} \bar{C}^1 \). In fact, with most utility functions this condition will not hold. If \( \bar{C}^3 > (1 + n)^{\frac{1}{2}} \bar{C}^1 \) then we shall, once again, have too little borrowing in the economy and an additional agent with negative net worth would be needed in order to attain efficiency. However if \( \bar{C}^3 < (1 + n)^{\frac{1}{2}} \bar{C}^1 \) then we shall have too much borrowing in the economy, and it will be possible to introduce a financial intermediary with positive net worth which will guarantee efficiency. The balance sheet of this intermediary will show I.O.U.’s of young people on the asset
side and nothing on the liabilities side. In order to see how this intermediary would operate, let us concentrate, once again, on period \( t \). In this period, a member of generation \( t \) offers to sell a quantity \( C^t \) of I.O.U.'s. The total supply of I.O.U.'s by generation \( t \) is therefore given by \((1 + n)^t C^t\). However, a member of generation \( t - 1 \) wishes to buy only \( C^{t-1}/(1 + n) \) in I.O.U.'s, so that the total quantity of I.O.U.'s demanded is \((1 + n)^{t-2} C^t\), which falls short of the quantity supplied. The intermediary now steps in and buys the excess supply, using as payment the resources which generation \( t - 1 \) pays in when it redeems its own I.O.U.'s, which it sells to the intermediary in the previous period. Stationarity insures that these resources will always be precisely adequate to buy the outstanding I.O.U.'s. The result will be that physical output will not be carried over from period to period, so that efficiency will be attained. The aggregate debt of the consumer sector to the financial intermediary will grow like a geometric progression, but each consumer by himself will be balancing expenditures and receipts over his lifetime; that is, at the end of his last period of life he will have zero net worth. Under these circumstances, the intermediary may very well be thought of as a privately owned, competitive institution.

X. Efficiency and Infinity

The possible inefficiency (or non-optimality) of the competitive mechanism, as demonstrated by Samuelson and Diamond, has given rise to a certain amount of speculation, mostly on an informal basis. Many people (including Samuelson [for example, 1958, p. 474; 1959, p. 522] and Diamond [1965, p. 1134]) seem to feel that this phenomenon has something to do with infinity. What apparently leads one to point an accusing finger at "infinity" is the fact that for the standard general equilibrium model (which is finite) we have theorems which tell us that the competitive mechanism always leads to an optimum (and, a fortiori, to efficiency). Nevertheless, the role played by "infinity" in leading the competitive mechanism astray has remained, at best, rather vague. In the present section, we wish to explore this question somewhat more systematically by trying to construct a finite model that resembles the infinite model of the foregoing discussion as closely as possible. As it turns out, inefficiency may well arise in such a finite model.

Consider an economy with \( m \) agents and \( m \) commodities (where \( m \geq 2 \)). Each agent is both a consumer and a producer. Let \( C^t_i \) be the amount of commodity \( j \) consumed by agent \( i \), and let \( Q^t_i \) be the amount of commodity \( j \) produced by agent \( i \). Agent \( i \) (for \( i = 1, 2, \ldots, m - 1 \)) is assumed to desire, for consumption, only two commodities: commodity \( i \) and commodity \( i + 1 \). (Agent \( m \) is assumed to desire commodity \( m \) and commodity 1.) Thus, agent \( i \)'s utility function is given by

\[
U_i = U(C^t_i, C^{t+i}_i) \quad i = 1, \ldots, m - 1, \\
U_m = U(C^m_1, C^{m+1}_m),
\]

where the function \( U \) is common to all. As for production possibilities, we assume that agent \( i \) can produce commodities \( i \) and \( i + 1 \) (with agent \( m \) producing commodities \( m \) and 1), but that he has a relative advantage in the production of commodity \( i \). More specifically, we shall assume that agent \( i \) can produce any combination of \( Q^t_i \) and \( Q^{t+i}_i \) satisfying

\[
Q^t_i \geq 0, \quad Q^{t+i}_i \geq 0, \quad Q^t_i + \frac{Q^{t+i}_i}{1 - s} \leq 1,
\]
where $\delta$ is some real number satisfying $0 < \delta < 1$.\(^{10}\)

Recall for a moment the infinite model of Section VI (with durable output). That model is easily shown to be mathematically equivalent to a model in which population is stationary, while the storing of output is subject to depreciation at some constant rate, say $\delta$. But the model which we have described in the foregoing paragraph is an exact finite analogue of this infinite model with depreciation. For, by chopping off a finite segment of the infinite model and then tying the two ends together to form a closed loop, one obtains the model that is being discussed here.

Let us, therefore, investigate the efficiency of the competitive mechanism in our finite model, to which we shall henceforth refer, for short, as the closed-loop model. This investigation turns on whether or not intermediation is permitted.

Case a: No intermediation.—Under the assumption that intermediation is altogether absent, it is evident that the closed-loop model is inefficient. For, without intermediation, an agent who wishes to trade, say, $x$ units of commodity $j$ for $y$ units of commodity $k$, must find, in order to be able to make the trade, an agent who wishes to trade $y$ units of commodity $k$ for $x$ units of commodity $j$. This is, of course, the well-known "double coincidence of wants" (see, for example, Samuelson, 1964, p. 51). Now, in the closed-loop model, things are tailored in such a way that whatever trade an agent might wish to engage in, at whatever prices, he can never find another agent wishing to engage in the same trade on the opposite side. In other words, the double coincidence of wants never occurs. In the absence of intermediation, therefore, each agent must satisfy all his wants under complete autarky, and the doctrine of comparative advantage tells us immediately that this is inefficient. More formally, the decentralized solution of the closed-loop model in the absence of intermediation is given by

$$C_i^t, Q_i^t \left\{ \begin{array}{l} C_i^{t+1} = Q_i^{t+1} \\
\text{for } i = 1, \ldots, m, \end{array} \right.$$ 

where $(Q_i^t, Q_i^{t+1})$ is chosen so as to

$$\text{maximize } U(Q_i^t, Q_i^{t+1}),$$

subject to $Q_i^t + \frac{Q_i^{t+1}}{1 - \delta} = 1$, $Q_i^t \geq 0$, $Q_i^{t+1} \geq 0$.

