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ON THE EXISTENCE OF AN OPTIMAL PLAN IN A CONTINUOUS-TIME ALLOCATION PROCESS[1]

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Every now and then, one encounters an allocation problem in which the "choice variable" is a point in an infinite-dimensional space. In such problems, the existence of an optimal choice among all permissible choices is not always assured. The present discussion is devoted to an investigation of this issue for a specific (but fairly common) class of allocation problems. Conditions for the existence of an optimal choice are derived and discussed.

1. INTRODUCTION

THE OBJECT of this discussion is to investigate the existence of a solution in a specific class of allocation problems. These problems may be characterized as follows: Let an objective function, U, be defined by

$$U(c) = \int_0^1 \alpha(t) g[c(t)]\, dt\,,$$

where α is a bounded nonnegative and continuous real function, defined on the interval $[0, 1]$, and g is a concave[2] real function, defined on the half-line $[0, \infty)$.

The integral in the definition of U is taken in the sense of Lebesgue, so that $U(c)$ is well defined for all bounded measurable functions c on the unit interval.

Now consider the set C such that $c \in C$ if and only if the following three conditions hold:

(i) c is a bounded measurable real function on $[0, 1]$;
(ii) $c(t) \geqslant 0$ for all t in $[0, 1]$;
(iii) $\int_0^1 c(t)\, dt \leqslant \lambda$,

where λ is a given real number such that $0 < \lambda < \infty$. It should be noted that we are not assuming here that the set C is uniformly bounded. In fact, given any real number K, no matter how large, there exist elements c of C such that $c(t) > K$ for some t.

PROBLEM: Is there a $c^* \in C$ such that $U(c^*) \geqslant U(c)$ for all $c \in C$?

The first thing to notice is that in the absence of further qualifications the answer to this question must, in general, be in the negative. For consider simply the case

[1] This investigation would not have reached completion were it not for the help of Joram Lindenstrauss of Yale University. I have also benefited from many discussions with William Brainard and Emmanuel Drandakis.

[2] I.e., g is such that $g[\lambda x + (1 - \lambda)y] \geqslant \lambda g(x) + (1-\lambda)g(y)$ for all x and y in the domain of g and for all λ in the unit interval.

where $g(x) = x$ for all $x \geqslant 0$ and α is a strictly decreasing function. Regardless of the value of λ, there is no $c^* \in C$ such that $U(c^*) \geqslant U(c)$ for all $c \in C$. In fact, $U(c) < \lambda\alpha(0)$ for all $c \in C$, but a sequence c_n can always be picked from C such that lim $U(c_n) = \lambda\alpha(0)$.[3] Hence, for any $c \in C$ there exists a $c' \in C$ such that $U(c') > U(c)$. One is therefore faced with the task of searching for conditions which ensure the existence of an optimal plan c^* in C.

The problem stated above occurs every now and then in discussions of allocation over time. Consider, for instance, the following interpretation: Let c be a consumption program and let C be the set of all admissible programs. U is a utility functional, with α having the interpretation of a subjective discount function and with g being a utility associated with the rate of consumption at every moment of time. The interval [0, 1] is the planning period, and the problem is one in which the consumer wishes to find the consumption program c^* which best allocates his total wealth, λ.[4] In this consumer's choice interpretation the problem appears, for instance, in Strotz [**8**]. Other variants of the problem, sometimes modified or generalized, can be found also in studies of allocation over time in production or capacity planning, such as Arrow-Karlin [**1**], Koopmans [**6**], and others.

In a recent article [**2**], S. Chakravarty investigates the existence of an optimal plan in allocation problems which are in many ways similar to the ones discussed here. Chakravarty is concerned principally with problems arising from the divergence of the value of the objective functional for some admissible plans. In the present discussion we concentrate on the fact that even when matters are fixed in such a way that the objective functional has a finite value for all admissible plans, the existence of an *optimal* plan is not assured. Mathematically speaking, the difficulty rests in the fact that given any of the usual topologies, the set C of admissible plans is noncompact.

A unique approach to the existence problem is contained in a contribution by S. Karlin [**5**, Vol. II, pp. 210–214]. The author develops a constructive method for characterizing the optimal plan and at the same time exhibiting its maximal property. Thus, existence is always assured. In Karlin's investigation, the function g (in our notation) is strictly concave and bounded. (Karlin also assumes that the discount function, α, is strictly decreasing.) We shall see that, indeed, whenever g is a bounded function, an optimal plan always exists. It can also be shown that if g is strictly concave, then the optimal plan is unique.

[3] For instance, consider the sequence whose typical element is given by

$$c_n(t) = n\lambda \quad \text{for} \quad 0 \leqslant t < \frac{1}{n},$$
$$= 0 \quad \text{for} \quad \frac{1}{n} \leqslant t \leqslant 1.$$

[4] The rate of interest is assumed to be identically equal to zero on the interval [0, 1]; otherwise, the constraint $\int c(t)dt \leqslant \lambda$ must be replaced by a more general linear constraint, say $\int r(t)c(t)dt \leqslant \lambda$. This will be done below.

