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ON THE CONSUMER'S LIFETIME ALLOCATION PROCESS*

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1. INTRODUCTION.

THE PURPOSE of this discussion is to study various aspects of consumer allocation over time with special emphasis on the notion that consumer preferences depend not only on the rate of consumption but also on terminal wealth (or bequests). This notion is, of course, not new. Marshall, for instance, refers to family affections as "the chief motive of saving" [8, (228)]. More recent authors, such as Fisher [4], Duesenberry [2], Friedman [5], and Modigliani and Brumberg [9], while aware of the bequest-motive, tend for various reasons to dismiss it from their theoretical discussions.[2]

Consider a consumer unit and let T be its *horizon*. T is a given positive real number. The closed interval $[0, T]$ will be referred to as the consumer's *lifetime* or, alternatively, as the consumer's *planning period*. The term "lifetime" is used because it may be reasonable to say that for some consumer units the lifetime of the head of the family is also the planning period.

The unit's *consumption plan* is a real-valued function c on the interval $[0, T]$. $c(t)$ is required to be non-negative for all t in $[0, T]$.

The unit's stream of *non-interest income* (or *contractual income*) is a real-valued function, to be denoted m, on $[0, T]$. We require that the integral of m exist, and for convenience we shall assume that $m(t)$ and $c(t)$ are measured in the same unit.

The consumer unit expects with certainty that a *rate of interest* of $r(t)$ per unit time will prevail at time t. r is assumed to be a bounded continuous real function on $[0, T]$.

Assuming that all assets (positive or negative) are held in the form of notes bearing interest at a rate of $r(t)$ at time t and that all loans are recontracted instantaneously, we have that the consumer's *bequests* (or *terminal assets*) are given by the quantity

$$(1)\qquad \left[\exp \int_0^T r(t)dt\right]S(0) + \int_0^T\left[\exp \int_t^T r(u)du\right][m(t) - c(t)]dt ,$$

where $S(0)$ is the consumer's initial assets. Mainly for typographical

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[1] I am indebted to Kenneth J. Arrow, Tjalling C. Koopmans, and Herbert E. Scarf for their comments.

[2] An exception is J. Tobin's unpublished dissertation [11].

convenience, we shall assume further that initial assets, $S(0)$, are zero, and that the rate of interest is constant and equal to some number j throughout the interval $[0, T]$. It can be shown that this last assumption can be made essentially without loss of generality so long as the function which describes the rate of interest is fixed and known in advance. Given these two assumptions, we can write bequests, to be denoted S, as follows

$$S = \int_0^T e^{j(T-t)}[m(t) - c(t)]dt \ . \tag{2}$$

It is convenient to express S in terms of the consumer's *lifetime wealth*, which will be denoted M,

$$M = \int_0^T e^{j(T-t)}m(t)dt \ , \tag{3}$$

which leads to

$$S = M - \int_0^T e^{j(T-t)}c(t)dt \ . \tag{4}$$

Turning now to the consumer's preferences, we assume that they can be represented by a utility function and, further, that this utility function, to be denoted V, is given by

$$V(c) = \int_0^T \alpha(t)g[c(t)]dt \ . \tag{5}$$

The function α has the interpretation of a *subjective discount* function, and it is assumed to be nonnegative and differentiable at least once. The function g is to be interpreted as the utility associated with the rate of consumption at every moment of time. g is defined on the half-line $[0, \infty)$, and we assume that it is twice differentiable and strictly concave.[3] Writing the utility function as in equation (5) involves the assumption that the consumer's preferences are independent over time.[4] This is, indeed, a very strong assumption. However, some such assumption must be made (with discrete time, authors usually assume homogeneity of the utility function) if specific implications are to be obtained.

When V is the relevant utility function, the consumer's decision problem is customarily stated as the maximization of V subject to the wealth constraint $S \geqq 0$. We shall refer to this problem as the *wealth constraint problem* (to be distinguished from the *bequest motive prob-*

[3] Strict concavity is not indispensable, but it guarantees that the optimal plan, if it exists, will be unique and continuous.

