Cowles Foundation Paper 232

Reprinted from the AMERICAN ECONOMIC REVIEW
Vol. LV, No. 4, September, 1965

SECOND ESSAY ON THE GOLDEN RULE OF ACCUMULATION

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Four years ago, I presented a theorem on maximal consumption in a golden age [7]. The same theorem was discovered and published by Allais [1], Desrousseaux [3], Mrs. Robinson [10], Swan [15], and von Weizsäcker [17].[1] The theorem established may be expressed as follows:

> If there exists a golden-age growth path[2] on which the social net rate of return to investment equals the rate of growth (hence, in one class of models, the fraction of output saved equals the capital elasticity of output)—or, in market terms, a golden-age path on which the competitive interest rate equals the growth rate and hence gross investment equals the gross competitive earnings of capital—then this golden age produces a path of consumption which is uniformly higher than the consumption path associated with any other golden age.

The consumption-maximizing golden age will be referred to in this paper, as in [7], as the Golden Rule or GR path.

The papers cited raise two sorts of questions. The first concerns the conditions for the existence of the GR path. Some of the papers (including my own) erroneously suggest that the GR path can exist only in "neoclassical" models, i.e., models in which capital and labor are continuously substitutable. Some of the papers leave the false impression that the GR path exists only if there is no technical progress, while my own paper errs with respect to the type of technical progress which permits a GR path. The first part of this paper examines in two kinds of models the conditions for the existence of the GR path. We show, as a few writers have indicated, that the GR path may exist in the un-

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[1] Mention should also be made of an unpublished paper by Beckmann [2] in which the theorem is proved for the Cobb-Douglas case and the dissertation of Srinivasan [14] in which the existence of a state of maximum per capita consumption with a growing labor force is shown. All these authors made the finding independently, circa 1960.

[2] By a golden-age path we mean a growth path in which literally every variable changes (if at all) at a constant relative rate. It follows immediately that if investment is positive then output, investment, and consumption must all grow at the same (constant) rate. Various other properties can be derived.

neoclassical Harrod-Domar model as well as in the neoclassical model. And we show that a positive-investment GR path can exist only if technical progress can be described as purely "labor-augmenting."

Question also arises as to the normative significance of the theorem. We called the saving rule prevailing on the consumption-maximizing golden-age path the Golden Rule of Accumulation because, on that path, each "generation" saves (on behalf of future generations as it were) that fraction of income which it would have past generations save, subject to the constraint that all generations past and present are to save the same fraction of income. But no proof of the "optimality" of the GR path was given nor was any suggestion of its optimality seriously intended. Society need not confine itself to golden-age paths (should they exist) nor aim to achieve golden-age growth asymptotically. And even if some golden-age path should be utility-maximizing (at least for some initial conditions) the rate of time preference may make that path different from the GR path. It was evidently reflections such as these which led Pearce [6] and Samuelson [11] to doubt whether the GR path has any important normative significance at all.

In the second part of this paper it will be shown, however, that, whether or not it is "optimal," the GR path has the following important normative property: Any growth path on which, at some point in time and forever after, the capital-output ratio always exceeds its GR level by at least some constant amount—equivalently, any path which eventually keeps the social net rate of return to investment (or competitive rate of interest) permanently below its GR value by at least some finite amount—is *dynamically inefficient* in the sense that there always exists another path which, starting from the same initial capital stock, produces more consumption at least some of the time and never less consumption. This is the proposition conjectured by the author in reply to Pearce [8]. Its proof here is based on a proof provided by Tjalling C. Koopmans. The significance of the theorem is this: no path which is dynamically inefficient can be optimal; hence no path which transgresses the GR path in the manner described can be optimal. (Warning: It is only paths which so transgress the GR path for *infinitely long time* that can be shown to be dynamically inefficient.)

Since the conditions for a GR path are stringent, this theorem is only of theoretical interest. But we are able to prove analogous theorems even when no golden-age path, and hence no GR path, need exist. Thus we show that the possibility of "excessive capital deepening," despite a continuously positive rate of interest, is quite general.

A fuller summary of the paper and some concluding remarks close the paper.

I. *Existence of the Golden-Rule Path*

Section A will study the neoclassical and the Harrod-Domar models on the postulate that technical progress can be described as solely labor-augmenting. Section B will show that a GR path can exist only if technical progress can be described as labor-augmenting.

A. *Labor-Augmenting Technical Progress*

In both the neoclassical and Harrod-Domar cases, output, $Q(t)$, is a continuous function of capital, $K(t)$, labor, $L(t)$, and time:

$$Q(t) = F[K(t), e^{\lambda t}L(t)], \qquad \lambda \geq 0. \tag{1}$$

It is assumed here that technical progress can be described as solely labor-augmenting—time enters only in the second (labor) argument of the function—and that labor augmentation occurs at the constant exponential rate λ. The function is supposed to be homogeneous of degree one (constant returns to scale).

We suppose that the labor force grows exponentially at rate γ:

$$L(t) = L_0 e^{\gamma t}, \qquad \gamma \geq 0. \tag{2}$$

Capital is taken to be subject to exponential decay at rate δ, so that if $I(t)$ denotes the rate of gross investment:

$$I(t) = \dot{K}(t) + \delta K(t), \qquad \delta \geq 0. \tag{3}$$

Finally, consumption, $C(t)$, is the difference between output and gross investment:

$$C(t) = Q(t) - I(t), \qquad C(t) \geq 0. \tag{4}$$

The neoclassical case. We suppose now that the production function has the following "neoclassical" properties: it is twice differentiable (smooth marginal products), it is strictly concave (diminishing marginal products), and it has everywhere positive first derivatives (marginal products). That is,

$$\begin{aligned} &\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial F}{\partial L} > 0; \\ &\frac{\partial^2 F}{\partial K^2} < 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0. \end{aligned} \tag{1a}$$

By virtue of constant returns to scale and (2):

$$Q(t) = L_0 e^{(\gamma+\lambda)t} F\left[\frac{K(t)}{L_0 e^{(\gamma+\lambda)t}}, 1\right]. \tag{5}$$

Hence, if we let $k(t)$ denote capital per unit "effective labor,"

$$k(t) = \frac{K(t)}{L_0 e^{(\gamma+\lambda)t}}, \tag{6}$$

and if we define

$$f(k(t)) = F[k(t), 1], \tag{7}$$

we can express the production function for all t as

$$Q(t) = L_0 e^{(\gamma+\lambda)t} f(k(t)), \qquad f'(k(t)) > 0, \qquad f''(k(t)) < 0. \tag{8}$$

We show now that if $k(t)$ is equal to any positive constant $k>0$, then the economy will grow in the manner of a golden age, provided of course that the constraint $I(t) \leq Q(t)$ is satisfied.