Under the assumption that $\delta > 0$, this solution is inefficient. The economy can produce more of every commodity.

Now let us go back for a moment to the model of Section VI. The inefficiency which beset the economy of that section is precisely the inefficiency which now besets our closed-loop model. All the arguments (indeed, the very words) of Section VI are applicable to the closed-loop model, with only minor changes which have to do with substituting depreciation for population growth.

Case b: The money economy.—The necessity for a double coincidence of wants disappears, as is well known, as soon as barter is abandoned in favor of the money economy. Thus, one might expect the introduction of "the contrivance of money" to cure the aforementioned inefficiency, just as it cured the inefficiency of the model of Section VI. Such, indeed, is the case. In the money economy, agent $i$ will produce commodity $i$ exclusively, sell some of it

\(^{10}\) We shall, from now on, neglect to write separate expressions for the case $i = m$. Let us agree, therefore, that whenever $i = m$, $i + 1$ is simply 1.
to agent \((i-1)\), and then use the money which he receives in return to buy some of agent \((i+1)’s\) output. More formally, let \(p_i\) be the money price of commodity \(i\). Then, the equilibrium of the money economy is described by

\[
p_i = p_j \quad \text{for all } i \text{ and } j,
\]

\[
Q_{i}^j = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases}
\]

and \((C_t, C_{t+1})\) chosen so as to maximize \(U(C_t, C_{t+1})\), subject to \(C_t + C_{t+1} = 1\), \(C_t \geq 0\), \(C_{t+1} \geq 0\).

This equilibrium is efficient, and it is precisely analogous to the equilibrium in the money economy that was discussed in Section VIII. People “buy” money in return for goods which they produce, and then “sell” it for goods which they want. Basically, money here provides intermediation. Agent \(i\) sells to agent \((i-1)\), but he is being paid in terms of output of agent \((i+1)\).

There are several ways in which money can be introduced into the closed-loop model. The simplest is as follows: Think of agent 1 selling a promissory note to agent 2, who, in turn, proceeds to sell it to agent 3, and so on. The note travels around the loop until it reaches agent \(m\), who redeems it from agent 1. All other agents, however, accept agent 1’s note not because they are interested in agent 1’s output, but simply because the note is negotiable. In other words, agent 1’s note will assume the role of money.

Case: The ordinary general equilibrium model.—The closed-loop model is, after all, a finite general equilibrium model, and for the latter we have theorems concerning optimality and efficiency. What role, then, does intermediation play in the standard general equilibrium theory? The answer to this question is straightforward: In all of general equilibrium theory it is assumed (mostly implicitly) that a central clearing house exists, through which trades are channeled. Only with such a central clearing house is it possible to define competitive equilibrium in terms of aggregate excess demands alone. If a central clearing house were to be introduced in the closed-loop model, competitive behavior would lead to efficiency, as may be verified directly (without appealing to general theorems). But the central clearing house is obviously an intermediary. It is, in fact, precisely the analogue, in the closed-loop model, of our negative net worth intermediary of Section VII.

XI. CONCLUDING REMARK

Before closing, we would like to add a brief comment concerning the relation of the topics we have been discussing here to the Modigliani-Brumberg (1955) “life-cycle” theory of saving. Modigliani and Brumberg have postulated that aggregate saving can be explained by the interaction of individual saving for retirement and changes in the population structure. It is interesting to note that under the Modigliani-Brumberg assumptions of a zero rate of interest and an exponentially growing population, providing for old age in a way which results in positive aggregate saving is inefficient.\(^{11}\)

\(^{11}\) We are indebted to James Tobin for this observation.
APPENDIX

Let \( x_t \) be the level of inventories carried over by the economy from period \( t \) into period \( t + 1 \). In Section VIII a necessary and sufficient condition for efficiency was stated. The following is an equivalent assertion:

**Theorem:** The economy is efficient if and only if

\[
\liminf_{t \to \infty} x_t = 0.
\]

**Proof.** (a) Necessity: Suppose \( \liminf_{t \to \infty} x_t > 0 \). Then there exists an \( \epsilon > 0 \) and an integer \( T \) such that \( x_t > \epsilon \) for \( t \leq T \). Let aggregate consumption in period \( T \) be increased by \( \epsilon \), and let consumption in all subsequent periods remain unchanged. Let the sequence of inventory levels under this new scheme be denoted \( \{x'_t\} \). Clearly, \( x'_t = x_t \) for \( t \leq T \) and \( x'_t = x_t - \epsilon \) for \( t > T \). All that must be verified to show that the new scheme is feasible is \( x'_t \geq 0 \) for all \( t \). But this follows from the hypothesis.

(b) Sufficiency is proved similarly: Assume \( \liminf_{t \to \infty} x_t = 0 \). We must show that it is impossible to increase aggregate consumption in any period while keeping aggregate consumption in subsequent periods unchanged. Let aggregate consumption in period \( T \) be increased by an amount \( \epsilon > 0 \), where \( T \) is an arbitrary integer. Denote the new sequence of inventory levels by \( \{x'_t\} \). Now for \( t > T \), we have \( x'_t = x_t - \epsilon \), but by hypothesis there exists a \( t > T \) such that \( x_t < \epsilon \). Hence \( x'_t \) would have to be negative; that is, the new scheme is not feasible.

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