[4] See, for example, Debreu [1].

lem, which is described below).

To incorporate bequests in the utility function, we add to V one more term and thus obtain a new functional, to be denoted U,

$$U(c) = \int_0^T \alpha(t)g[c(t)]dt + \varphi[S] , \tag{6}$$

where φ is a real function defined for all real values and assumed to be twice differentiable and concave. φ is to be interpreted as the utility of bequests. As an alternative to the wealth constraint problem (i.e., the maximization of V subject to the wealth constraint), we shall consider the maximization of U subject only to the requirement that consumption be nonnegative. This problem will be referred to as the *bequest motive problem*.

2. OPTIMAL BEHAVIOR

In this section we shall attempt to characterize optimal behavior both in the framework of the wealth constraint problem and in that of the bequest motive problem. This will be followed by a brief comparison of the two problems.

2.1. *The wealth constraint problem*: Our aim is to maximize V over the set of all consumption plans which are nonnegative and satisfy the wealth constraint. In other words, we wish to maximize

$$\int_0^T \alpha(t)g[c(t)]dt \tag{7}$$

subject to:

$$c(t) \geqq 0 \qquad \text{for} \quad 0 \leqq t \leqq T ,$$

$$\int_0^T e^{j(T-t)}c(t)dt \leqq M .$$

In general, a solution for this problem may not exist. We shall proceed, however, on the assumption that a solution does exist. The first thing to notice is that without loss of generality one may assume that the wealth constraint $\int e^{j(T-t)}c(t)dt \leqq M$ actually holds with equality. For if it does not, then the determination of the optimal plan is a trivial affair. Next, it can be shown that without loss of generality one can take the optimal plan to be, say, *right-continuous*. In order to find a first-order condition which the optimal plan must satisfy, one proceeds as follows: Let c^* be the optimal plan and let R be the set on which c^* is positive. R must be a union of intervals. Now let x be a real valued function on $[0, T]$ satisfying the following conditions:

(8a) $$|x(t)| \leqq Kc^*(t) \qquad \text{for } 0 \leqq t \leqq T \text{ and for some } K > 0\ ;$$

(8b) $$\int_0^T e^{j(T-t)}x(t)dt = 0\ .$$

Note that condition (8a) implies that x must vanish outside the set R. Consider now the consumption plan $\bar{c}$ given by

(9) $$\bar{c}(t) = c^*(t) + \varepsilon x(t) \qquad \text{for all } t\ .$$

For small ε, the plan $\bar{c}$ is certainly admissible. The utility of the plan $\bar{c}$, i.e., $V(\bar{c})$, may be approximated for small ε by

(10) $$V(\bar{c}) = V(c^*) + \varepsilon \int_0^T \alpha(t)g'[c^*(t)]x(t)dt\ .$$

A necessary condition for c^* to be maximal is that the second term on the right of (10) vanish for all admissible choices of the function x. This is equivalent to saying that

(11) $$\int_0^T \{e^{j(t-T)}\alpha(t)g'[c^*(t)]\}\{e^{j(T-t)}x(t)\}dt = 0$$

for all choices of x satisfying (8).

A necessary and sufficient condition for (11) *to hold for all choices of x is given by*

(12) $$e^{j(t-T)}\alpha(t)g'[c^*(t)] = k \qquad \textit{for } t \textit{ in } R \textit{ and for some } k > 0\ .$$

PROOF. Sufficiency is immediate, in view of condition (8). To show necessity, consider two points, t_0 and t_1, in R and assume that neither of them is T. By right-continuity, we can find a number $h > 0$ such that the intervals $[t_0, t_0 + h]$ and $[t_1, t_1 + h]$ are both contained in R, and such that the quantity $e^{j(t-T)}\alpha(t)g'[c^*(t)]$ is continuous in t on both intervals. Now construct the function x as follows:

(13) $$x(t) = \begin{cases} \beta e^{j(t-T)} & \text{for } t \text{ in } [t_0, t_0 + h) \\ -\beta e^{j(t-T)} & \text{for } t \text{ in } [t_1, t_1 + h) \\ 0 & \text{otherwise}\ , \end{cases}$$

where $\beta > 0$ is selected so as to satisfy (8). Given this definition for x, reconstruct equation (11) and then let $h \to 0$. This will lead to equation (12).