Clearly, output will grow exponentially at rate $g=\gamma+\lambda$,

$$Q(t) = L_0 e^{(\gamma+\lambda)t} f(k) = Q(0)e^{gt}, \tag{9}$$

as will the capital stock:

$$K(t) = L_0 e^{(\gamma+\lambda)t} k = K(0)e^{gt}. \tag{10}$$

Hence, from (3) and the relation $\dot{K}(t)=gK(t)$, investment will also grow at the rate g:

$$I(t) = (g+\delta)K(0)e^{gt} = (g+\delta)L_0 k e^{gt}. \tag{11}$$

Since investment and output will grow at the same rate, g, so will consumption, $C(t)$, (where $C(t)=Q(t)-I(t)$)

$$C(t) = [Q(0) - (g+\delta)K(0)]e^{gt} = [f(k) - (g+\delta)k]L_0 e^{gt}. \tag{12}$$

The gross investment-output ratio, s, will be constant:

$$s = \frac{I(t)}{Q(t)} = \frac{(g+\delta)K(0)}{Q(0)} = \frac{(g+\delta)k}{f(k)}. \tag{13}$$

So will the marginal productivity of capital,

$$\frac{\partial F(K(t), e^{\lambda t}L(t))}{\partial K}:$$

$$\frac{\partial F}{\partial K} = \frac{\partial F\left(\dfrac{K(t)}{L_0 e^{(\gamma+\lambda)t}}, 1\right)}{\partial\left(\dfrac{K(t)}{L_0 e^{(\gamma+\lambda)t}}\right)} = f'(k). \tag{14}$$

And so will the share of gross output going to capital, a, if capital receives its marginal product:

$$a = \frac{\partial F}{\partial K} \frac{K(t)}{Q(t)} = \frac{f'(k)k}{f(k)}. \tag{15}$$

Conversely, it can be shown that every golden-age path in which investment is positive implies a constant value of $k(t) > 0$ and a growth rate equal to $\gamma + \lambda$.[3] Therefore, a golden age with positive investment occurs if and only if $k(t) = k$, a constant.

Hence, in every golden age with positive investment, the growth rate of output, investment, and consumption is $\gamma + \lambda$. These golden-age consumption paths are therefore logarithmically parallel. Associated with each golden age is a certain value of s, of $\partial F / \partial K$, of $K(0)$ and of k. Let us assume for the moment (we drop this assumption later) that the golden age yielding the maximal consumption path, if such exists, is one in which k, and hence $K(0)$, is greater than zero. We assume, in other words, that if a maximum exists, it is an *interior* one rather than a corner maximum at $k = 0$. Then, for every t, the derivative of $C(t)$ with respect to $K(0)$ in (12) must be zero on the GR path:

$$\frac{\partial C(t)}{\partial K(0)} = \frac{\partial F}{\partial K} - (g + \delta) = 0. \tag{16}$$

Equivalently, one can differentiate (12) with respect to k to obtain:

$$f'(k) - (g + \delta) = 0. \tag{16a}$$

That is, on this assumption, the marginal product of capital will equal $g + \delta$ on the GR path (if it exists).[4] Transposing terms in (16), we have:

$$\frac{\partial F}{\partial K} - \delta = g. \tag{17}$$

The left-hand side of (17) is the social net rate of return to investment.[5] Hence this result states that if an *interior* golden-age consumption maxi-

[3] In a golden age, if investment is positive, then investment, consumption, and output must all grow at the same constant relative rate, denoted g. Hence $Q(t) = Q(0)e^{gt}$ and $I(t) = I(0)e^{gt}$. And capital must grow at some constant relative rate, denoted h. Hence $\dot{K}(t) = hK(t)$. Therefore, by (3), $I(t) = (h + \delta)K(t)$ which implies $h = g$. But if $K(t) = K(0)e^{gt}$ then, from (1) and the postulate that $\partial F / \partial L > 0$, if follows that $g = \gamma + \lambda$, hence that $k(t)$ is constant.

[4] A common-sense explanation of (16) has been provided by Solow [13]. Imagine that capital is initially free but that we are to invest so as to maintain a golden age once the initial capital stock has been chosen. Consider a small increase of initial capital, $\Delta K(0)$. The rules of the game require that we then increase the rate of investment by $\Delta I(0) = (g + \delta)\Delta K(0)$ to make capital grow at rate g. The increase of initial capital will increase output by $\Delta Q(0) = (\partial F / \partial K)\Delta K(0)$. Hence consumption will increase by $\Delta C(0) = \Delta Q(0) - \Delta I(0) = [(\partial F / \partial K) - (g + \delta)]\Delta K(0)$. As long as $(\partial F / \partial K) > g + \delta$ it pays to accept more capital. The consumption-maximizing golden age is reached when $K(0)$ has increased to the point where $(\partial F / \partial K) - (g + \delta) = 0$.

mum exists, it is where the social net rate of return to investment equals the golden-age growth rate. This is the first (and most general) way to characterize the GR path in purely technological terms.

The other technological characterization is obtained by multiplying both sides of (17) by $K(t)/Q(t)$ and rearranging terms:

$$\frac{\partial F}{\partial K}\frac{K(t)}{Q(t)} = (g+\delta)\frac{K(t)}{Q(t)} = \frac{I(t)}{Q(t)}\,. \tag{18}$$

Hence

$$s = \frac{\partial F}{\partial K}\frac{K(t)}{Q(t)}\,. \tag{19}$$

This states that on the interior GR path the saving ratio is equal to the elasticity of output with respect to capital. (This was the characterization of the GR path employed by Swan and the present author; of course, such a capital elasticity exists only in one-commodity models in which output is a function of "capital.")

Conditions (17) and (19) can be translated into "market" terms if the economy is purely competitive and free of externalities in production. On these assumptions, $\partial F/\partial K$ is the gross rental rate of capital and $(\partial F/\partial K)-\delta$ is the (equilibrium) rate of interest. Then (17) implies that on the interior GR path the interest rate is equal to the golden-age growth rate. (19) implies that the saving ratio equals capital's gross relative share, or that net investment equals net profits.

Now we shall investigate the conditions for the existence of the GR path. For this purpose we adapt, in Figure 1, a diagram first presented by Pearce [6] and later employed by Koopmans [5]. It is a diagram of the relation between $K(0)$ and $C(0)$ in a golden age as given by (1) and (12):

$$C(0) = F[K(0), L_0] - (g+\delta)K(0). \tag{20}$$

(Some readers may wish to diagram $c=f(k)-(g+\delta)k$ where c is consumption per unit effective labor force and observe that c is maximal where k is such that $f'(k)=g+\delta$.)

Figure 1 depicts a golden-age consumption maximum at $K(0)=\hat{K}(0)$ where $(\partial F/\partial K)=g+\delta$. It is easy to see from the diagram, however, that there are two cases in which no such interior GR maximum exists.

[5] By the (instantaneous) social net rate of return to investment at time t we mean

$$\lim_{h\to 0}\left\{\left[\frac{\partial C(t+h)}{\partial C(t)}-1\right]\Big/ h\right\}.$$

For a discussion of the rate of return to investment see Solow [12].

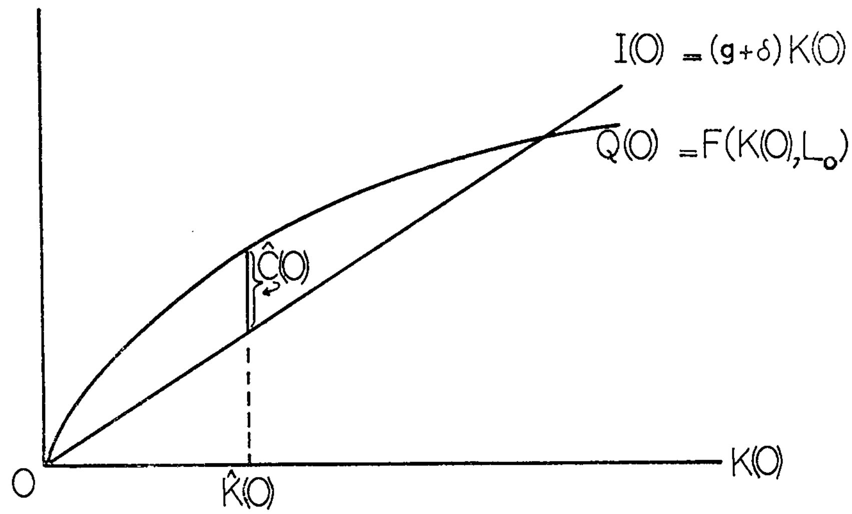

FIGURE 1

In one case, neither an interior nor a corner maximum exists. This is the case in which

$$\lim_{K\to\infty} \frac{\partial F}{\partial K} > g + \delta;$$

then the $Q(0)$ curve is everywhere steeper than the $I(0)$ line so that the distance between them always increases with $K(0)$. It is easy to see that this case implies

$$\lim_{K\to\infty} \frac{Q(0)}{K(0)} \geq g + \delta.$$

While our assumptions on the production function do not exclude this possibility, it can be shown however that, if $g+\delta>0$, this case can arise only if positive output can be produced without labor. Proof: $(Q/K)=F(1, L/K)$. Hence

$$\lim_{K\to\infty} \frac{Q}{K} = F(1, 0).$$

But $F(1, 0)=0$ if $F(K, 0)=0$. Hence

$$\lim_{K\to\infty} \frac{Q}{K} \geq g + \delta > 0$$

only if labor is not required for positive production.

The other case in which no interior maximum exists occurs when

$$\lim_{K\to 0} \frac{\partial F}{\partial K} \leq g + \delta;$$

then the $Q(0)$ curve is everywhere flatter than the $I(0)$ line so that a corner maximum exists at $K(0)=0$. We shall show that, in this case, $K(t)=0$ can be considered the GR path.

There are two sub-cases to consider. Suppose first that $F(0, L_0)>0$. Then

$$\lim_{K\to 0} \frac{\partial F}{\partial K}\cdot\frac{K(t)}{Q(t)} = 0$$

since $Q(t)$ does not go to zero in the limit. Hence, when $K(t)=0$;

$$\frac{\dot{Q}(t)}{Q(t)} = \left[\lim_{K\to 0} \frac{\partial F}{\partial K}\cdot\frac{K(t)}{Q(t)}\right]\frac{\dot{K}(t)}{K(t)} + \left[1 - \lim_{K\to 0} \frac{\partial F}{\partial K}\,\frac{K(t)}{Q(t)}\right](\gamma+\lambda) = \gamma+\lambda.$$

That is, output grows at the usual golden-age rate, or "natural" rate $\gamma+\lambda$. So does consumption. This golden-age path, $C(t)=F(0, L_0)e^{gt}$, is maximal and hence it is the GR path since

$$\lim_{K\to 0} \frac{\partial F}{\partial K} \leq g + \delta;$$

investment would have to increase more than output to maintain a golden age with positive $k(t)$.[6]

The other sub-case is $F(0, L_0)=0$. In this case the $Q(0)$ curve lies uniformly below the $I(0)$ line (since they both start from the origin and $I(0)$ rises more steeply from the start). This implies that no golden age with $K(0)>0$ is possible for it would require $I(t)>Q(t)$. But $K(t)=0$ clearly implies a "golden age" for then $C(t)=Q(t)=I(t)=F(0, L_0e^{(\lambda+\gamma)t})=0$. Since this is the only golden age that exists, it is the maximal golden age and hence the GR path.

Summarizing, if labor is required for positive output and $g+\delta>0$ then a GR path always exists in the model under consideration. If there exists a golden-age capital path $K(t)=K(0)e^{gt}$ such that $(\partial F/\partial K)=g+\delta$ then this is the GR path; if there does not exist such a path then $K(t)=0$ is the GR path. In short, $K(t)=K(0)e^{gt}$ produces the GR path if $(\partial F/\partial K)=g+\delta$ for some $K(0)>0$ or if $(\partial F/\partial K)\leq g+\delta$ when $K(0)=0$.

The Harrod-Domar case. To illustrate the fact that no neoclassical assumptions are required for the existence of the GR path, we now drop the assumptions of twice differentiability, strict concavity and everywhere positive marginal products and specialize (1) to the Harrod-

[6] Note that on this GR path, where $K(t)=0$, the saving ratio and capital's relative share are equal, since they are both equal to zero. But the interest rate may be less than the growth rate.

Domar case:

$$Q(t) = \min\left[\alpha K(t), \beta e^{\lambda t} L(t)\right]. \tag{1b}$$

We retain equations (2), (3), and (4).

By virtue of (2) and the constant returns to scale implied by (lb):

$$Q(t) = L_0 e^{(\gamma+\lambda)t} \min\left[\alpha \frac{K(t)}{L_0 e^{(\gamma+\lambda)t}}, \beta\right] \tag{21}$$

or

$$Q(t) = L_0 e^{(\gamma+\lambda)t} \min\left[\alpha k(t), \beta\right]. \tag{21a}$$

It is easy to show again that if $k(t)$ is equal to any constant $k>0$ then, provided the restraint $I(t) \leq Q(t)$ is satisfied, golden-age growth results. Clearly output, capital and investment will grow at the constant rate $g=\gamma+\lambda$; hence, so will consumption. As before, $s=(g+\delta)K(0)/Q(0)$; if $\alpha K(0) \leq \beta L_0$ (meaning that capital is not in surplus) then $(K(0)/Q(0)) = 1/\alpha$ and if $\alpha K(0) > \beta L_0$ then $(K(0)/Q(0)) = (K(0)/\beta L_0)$. $\partial F/\partial K$ will be constant, either equal to α (if labor is in surplus) or zero (if capital is in surplus).