It can easily be verified that for t outside the set R the inequality

(14) $$e^{j(t-T)}\alpha(t)g'[c^*(t)] \leqq k$$

must hold. Thus, the marginal utility condition for the wealth constraint problem is given by

$$(15)\qquad \begin{aligned} e^{j(t-T)}\alpha(t)g'[c^*(t)] &\leqq k && \text{for all } t^{5} \\ c^*(t) &= 0 && \text{whenever} < \text{holds}\,. \end{aligned}$$

The constant k is to be interpreted as the marginal utility of wealth as of time T.

Will the optimal plan c^* (if it exists) be continuous? The answer to this question is as follows:

As long as the first derivative of the function g is strictly decreasing, the optimal plan c^* is continuous.

PROOF. Suppose that a discontinuity occurs in c^* at the point t. This means (by right-continuity) that $c^*(t) \neq c^*(t-0)$. If g has a strictly decreasing first derivative, then this inequality can occur only if either

$$(16)\qquad \begin{aligned} e^{j(t-T)}\alpha(t)g'[c^*(t)] &= k\,, \\ e^{j(t-T)}\alpha(t)g'[c^*(t-0)] &< k\,, \end{aligned}$$

or

$$(17)\qquad \begin{aligned} e^{j(t-T)}\alpha(t)g'[c^*(t)] &< k\,, \\ e^{j(t-T)}\alpha(t)g'[c^*(t-0)] &= k\,. \end{aligned}$$

If (16) were to hold, then we would have

$$(18)\qquad g'[c^*(t-0)] < g'[c^*(t)]\,,$$

but we would also have, in view of (15), that

$$(19)\qquad c^*(t) > 0 = c^*(t-0)\,.$$

The inequalities in (18) and (19) contradict the concavity of g. A similar argument would hold if (17) were true.

Next, we wish to investigate the differentiability of the optimal plan c^*. The following statement can be made:

If the function g has a continuous and negative second derivative, then the optimal plan c^* is differentiable in the interior of the set R.

PROOF. Equation (12) implies that $dg'[c^*(t)]/dt$ exists for t in the interior of R. By the Mean Value Theorem, we can write

$$(20)\qquad \frac{g'[c^*(t+h)] - g'[c^*(t)]}{h} = \frac{c^*(t+h) - c^*(t)}{h} g''[c^*(t) + \delta]$$

where $\delta \rightarrow 0$ as $h \rightarrow 0$. Since the limit as $h \rightarrow 0$ of the left-hand side

[5] No problem arises at $t = T$, because c^* can be adjusted at T so as to satisfy (15).

of (20) exists, the limit of the right-hand side must also exist. Hence, if $g'' < 0$, the derivative of c^* at t must exist.

Thus, we see that it is permissible to differentiate equation (12) with respect to t and thereby obtain a basic differential equation in the optimal plan c^*. This differential equation turns out to be

$$\dot{c}^*(t) = -\left\{j + \frac{\dot{\alpha}(t)}{\alpha(t)}\right\}\frac{g'[c^*(t)]}{g''[c^*(t)]} \qquad \text{for } t \text{ in the interior of } R.[6] \tag{21}$$

Equation (21) implies that the sign of $\dot{c}^*(t)$ is the same as the sign of the quantity $[j + \dot{\alpha}(t)/\alpha(t)]$. Now $-\dot{\alpha}(t)/\alpha(t)$ is in fact the *relative rate of subjective discount* at time t. Thus, if the rate of interest is greater than the rate of subjective discount, then the optimal consumption plan is increasing, and if the rate of subjective discount is greater than the rate of interest, then the optimal plan is decreasing.