Conversely, $k(t)$ is constant in every golden age with positive investment. If investment (hence output and consumption) is growing at some constant rate, g, and capital is growing exponentially then capital must also be growing at rate g. Now if g were less than $\gamma+\lambda$, then labor would become redundant (if it was not initially) and the unemployment ratio would grow nonexponentially, which contradicts the notion of a golden age; if g were greater than $\gamma+\lambda$, then labor would eventually become scarce (if it were not initially) and growth of output at the rate g would then be impossible Hence, in a golden age with positive investment, capital grows at the rate $\gamma+\lambda$ and $k(t)$ is therefore constant. Therefore, golden-age growth with positive investment occurs if and only if $k(t)$ is constant.

To investigate the GR path we use Figure 2 which differs from Figure 1 only in that, in (20), we have substituted the Harrod-Domar function min $[\alpha K(0), \beta L_0]$ for $F[K(0), L_0]$:

$$C(0) = \min\left[\alpha K(0), \beta L_0\right] - (g + \delta) K(0). \tag{20'}$$

The diagram depicts an interior golden-age consumption maximum at $K(0) = \beta L_0/\alpha$. At this point the capital stock is just large enough to employ the entire labor force. A larger capital stock would put capital in surplus; a smaller stock would cause a surplus of labor. In the Harrod-Domar model, therefore, the interior GR path, if it exists, is the golden-age path in which there is full employment of both labor and capital.

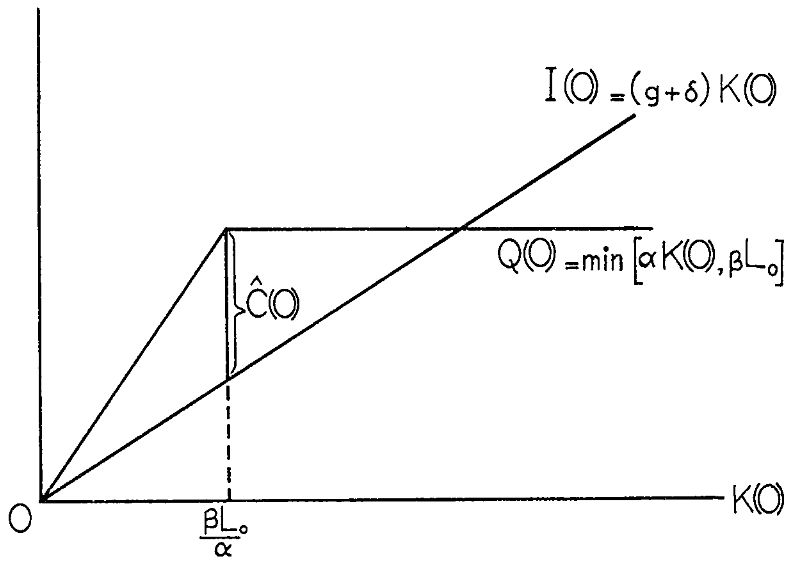

FIGURE 2

What of the usual characterizations of the GR path in terms of the interest rate and capital's relative share? On the interior GR path the saving ratio is $(g+\delta)/\alpha$ and the growth rate $\gamma+\delta$. But relative shares and the rate of interest are indeterminate: we can say only that capital's share is between zero and one and that the interest rate is between zero and $\alpha-\delta$. But it is true that this interior GR path is the only golden-age path with positive investment in which it is possible that the saving ratio equal capital's share and the interest rate equal the growth rate; for in all other positive-investment golden ages capital's relative share and the interest rate are determinate and do not satisfy these equalities. Thus it remains valid that if there exists a golden age in which the interest rate equals the growth rate and the saving ratio equals capital's relative share, then this golden-age path is the GR path. Hence the Golden Rule theorem applies to the Harrod-Domar model as well as to the neoclassical model. (See Robinson [10] and Samuelson [11] for similar comments on more complicated fixed-coefficient models.)

As in the neoclassical case, however, an interior GR path may not exist. Figure 2 shows that if $\alpha<g+\delta$ then no golden age with positive investment exists, hence no interior GR path. In this case the "golden age" $K(t)=Q(t)=I(t)=C(t)=0$ is the only possible golden age; hence it can be regarded as the GR path.

Note that, in the Harrod-Domar case, either an interior or a corner GR path must exist since positive labor input is required for positive output.

B. *The Necessity that Technical Progress Be Labor-Augmenting*

The necessity that any technical progress be describable as labor-augmenting for the existence of a positive-investment (interior) GR path follows from analyses of technical progress by Diamond [4], Robinson [9] and Uzawa [16]. I shall merely indicate one line of proof.

The existence of an interior GR path depends upon the existence of a continuum of logarithmically parallel growth paths on which output, consumption, and investment all grow at some common exponential rate, say g. Since investment grows at rate g and has always been doing so, capital also grows at rate g on each of the paths.

Suppose that the production function is neoclassical (see (1a)) and is homogeneous of degree one:

$$Q(t) = F[K(t), L(t); t]. \tag{22}$$

Differentiating this totally with respect to time and dividing the resulting equation by $Q(t)$ yields

$$\frac{\dot{Q}(t)}{Q(t)} = a(t)\frac{\dot{K}(t)}{K(t)} + [1 - a(t)]\frac{\dot{L}(t)}{L(t)} + \frac{F_t}{Q(t)} \tag{23}$$

where $a(t)$, capital's relative share at t, denotes $(\partial F/\partial K)/(K(t)/Q(t))$, so that, by Euler's theorem, $1-a(t)=(\partial F/\partial L)/(L(t)/Q(t))=$ labor's share. $F_t/Q(t)=(\partial F/\partial t)/Q(t)$ is the relative rate of technical progress at time t.

If the aforementioned parallel paths exist, we may substitute g for $\dot{Q}(t)/Q(t)$ and $\dot{K}(t)/K(t)$ and obtain

$$\frac{F_t/Q(t)}{1 - a(t)} = g - \frac{\dot{L}(t)}{L(t)} = \Phi(t). \tag{24}$$

Hence, the rate of technical progress expressed as a ratio to labor's relative share is a function solely of time (independent of the capital-labor ratio) if these parallel paths exist.

Diamond [4] has shown the equivalence of the property expressed in (24) and the Harrod-neutrality, for all K, L, and t, of the technical progress represented by $F[K(t), L(t); t]$. (By definition, progress is Harrod-neutral if and only if relative shares or the capital-output ratio are constant over time for a constant rate of interest or marginal product of capital.)

Now the Robinson-Uzawa theorem [9] [16] proves that if technical progress is everywhere Harrod-neutral then technical progress can be described as purely labor-augmenting:[7]

[7] There are cases in which Harrod-neutral progress can be described as capital-augmenting. The Cobb-Douglas function is such a case (and the only case under constant returns to scale) for the function $K^{\alpha}[A(t)L]^{1-\alpha}$ can be written $[B(t)K]^{\alpha}L^{1-\alpha}$.

$$Q(t) = F[K(t), L(t); t] = G[K(t), A(t)L(t)]. \tag{25}$$

All this proves that, if an interior GR path exists, any technical progress present must be describable as labor-augmenting.

Note that if $Q(t)$ and $K(t)$ both grow exponentially at rate g then, by constant returns to scale, $A(t)L(t)$ or "effective labor" must also grow exponentially at rate g. (It is not essential that $A(t)$ and $L(t)$ each grow exponentially.)

Labor augmentation is, of course, a very restrictive type of technical progress. But the notion of the Golden Rule path has considerable heuristic value even if progress cannot be described as labor-augmenting or even as "factor-augmenting" in general. It will be shown in the next part of this paper that there still exists in these cases a critical path—which we call the Quasi-Golden-Rule path—having, in one respect, the same normative significance as the GR path.

II. *Inefficient Growth Paths*

The preceding analysis can be made to show immediately that some golden-age paths are inefficient. Consider any golden age in which the capital-effective labor ratio forever exceeds its GR value. It will be dominated by a policy of immediately gobbling up the "excess" capital and forever after maintaining the capital-effective labor ratio at its GR value, i.e., following the GR path; such a policy will clearly make consumption higher at every point in time. It follows that any investment policy which at some point permanently fixes the capital-effective labor ratio at a level exceeding the GR level is inefficient and therefore cannot be optimal (since a policy to be optimal must be optimal at every stage).

In the author's reply to Pearce an obvious generalization of this result to non-golden-age paths was conjectured: "Any policy which causes the capital-output ratio [equivalently, the capital-effective labor ratio, since the one ratio is a monotonically increasing function of the other] permanently to exceed—always by some minimum finite amount—its GR level is inefficient and hence cannot be optimal" [8, p. 1099]. A proof of this conjecture was later communicated to the author by Tjalling Koopmans. In what follows we present what is essentially Koopmans' proof and then employ the technique to prove an analogous theorem for the case in which technical progress must be described as (at least partially) capital-augmenting, for the case of nonexponential labor growth and factor augmentation, and finally for the case in which technical progress cannot necessarily be described as factor-augmenting.

We confine our analysis to the neoclassical production function, although the theorems proved clearly carry over to the Harrod-Domar production function.

A. *Pure Labor Augmentation at a Constant Rate*

Suppose first that technical progress can be described as solely labor-augmenting and that the rate of labor augmentation is a constant, λ. Then, as was shown above, when $k(t)$ is fixed, the consumption path is given by the equation

$$C(t) = [f(k) - (\gamma + \lambda + \delta)k]L_0e^{(\gamma+\lambda)t} \tag{12}$$

where $f'(k) > 0, f''(k) < 0$.

We show now that if $k(t)$ is not fixed, then the consumption path is given by the equation

$$C(t) = [f(k(t)) - (\gamma + \lambda + \delta)k(t) - \dot{k}(t)]L_0e^{(\gamma+\lambda)t} \tag{26}$$

Proof: From (3), (4), and (9) we have

$$C(t) + \dot{K}(t) + \delta K(t) = L_0e^{(\gamma+\lambda)t}f(k(t)) \tag{27}$$

or

$$\frac{C(t)}{L_0e^{(\gamma+\lambda)t}} = f(k(t)) - \delta k(t) - \frac{\dot{K}(t)}{L_0e^{(\gamma+\lambda)t}}\cdot \tag{28}$$

Now, differentiating $k(t)$ with respect to time, we have

$$\dot{k}(t) = \frac{\dot{K}(t)}{L_0e^{(\gamma+\lambda)t}} - (\gamma + \lambda)\frac{K(t)}{L_0e^{(\gamma+\lambda)t}} \tag{29}$$

or

$$\frac{\dot{K}(t)}{L_0e^{(\gamma+\lambda)t}} = \dot{k}(t) + (\gamma + \lambda)k(t). \tag{30}$$

Substituting (30) into (28) yields (26).

Assume now that there exists a GR path, hence a GR value of $k(t)$, say $\hat{k}$. For simplicity only, we assume that the GR maximum is an interior one so that $\hat{k}$ is determined by the equation, derived from (12) (see also (16a)):

$$f'(\hat{k}) = \gamma + \lambda + \delta. \tag{31}$$

As a consequence of (31), the expression $f(k) - (\gamma+\lambda+\delta)k$ is monotonically increasing in k up to $k = \hat{k}$ and monotonically decreasing in k for all $k > \hat{k}$.

Consider now any capital-path which "violates" the Golden Rule in that, at some point in time (perhaps initially) and thereafter, it keeps the capital-effective labor ratio in excess of its GR value by at least some positive, constant amount. That is, consider any path $k(t)$ such that, for all $t \geq t_0 \geq 0$,

$$k(t) \geq \hat{k} + \epsilon, \qquad \epsilon > 0 \text{ and independent of } t. \tag{32}$$

Then the following theorem can be proved:

> Any path satisfying (32) is "dynamically inefficient" or (equivalently) "dominated," for there always exists another path which, starting from the same initial capital stock, provides more consumption at least some of the time and never less consumption.

Proof: Define another path, $k^*(t)$, such that

$$k^*(t) = \begin{cases} k(t), & 0 \leq t < t_0; \\ k(t) - \epsilon, & t \geq t_0. \end{cases} \tag{33}$$

In the first interval, $0 \leq t \leq t_0$, the two paths are identical so that $C^*(t) = C(t)$ in this interval (which will not exist if $t_0 = 0$). At $t = t_0$, the starred path gives a discontinuous consumption bonus, for an amount of capital equal to $\epsilon L_0 e^{gt_0}$ is instantly consumed so as to make $k^*(t) = k(t) - \epsilon$ at $t = t_0$. In the remaining interval, $t > t_0$, the difference between the consumption rate offered by the starred path and the path specified in (32) is implied by (26) to be

$$\begin{aligned} C^*(t) - C(t) = \{&[f(k^*(t)) - (\gamma + \lambda + \delta)k^*(t) - \dot{k}^*(t)] \\ &- [f(k(t)) - (\gamma + \lambda + \delta)k(t) - \dot{k}(t)]\} L_0 e^{(\gamma+\lambda)t}. \end{aligned} \tag{34}$$

But observe that, for all $t > t_0$, $\dot{k}^*(t) = \dot{k}(t)$ since the two paths differ after t_0 by only a constant, ϵ. Hence (33) and (34) imply

$$\begin{aligned} C^*(t) - C(t) = \{&[f(k^*(t)) - (\gamma + \lambda + \delta)k^*(t)] \\ &- [f(k(t)) - (\gamma + \lambda + \delta)k(t)]\} L_0 e^{(\gamma+\lambda)t}. \end{aligned} \tag{35}$$

The righthand side of (35) is strictly positive for all $t > t_0$ since $k^*(t) \geq \hat{k}$, $k(t) > k^*(t)$ and $f(k) - (\gamma + \lambda + \delta)k$ is strictly decreasing in k for all $k > \hat{k}$. Hence, in the interval $t > t_0$ the starred path gives more consumption at every point in time. Therefore, the starred path dominates the other path for it is never worse and is better for all $t \geq t_0$.