2.2. *The bequest-motive problem*: Suppose now that the utility function is U rather than V, and that it is to be maximized over all nonnegative consumption plans. In other words, we wish to maximize

$$\int_0^T \alpha(t)g[c(t)]dt + \varphi(S) \tag{22}$$

subject to:

$$c(t) \geqq 0 \qquad \text{for all } t\,.$$

It is not difficult (in fact, it is even easier than in the wealth constraint problem) to derive a marginal utility condition for this problem. As might be expected, a necessary condition for a consumption plan c^* to be optimal is

$$\begin{aligned} &e^{j(t-T)}\alpha(t)g'[c^*(t)] \leqq \varphi'(S^*) \\ &c^*(t) = 0 \qquad \text{whenever} < \text{holds}\,, \end{aligned} \tag{23}$$

where S^* is the level of bequests under the consumption plan c^*. This condition is the same as condition (15) of the wealth-constraint problem, except that the constant on the right-hand side is now simply the marginal utility of bequests.

The conditions for continuity and differentiability of the optimal consumption plan remain the same as in the wealth-constraint approach. The optimal consumption plan (if it exists) must once again satisfy the differential equation (21) wherever it is positive.

We conclude this section with a few words on the differences between the wealth-constraint approach and the bequest-motive approach.

Some authors, for example, Modigliani and Brumberg [9 (Assump-

[6] Dots above symbols indicate time derivatives.

tion I)], describe the absence of a bequest motive as a simplifying assumption. The foregoing paragraphs show that in the present framework such an assumption does not give rise to any gain in simplicity. If anything, the wealth-constraint approach is slightly more complicated than the bequest-motive approach, because of an additional constraint in the maximization. It seems, therefore, that there should be no major difficulty in taking theoretical account of the empirical fact that the terminal assets of households (plus transfers made to children along the way) are often unequal to initial assets.

As will be seen below, under the wealth-constraint approach, the elasticities of consumption with respect to wealth cannot all be less than one. This may be another reason for preferring the bequest-motive approach.

Finally, it might be mentioned that the wealth constraint approach can in a sense be looked upon as a limiting case of the bequest motive appraoch. To make this statement more precise, define the utility-of-bequests function, φ, as follows:

$$\varphi(x) = \delta u(x) , \tag{24}$$

where δ is an arbitrary positive real number and u is a concave real function satisfying the requirement

$$\lim_{x \to 0} u'(x) = +\infty . \tag{25}$$

Now solve the bequest motive problem. The optimal plan (if it exists) will depend on δ. Denote it c_δ^*. It can be shown that as $\delta \to 0$, the plans c_δ^* approach a limit, say c^*, and that c^* is in fact a solution for the corresponding wealth constraint problem.[7]

3. WEALTH EFFECTS AND INCOME EFFECTS

By "wealth effects" we have in mind the effects which differences in lifetime wealth, M, have on the consumer's optimal plan. We shall use the bequest motive problem as our framework.

For every t in the interval $[0, T]$, let $c_M(t)$ be defined as the partial derivative $\partial c^*(t, M)/\partial M$, where c^* is the optimal consumption plan. It can be shown that $c_M(t)$ exists for every t, provided that g possesses a continuous and negative second derivative. Now define a quantity S_M as follows:

$$S_M = 1 - \int_0^T e^{j(T-t)} c_M(t) dt . \tag{26}$$

S_M is clearly the derivative of S^* (the optimal level of bequests) with

[7] In the sense of convergence in the mean (i.e., L_1-convergence).

respect to M. Finally, let f denote the maximum attainable utility, taken here as a function of M:

$$f(M) = \int_0^T \alpha(t)g[c^*(t)]dt + \varphi(S^*) . \tag{27}$$

Differentiate f with respect to M:

$$f'(M) = \int_0^T \alpha(t)g'[c^*(t)]c_M(t)dt + S_M\varphi'(S^*) . \tag{28}$$

Let R be the set on which c^* is positive and let $\bar{R}$ be the closure of R. It is easy to show, using condition (23), that for t outside $\bar{R}$, $c_M(t)$ is zero. It follows that we can substitute (23) in (28) as if (23) held everywhere with equality. This leads to

$$f'(M) = \varphi'(S^*) = e^{j(t-T)}\alpha(t)g'[c^*(t)] , \tag{29}$$

the second of these holding for t in $\bar{R}$. By differentiating (28), again with respect to M, one obtains (following simplification)

$$f''(M) = \int_0^T \alpha(t)g''[c^*(t)]c_M^2(t)dt + S_M^2\varphi''(S^*) , \tag{30}$$

which is always negative. It is now possible to differentiate (29) with respect to M and use the fact that $f''(M)$ is negative. This leads to

$$S_M > 0 \tag{31}$$

and to

$$c_M(t)\begin{cases} \geqq 0 & \\ > 0 & \text{for } t \text{ in } \bar{R} . \end{cases} \tag{32}$$

Equations (31) and (32) together imply that

$$0 < \int_0^T e^{j(T-t)}c_M(t)dt < 1 . \tag{33}$$

The quantity S_M is the consumer's *lifetime marginal propensity to save*. If an economy existed in which all consumer units were like our unit here (except for differences in wealth) then, abstracting from the dynamic effects of changes in the population structure, S_M should be the cross-section estimate of the marginal propensity to save. But S_M is the derivative of a *stock* with respect to a *stock*, while we usually think of the marginal propensity to save as the derivative of a *flow* with respect to a *flow*. For this reason, many economists have tried to define flow variables which would adequately represent the underlying stock variables. The main problem is in defining a flow variable which will give adequate representation to wealth. Probably the best known definition in this area is Hicks' definition (in [7]) of income as the flow which would have to be consumed if capital (i.e., wealth) were to remain intact. Friedman, in [5] and in [6], accepts Hicks' definition of income on the theoretical level, but on the practical level he defines "permanent income" in a somewhat different manner.

The fact that S_M is, to use Friedman's word, the "permanent" marginal propensity to save, can be used to derive a natural definition for an income variable which would represent wealth adequately. Suppose y is that income variable. Let $c_y(t)$ be the derivative $\partial c^*(t)/\partial y(t)$. Then $1 - c_y(t)$ is the short-run marginal propensity to save at time t. If y is to be defined properly, then the long-run marginal propensity to save, S_M, should be a weighted average of the short-run propensities, $1 - c_y(t)$ for $0 \leqq t \leqq T$, with weights proportional to e^{-jt} (since interest causes early saving to have more weight than late saving in total lifetime savings). It is easy to see, using equation (26), that the only definition of y which satisfies this requirement is

$$y(t) = \frac{jM}{e^{jT} - 1} \qquad \text{for all } t \,. \tag{34}$$

Thus, y is that constant stream which, if compounded at the rate of interest j, would in T years accumulate to become equal to M. This is precisely what Farrell [3] calls *normal income*. If the consumer's contractual income, m, happens to be constant over his lifetime then contractual income and permanent (or normal) income coincide. It seems that this is in keeping with what Friedman has in mind.

Note that permanent income, as defined in (34), does not in general conform with Hicks' definition of income. According to Hicks' definition, income is given by the quantity

$$je^{-jT}M \tag{35}$$

for all t. If we let $\bar{y}$ be permanent income of equation (34), i.e., $\bar{y} = jM/(e^{jT} - 1)$, then Hicks' income is equal to $(1 - e^{-jT})\bar{y}$. Hicks' income coincides with $\bar{y}$ only if the horizon, T, is infinite.

Let us turn now to wealth elasticities. Define $\eta_c(t)$ and η_S as follows:

$$\begin{aligned} \eta_c(t) &= \frac{Mc_M(t)}{c^*(t)} && \text{for } t \text{ in } R \,, \\ \eta_S &= \frac{MS_M}{S^*} && \text{for } S^* > 0^{8} \end{aligned} \tag{36}$$

where R is the set on which $c^*(t) > 0$. It should be noted that $\eta_c(t)$

[8] As usual, we define elasticities only for the case where both quantities are positive.

and η_S are wealth elasticities as well as income elasticities, so long as income is defined as some function of t times M.