To elaborate a little on the last step of the proof, note that $k^*(t) \geq \hat{k}$ because $k^*(t)$ is only ϵ smaller than $k(t)$ and the latter is at least ϵ larger than $\hat{k}$ for all t. Figure 3 illustrates why $f(k^*(t)) - (\gamma + \lambda + \delta)k^*(t) > f(k(t)) - (\gamma + \lambda + \delta)k(t)$ for any $t > t_0$.

The theorem can be expressed in another way. Since the social net rate of return to investment (and the competitive rate of interest), $f'(k(t)) - \delta$, is a monotonically decreasing function of $k(t)$ and independent of time, an equivalent proposition is that any growth path which keeps the rate of return to investment forever and finitely below its GR value (the golden-age growth rate on the assumption expressed by (31)

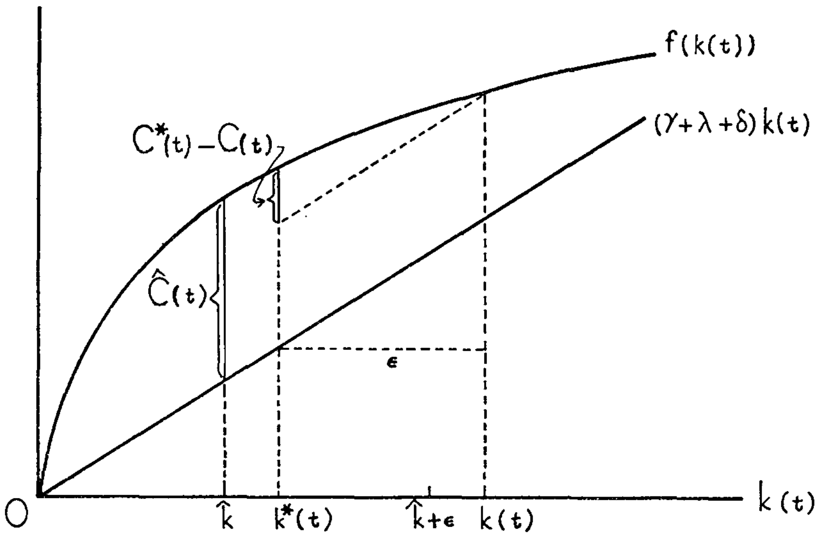

FIGURE 3

is dynamically inefficient. Or the proposition can be expressed in terms of the capital-output ratio, as we first conjectured it.

Another remark is that the neoclassical assumptions $f'(k)>0$ and $f''(k)<0$ for all k are far stronger than necessary for the theorem. If $f''(k)=0$ for all $k>\hat{k}$, for example (where $\hat{k}$ is now defined as the smallest k for which $f'(k)=(\gamma+\lambda+\delta)$), then, while the two paths will yield the same consumption path after t_0, the starred path still offers the consumption bonus at t_0, and hence dominates the other path. Secondly, the theorem is trivial in the Harrod-Domar case, where $f'(k)=0$ for $k>\hat{k}$, for it simply means that any path which keeps capital permanently in surplus is inefficient, and this hardly needs proving.

B. *Factor-Augmenting Progress*

We turn now to the case in which technical progress can be described as factor-augmenting and may be partially or wholly capital-augmenting. Suppose that the rate of capital augmentation is a constant, $\mu\geq 0$. And suppose once again that we have a neoclassical production function. Then

$$Q(t) = F[e^{\mu t}K(t), e^{\lambda t}L(t)], \qquad \mu \geq 0,\ \lambda \geq 0. \tag{36}$$

In the spirit of the first part of this paper, we define

$$k(t) = \frac{K(t)}{L_0 e^{(\gamma+\lambda-\mu)t}}, \tag{37}$$

which is the ratio of "effective capital" to "effective labor," and

$$f(k(t)) = F[k(t), 1] \tag{38}$$

to obtain, by virtue of constant returns to scale,

$$Q(t) = L_0 e^{(\gamma+\lambda)t} f(k(t)). \tag{39}$$

To obtain the consumption path as a function of $k(t)$, we follow the same procedure used to obtain (26). From

$$C(t) + \dot{K}(t) + \delta K(t) = L_0 e^{(\gamma+\lambda)t} f(k(t)) \tag{40}$$

we have

$$\frac{C(t)}{L_0 e^{(\gamma+\lambda-\mu)t}} = e^{\mu t} f(k(t)) - \delta k(t) - \frac{\dot{K}(t)}{L_0 e^{(\gamma+\lambda-\mu)t}} \cdot \tag{41}$$

From

$$\dot{k}(t) = \frac{\dot{K}(t)}{L_0 e^{(\gamma+\lambda-\mu)t}} - (\gamma + \lambda - \mu) \frac{K(t)}{L_0 e^{(\gamma+\lambda-\mu)t}} \tag{42}$$

we have

$$\frac{\dot{K}}{L_0 e^{(\gamma+\lambda-\mu)t}} = \dot{k}(t) + (\gamma + \lambda - \mu) k(t). \tag{43}$$

Hence, from (41) and (43),

$$C(t) = \{e^{\mu t} f(k(t)) - (\gamma + \lambda + \delta - \mu) k(t) - \dot{k}(t)\} L_0 e^{(\gamma+\lambda-\mu)t}. \tag{44}$$

(If $\mu = 0$, we obtain (26) again.)

Now we define $\hat{k}(t)$ as the value of $k(t)$ which, for fixed $\dot{k}(t)$ and a particular t, maximizes $C(t)$. For simplicity only we assume an interior maximum is attained so that $\hat{k}(t)$ is defined by[8]

$$e^{\mu t} f'(\hat{k}(t)) = \gamma + \lambda + \delta - \mu. \tag{45}$$

Of course, $e^{\mu t} f'(k(t))$ is just the marginal productivity of capital at time t.[9] Hence the path $\hat{k}(t)$ defined by (45) is a constant interest-rate path in which the (competitive) interest rate is $e^{\mu t} f'(k(t)) - \delta = \gamma + \lambda - \mu$.

We know that $\hat{k}(t)$ is not the GR path; no family of golden-age paths exist when $\mu > 0$, and hence no GR path exists. Nevertheless we shall dub this path the Quasi-Golden-Rule path. For we shall demonstrate

[8] If $f'(k) > 0$ for all k, as we assume, then $\gamma + \lambda - \mu > 0$ is required for the existence of such a value of $k(t)$.

Note that $\hat{k}(r)$ must be increasing over time if $\mu > 0$; and if $\lambda + \delta - \mu > 0$, then so must $\hat{K}(t)$, by (37).