Let us return for a moment to the wealth-constraint problem. The wealth constraint states that

$$\int_R e^{j(T-t)} c^*(t)dt = M , \tag{37}$$

from which it follows, upon differentiation and conversion to elasticities, that

$$\int_R e^{j(T-t)} c^*(t)[1 - \eta_c(t)]dt = 0 . \tag{38}$$

Thus, either $\eta_c(t) = 1$ for all t (except possibly on a set of measure zero), or else there must exist a set of positive measure on which $\eta_c(t) > 1$. In particular, $\eta_c(t) < 1$ for all t is an impossibility.[9] If one adopts the wealth-constraint approach, one commits oneself in advance to the statement that income elasticities of consumption are either all equal to one, or else some of them are strictly greater than one. In the bequest-motive case the situation is different. The equation, which is analogous to (38), is given by

$$\int_R e^{j(T-t)} c^*(t)[1 - \eta_c(t)]dt = (\eta_S - 1)S^*, \tag{39}$$

so it is entirely possible that $\eta_S > 1$ and $\eta_c(t) < 1$ for all t.

It is of course possible to take the position that the statement

$$\eta_c(t) = 1 \qquad \text{for all } M \text{ and for } t \text{ in } R \tag{40}$$

is an entirely reasonable one. Indeed, there exists empirical evidence to support this view. It is, therefore, of interest to explore the consequences of this statement. First of all, it is clear from equation (39) that if (40) holds then $\eta_S = 1$ holds as well. Secondly, the fact that all the income-elasticities are equal to one means that

$$\begin{aligned} c^*(t) &= M\gamma(t) , \\ S^* &= \lambda M \end{aligned} \tag{41}$$

for some nonnegative function γ and for some positive real number λ. By differentiating the marginal utility condition (23) with respect to M, one obtains (upon setting all the η's equal to 1)

$$\frac{M\gamma(t)g''[M\gamma(t)]}{g'[M\gamma(t)]} = \frac{\lambda M\varphi''(\lambda M)}{\varphi'(\lambda M)} \tag{42}$$

for all t in R and for all M. Now, if the consumption plan is not

[9] Herbert Scarf has a somewhat similar result in [10].

constant, i.e., if $e^{jt}\alpha(t)$ is not constant on $[0, T]$, then it is possible to simplify the right-hand side of (42), making use of the differential equation (21). As a result, one obtains

$$(43) \qquad -\frac{\gamma(t)}{\dot{\gamma}(t)}\left\{j + \frac{\dot{\alpha}(t)}{\alpha(t)}\right\} = \frac{\lambda M\varphi''(\lambda M)}{\varphi'(\lambda M)}$$

for all M and for those t in the interior of R for which $j + \dot{\alpha}(t)/\alpha(t) \neq 0$. But the left-hand side of (43) is independent of M and the right-hand side of (43) is independent of t. So, by varying M for a fixed t we obtain that φ must be such that $x\varphi''(x)/\varphi'(x)$ is constant for all x, and by varying t for a fixed M we obtain that $xg''(x)/g'(x)$ must be constant for all x in the range of c^*. But this can only be true if both g and φ are of the form $\log x$ or x^β for some $0 < \beta < 1$. To sum up, we have the following statement: If income elasticities are all equal to unity and if the optimal consumption plan is not constant on $[0, T]$, then the utility function must be such that both g and φ are either the logarithm or some power function (or some linear transformations of same).

4. RATE OF INTEREST EFFECTS

We come now to the question of how differences in the rate of interest affect the consumer's optimal plan.

One usually thinks of a change in the rate of interest as having two effects: first, it changes interest earnings; second, it causes a revaluation of existing assets and thus gives rise to capital gains or capital losses. In the present framework we are assuming instantaneous recontracting so that the distinction between interest earnings and capital gains vanishes. Assets are automatically revalued at every instant. This is, of course, a shortcoming of the present model, because it does not permit the distinction between the value of assets and their earning power. Another question concerns the assumption that the rate of interest is constant on the interval $[0, T]$. It can be shown, however, that relaxing this assumption is practically of no value unless we make additional assumptions having to do with the time-shape of the stream of contractual earnings, m.