[9] $$\frac{\partial F(e^{\mu t} K(t), e^{\lambda t} L(t))}{\partial K(t)} = e^{\mu t} \frac{\partial F\left(\frac{e^{\mu t} K(t)}{e^{(\gamma+\lambda)t} L_0}, 1\right)}{\partial \left(\frac{e^{\mu t} K(t)}{e^{(\gamma+\lambda)t} L_0}\right)} = e^{\mu t} f'(k(t)).$$

that it is like the GR path in the following respect: Any path which, at some point in time and forever after, keeps the ratio of effective capital to effective labor in excess of the Quasi-GR value of that ratio, $\hat{k}(t)$, is dynamically inefficient.[10]

Such a path is one which causes $k(t)$ to satisfy, for all $t \geq t_0 \geq 0$,

$$k(t) \geq \hat{k}(t) + \epsilon, \qquad \epsilon > 0 \quad \text{and} \quad \text{constant.} \tag{46}$$

We show now that the following path dominates any such path:

$$k^*(t) = \begin{cases} k(t), & 0 < t < t_0; \\ k(t) - \epsilon, & t \geq t_0. \end{cases} \tag{47}$$

Comparing the associated consumption paths, we observe first that the two paths yield identical consumption paths until t_0. At this point the starred path yields a consumption bonus, unlike the other path. Subsequently, $\dot{k}^*(t) = \dot{k}(t)$, since, for $t > t_0$, $k^*(t)$ and $k(t)$ differ only by the constant, ϵ. Hence for all $t > t_0$,

$$\begin{aligned} C^*(t) - C(t) = \{ & [e^{\mu t} f(k^*(t)) - (\gamma + \lambda + \delta - \mu) k^*(t)] \\ & - [e^{\mu t} f(k(t)) - (\gamma + \lambda + \delta - \mu) k(t)] \} L_0 e^{(\gamma+\lambda-\mu)t}. \end{aligned} \tag{48}$$

The right-hand side of (48) must be positive for every t, since $k(t) > k^*(t) \geq \hat{k}(t)$, and $e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)k(t)$ is, for every t, monotonically decreasing in $k(t)$ in the range $k(t) > \hat{k}(t)$ (since $\hat{k}(t)$ is maximal and $f''(k(t)) < 0$). Hence, the starred path dominates the path which transgresses the Quasi-Golden-Rule path. Therefore, any path which violates the Quasi-Golden-Rule path in the manner described in (46) is dynamically inefficient.[11]

[10] While the Quasi-GR path does not dominate other constant interest-rate paths, it does dominate all $k(t)$ paths parallel to it so it is in fact a Generalized Golden Rule path.

[11] We have just shown that (46), that is, $k(t) \geq \hat{k}(t) + \epsilon$, is a sufficient condition that a $k(t)$ path be dominated by another path on which $k(t)$ is smaller by a constant amount. We show here that $k(t) > \hat{k}(t)$ is necessary that a path $k(t)$ be dominated *in this way*; but that $k(t) > \hat{k}(t)$ is not sufficient for such dominance.

First we show that every $k(t)$ path so dominated is a path along which $e^{\mu t} f'(k(t)) < \gamma + \lambda + \delta - \mu$, and hence $k(t) > \hat{k}(t)$, for all $t \geq t_0$.

Proof: Choose any path $k(t) \geq 0$ and suppose that it is dominated by another path $k^*(t) = k(t) - \epsilon$, $\epsilon > 0$ for $t \geq t_0$. Then, for every $t \geq t_0$ we have

$$\begin{aligned} C^*(t) - C(t) = \{ & [e^{\mu t} f(k(t) - \epsilon) - (\gamma + \lambda + \delta - \mu)(k(t) - \epsilon)] \\ & - [e^{\mu t} f(k(t)) - (\gamma + \lambda + \delta - \mu) k(t)] \} L_0 e^{(\gamma+\lambda-\mu)t} \geq 0. \end{aligned}$$

Then it is immediately clear that, for every $t \geq t_0$, $k(t)$ must exceed $\hat{k}(t)$; that is, $k(t)$ must lie on the right side of the hill whose peak occurs at $k(t) = \hat{k}(t)$, i.e., where $e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)k(t)$ is at a maximum.

This proves that $k(t) > \hat{k}(t)$ is a necessary condition that a path be dominated in the manner described. We show next that $k(t) > \hat{k}(t)$ is not a sufficient condition. Consider a path $k(t) > \hat{k}(t)$ with

$$\lim_{t \to \infty} [k(t) - \hat{k}(t)] = 0.$$

We can relax without difficulty the assumptions that the labor force and the technology increase at constant rates. Further, we may allow the depreciation rate at time t, $\delta(t)$, (the same for capital goods of every age) to vary with time. Write

$$Q(t) = F[B(t)K(t),\ A(t)L(t)] \tag{49}$$

where $A(t)$, $B(t)$ and $L(t)$ are continuously differentiable functions of time. Then, defining $k(t)=B(t)K(t)/A(t)L(t)$, one can easily derive

$$C(t) = \left\{B(t)f(k(t)) - \left[\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} + \delta(t) - \frac{\dot{B}(t)}{B(t)}\right]k(t) - \dot{k}(t)\right\} \cdot \frac{A(t)L(t)}{B(t)} \tag{50}$$

where $f(k(t))=F[k(t),\ 1]$.

Next we define the Generalized Quasi-GR path, $\hat{k}(t)$, by

$$B(t)f'(\hat{k}(t)) = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} + \delta(t) - \frac{\dot{B}(t)}{B(t)} \cdot \tag{51}$$

This may be a variable interest-rate path.

It can then be shown, in precisely the same manner as before, that any path which makes $k(t) \geq \hat{k}(t)+\epsilon$, $\epsilon>0$, is dynamically inefficient.[12]

Then, for any $\epsilon>0$ and sufficiently large t,

$$C^*(t) - C(t) = \{[e^{\mu t}f(k(t) - \epsilon) - (\gamma + \lambda + \delta - \mu)(k(t) - \epsilon)] - [e^{\mu t}f(k(t)) - (\gamma + \lambda + \delta - \mu)k(t)]\} L_0 e^{(\gamma+\lambda-\mu)t} < 0$$

since, for any $\epsilon>0$,

$$\lim_{t\to\infty} [k(t) - \epsilon - \hat{k}(t)] < 0$$

and

$$[e^{\mu t}f(k(t)) - (\gamma + \lambda + \delta - \mu)k(t)] < [e^{\mu t}f(\hat{k}(t)) - (\gamma + \lambda + \delta - \mu)\hat{k}(t)] \quad \text{whenever } k(t) < \hat{k}(t).$$

Hence, $k(t)>\hat{k}(t)$ is not a sufficient condition that the path $k(t)$ be dominated.

It does not follow that (46) is necessary for a path to be dominated by a path described in (47), although that can probably be shown, at least on certain additional assumptions. In any case, it should be emphasized, however, that (46) is not a necessary condition for a $k(t)$ path to be dominated in *any* way. In other words, it is not argued that (46) is a necessary condition for dynamical inefficiency; it has only been suggested in the present paragraph that (46) is a necessary condition for a path to be dominated by a path which relates to it in the particular way specified in (47).