As usual, we shall let c^* be the optimal consumption plan. For each t in $[0, T]$, let $c_j(t)$ be defined as the partial derivative $\partial c^*(t, j)/\partial j$. Given our assumptions on g and on φ, it can be shown that this partial derivative exists, and that it vanishes outside the closure of the set R, where R is the set on which $c^*(t) > 0$. Next, let S_j be the derivative of S^* with respect to j. S_j is given by

$$(44) \qquad S_j = \int_0^T (T - t)e^{j(T-t)}[m(t) - c^*(t)]dt - \int_0^T e^{j(T-t)}c_j(t)dt \; .$$

Equation (44) shows the two-fold effect which a change in the rate of interest has on S^*. First, there is an effect which occurs even in the absence of any adjustment of the consumer's plans. This effect is given by the derivative of S^* with respect to j, holding c^* constant. For convenience, we shall use the symbol S^j to denote this derivative,

$$(45) \qquad S^j = \int_0^T (T - t)e^{j(T-t)}[m(t) - c^*(t)]dt \; .$$

Second, the effect on S^* which is brought about by the adjustment of the consumer's optimal plan is given by the second term in the right-hand side of (44).

It should be noted here that S^j depends upon the time shape of the stream of contractual earnings, m. The consumer's decisions in themselves do not determine S^j, or even its sign. It is always possible to find two earnings streams, say m and m', such that both of them will lead to the optimal plan c^*, but one will lead to a positive S^j while the other will lead to a negative S^j. It will, however, be of interest to see if, given the sign of S^j, anything can be said about the sign of the total derivative, S_j.

Differentiation of the marginal utility condition (23) with respect to j leads to the following equation:

$$(46) \qquad c_j(t) = (T - t)\frac{g'[c^*(t)]}{g''[c^*(t)]} + \frac{e^{j(T-t)}\varphi''(S^*)}{\alpha(t)g''[c^*(t)]}S^j$$

for t in the set R. Assuming that T is in the set R, we can write

$$(47) \qquad c_j(T) = \frac{\varphi''(S^*)}{\alpha(T)g''[c^*(T)]}S_j \; ,$$

which implies that $c_j(T)$ and S_j have the same sign. In other words, the effect of a change in the rate of interest on bequests is in the same direction as its effect on consumption in the latter periods of the consumer's lifetime.

Suppose now that $S_j \leqq 0$. Then, by (46), $c_j(t) < 0$ for all t in R. But if this is true, then, by (44), we must have $S^j < 0$. Thus, by contraposition, we obtain the following:

$$(48) \qquad \text{If} \quad S^j \geqq 0 \quad \text{then} \quad S_j > 0 \; .$$

If $S^j \geqq 0$, then a rise in the rate of interest stimulates bequests, but it is also likely to stimulate consumption, especially in periods near the horizon T, and possibly in all periods.

Similarly, it can be shown that

$$\text{(49)} \qquad \text{If} \quad S^j < 0 \quad \text{then} \quad c_j(t) < 0 \quad \text{for some } t \text{ in } R\,.$$

So, when $S^j < 0$ a rise in the rate of interest depresses consumption in some periods, but it is possible that it also depresses bequests, in which case, by equation (46), it depresses consumption in *all* periods. Thus, it is entirely possible for a rise in the rate of interest to lower consumption in all periods and at the same time to lower bequests.

The foregoing observations serve to show that statements like "a rise in the rate of interest depresses consumption and stimulates saving" may often be rather misleading.

Finally, it may be of interest to see whether a given change in the rate of interest makes the consumer better off or worse off. Let f be total attainable utility, written as a function of the rate of interest,

$$\text{(50)} \qquad f(j) = \int_0^T \alpha(t)g[c^*(t)]dt + \varphi(S^*)\,.$$

Differentiating with respect to j and simplifying, one obtains

$$\text{(51)} \qquad f'(j) = S^j\varphi'(S^*)\,,$$

which implies that the sign of $f'(j)$ is the same as the sign of S^j. Thus, if a change in the rate of interest in and of itself (i.e., with the consumption plan held constant) causes lifetime savings to be greater, then the consumer should hope for such a change. If, on the other hand, such a change causes lifetime savings to decline, then the consumer should hope for the opposite change.

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