[12] In the purely labor-augmenting case, our theorems imply that all paths which keep the interest rate always finitely below the GR or Quasi-GR value are dynamically inefficient, provided that $(\dot{L}(t)/L(t)+(\dot{A}(t)/A(t))+\delta(t)$ has an upper bound. For if, for all t, $r(t)\leq\hat{r}(t)-\eta$, $\eta>0$, where $r(t)=f'(k(t))-\delta(t)$ and $\hat{r}(t)=f'(\hat{k}(t))-\delta(t)=(\dot{L}(t)/L(t))+(\dot{A}(t)/A(t))$, then $f'(\hat{k}(t))-f'(k(t))\geq\eta$; but if $f''(k)<0$ and $f'(\hat{k}(t))$ is bounded from above (because $(\dot{L}(t)/L(t))+(\dot{A}(t)/A(t))+\delta(t)$ is bounded), then it follows that $k(t)\geq\hat{k}(t)+\epsilon$ for some constant $\epsilon>0$.

But if there is capital-augmenting progress and $B(t)\to\infty$ as $t\to\infty$, then our theorems do

Note that if technical progress is Hicks-neutral, so that

$$Q(t) = A(t)\, F[K(t),\, L(t)]$$

then, since (by constant returns to scale)

$$A(t)\, F[K(t),\, L(t)] = F[A(t)\, K(t),\, A(t)\, L(t)],$$

we have $B(t) = A(t)$ and $(\dot{B}(t)/B(t)) = (\dot{A}(t)/A(t))$ in (51). In this case, the interest rate path corresponding to the Generalized Quasi-GR path is the same as for the case of no technical progress; the interest rate at t equals $\dot{L}(t)/L(t)$.

This observation suggests that if rates of factor augmentation are not defined then the Generalized Quasi-GR interest rate path is just the path of $\dot{L}(t)/L(t)$. We now demonstrate this.

C. *Nonfactor-Augmenting Progress*

Here we write the neoclassical production function in the form

$$Q(t) = F[K(t),\, L(t);\, t]. \tag{52}$$

Then, by constant returns to scale,

$$Q(t) = L(t) f(k(t);\, t) \tag{53}$$

where

$$k(t) = \frac{K(t)}{L(t)} \tag{54}$$

and

$$f(k(t);\, t) = F\left[\frac{K(t)}{L(t)},\, 1;\, t\right]. \tag{55}$$

From (53), (3) and (4) we have

$$\frac{C(t)}{L(t)} = f(k(t);\, t) - \delta k(t) - \frac{\dot{K}(t)}{L(t)}. \tag{56}$$

From (54) we have

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)} k(t). \tag{57}$$

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not imply that all paths which keep the interest rate finitely below the Quasi-GR value are dynamically inefficient. To see this, consider a path such that $r(t) \leq \hat{r}(t) - \eta$, $\eta > 0$, where now $r(t) = B(t) f'(k(t)) - \delta(t)$ and $\hat{r}(t) = B(t) f'(\hat{k}(t)) - \delta(t) = (\dot{L}(t)/L(t)) + (\dot{A}(t)/A(t)) - (\dot{B}(t)/B(t))$. Then $f'(\hat{k}(t)) - f'(k(t)) \geq \eta / B(t)$. If $B(t) \to \infty$ as $t \to \infty$ then, while $k(t) > \hat{k}(t)$ for all t, $k(t) \to \hat{k}(t)$ as $t \to \infty$ is possible. Hence "$k(t) \geq \hat{k}(t) + \epsilon$, $\epsilon > 0$" is not necessarily true of such a path, so the inefficiency of all such low-interest-rate paths is not implied. For $k(t) > \hat{k}(t)$ is not a sufficient condition that a path $k(t)$ be dominated, as the preceding footnote showed.

Equations (56) and (57) yield

$$C(t) = \left\{f(k(t);t) - \left[\frac{\dot{L}(t)}{L(t)} + \delta\right]k(t) - \dot{k}(t)\right\} L(t). \tag{58}$$

It is clear now that the Generalized Quasi-GR path, $\hat{k}(t)$, is defined by

$$f_k(\hat{k}(t);t) = \frac{\dot{L}(t)}{L(t)} + \delta. \tag{59}$$

It can be shown, by the same method that we have been using, that any path which, at t_0 and forever after, keeps $k(t) \geq \hat{k}(t) + \epsilon$ is dominated by a path $k^*(t) = k(t)$, $t < t_0$, $k^*(t) = k(t) - \epsilon$, $t \geq t_0$, so that such a path is dynamically inefficient.

Note that the interest rate, $f_k - \delta$, associated with the Generalized Quasi-GR path is the path of $\dot{L}(t)/L(t)$ which is independent of t. Hence if technical progress cannot be described in purely input-augmenting terms then the critical interest rate path is just the path of $\dot{L}(t)/L(t)$.

III. *Concluding Remarks*

It was demonstrated that a Golden-Rule path, that is, a consumption maximizing golden-age path, always exists in the neoclassical and Harrod-Domar models if the labor force increases at a constant rate, the depreciation rate is constant, technical progress, if any, is purely labor-augmenting, labor augmentation occurs at a constant rate, and positive labor is required for positive output. It was also demonstrated that a positive-investment GR path exists only if any technical progress present can be described as purely labor-augmenting.

It was then shown that any path which permanently deepens capital in excess of the GR path is dynamically inefficient—it is dominated with respect to consumption by another path. Further, if labor augmentation or labor-force growth is nonexponential or if technical progress cannot be described as purely labor-augmenting, then, while no GR path will exist, there may exist a Generalized Quasi-GR path having the same property, namely, that any path which permanently deepens capital in excess of that path is dynamically inefficient. (Note that such paths do not exhaust the class of dynamically inefficient paths. For example, even if no Quasi-GR path exists, the growth path produced by a permanently unitary saving ratio is clearly dynamically inefficient.)

Concerning the significance of these findings, I believe that it is of considerable theoretical interest to know that certain growth paths, even growth paths with continuously positive interest rate and less-than-unitary saving ratio, are dynamically inefficient. The practical importance of these findings is arguable. Beware of the weakness of what has been proved here. The growth paths shown to be dynamically inefficient are paths on which capital is excessive *forever*, that is, for infinite time. Whatever a nation does over a finite time cannot be shown to be dynamically inefficient in the sense of this paper; for what the nation does subsequently may save the entire growth path from being dominated.[13] At best, the economist armed with this paper can say to a country—be it a Soviet-type economy or a capitalist economy—that its public policies and private propensities are such that, if not *eventually* changed, dynamical inefficiency will result. But he cannot say that these policies must be changed within the year or in the next billion years. Such wisdom is not without practical value, I think. But it is to be hoped that some day economists will have stronger recommendations to make in the area of growth policy.

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[13] This observation leads to another qualification. In a world of uncertainty, as Pearce has observed [6], an economy may rationally deepen capital "excessively" in order to possess a "war chest" of capital for consumption in the event of an earthquake, a war, and other probabilistic phenomena. If these events never occur, so that the war chest is never consumed and capital is always "excessive," then, while the war chest strategy will be regretted from hindsight, it cannot be said to be irrational. But I doubt that such uncertainties are of sufficient quantitative importance to justify an appreciable war chest.

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