The objective of the present investigation is to isolate the existence aspect of the problem, while dropping the requirement that g be strictly concave and bounded. Once the existence of an optimal plan has been assured, one could presumably turn to classical variational methods for a characterization of this plan, taking proper account of nonnegativity constraints.

2. PRELIMINARIES

In this section a brief argument will be made to the effect that without loss of generality we may take the function g to be nondecreasing, and this will permit us to replace the constraint $\int c(t)\,dt \leqslant \lambda$ by $\int c(t)\,dt = \lambda$. Both of these facts are well known.

LEMMA 1: *Suppose that there exists some* $K \geqslant 0$ *such that* g *is nondecreasing on* $[0, K)$ *and strictly decreasing on* (K, ∞). *Let* $\hat{g}$ *and* $\hat{U}$ *be defined as follows*:

$$(1) \qquad \begin{aligned} \hat{g}(x) &= g(x)\,, & 0 \leqslant x \leqslant K\,; \\ &= g(K)\,, & K \leqslant x < \infty\,, \end{aligned}$$

and

$$(2) \qquad \hat{U}(c) = \int_0^1 \alpha(t)\,\hat{g}[c(t)]\,dt\,, \qquad c \in C\,.$$

If $\max_{c \in C} \hat{U}(c)$ *is attained, then* $\max_{c \in C} U(c)$ *is also attained.*

PROOF: Suppose $\max_{c \in C} \hat{U}(c)$ is attained at $\bar{c}$. In other words, $\bar{c}$ is in C and

$$(3) \qquad \hat{U}(\bar{c}) \geqslant \hat{U}(c) \quad \text{for all} \quad c \in C\,.$$

Define $\hat{c}$ as follows:

$$(4) \qquad \hat{c}(t) = \min\,(\bar{c}(t), K) \qquad 0 \leqslant t \leqslant 1\,.$$

Clearly, $\hat{U}(\hat{c})$ is also a maximum:

$$(5) \qquad \int_0^1 \alpha(t)\,\hat{g}[\hat{c}(t)]\,dt \geqslant \int_0^1 \alpha(t)\,\hat{g}[c(t)]\,dt \quad \text{for all} \quad c \in C\,.$$

We also know, since $\hat{g} \geqslant g$ and α is nonnegative, that

$$(6) \qquad \int_0^1 \alpha(t)\,\hat{g}[c(t)]\,dt \geqslant \int_0^1 \alpha(t)\,g[c(t)]\,dt \quad \text{for all} \quad c \in C\,.$$

But, by (4), $\hat{c}(t) \leqslant K$ for all t, so that $\hat{g}[\hat{c}] = g[\hat{c}]$. Hence

$$(7) \qquad \int_0^1 \alpha(t)\,g[\hat{c}(t)]\,dt \geqslant \int_0^1 \alpha(t)\,g[c(t)]\,dt \quad \text{for all} \quad c \in C\,,$$

i.e., $U(\hat{c}) \geqslant U(c)$ for all $c \in C$, as was to be shown.

Next, having taken g to be nondecreasing, one observes that the set C of admissible plans may now be restricted to a subset C_λ, defined by

$$(8) \qquad C_\lambda = \left\{ c \colon c \in C, \int_0^1 c(t)dt = \lambda \right\} .$$

This fact is an immediate consequence of

LEMMA 2: *If g is nondecreasing, then for every* $c \in C$ *there exists a* $c' \in C_\lambda$ *such that* $U(c') \geqslant U(c)$.

The proof is immediate, and will therefore be omitted.

3. THE CASE OF A NONINCREASING DISCOUNT FUNCTION

In many specific examples it is reasonable to assume that the function α is nonincreasing. Starting the discussion with this case, we find that one result is readily obtainable.

REMARK: *If* α *is constant, then an optimal plan always exists.*

PROOF: Set $\alpha(t) = 1$ for $0 \leqslant t \leqslant 1$.[5] By concavity of g, one obtains

$$(9) \qquad U(c) = \int_0^1 g[c(t)]\,dt \leqslant g[\int_0^1 c(t)\,dt] = g(\lambda)$$

for all $c \in C_\lambda$. Letting $c^*(t) = \lambda$ for all t in $[0, 1]$ leads to

$$(10) \qquad U(c^*) = \int_0^1 g(\lambda)\,dt = g(\lambda) ,$$

so that $U = g(\lambda)$ is attained.

We now turn to the less trivial case, where α is not a constant. To do this we need to devote a short digression to the notion of a monotone rearrangement of a function.

Monotone Rearrangements

The literature in economics contains occasional reference to the operation of rearranging a function in a monotonically decreasing manner. The explanation for a downward sloping marginal-efficiency-of-investment schedule in a case in point. Unfortunately, more often than not authors neglect to provide a rigorous defini-

[5] There is no loss of generality in setting $\alpha \equiv 1$, and in general in normalizing α so that, say, $\int_0^1 \alpha(t)\,dt = 1$. This follows from the well known fact that the function g is arbitrary up to a linear transformation.

tion of this notion. Given any measurable function on the unit interval, there are several paths which one could take in order to construct its monotone rearrangement. The present section follows the footsteps of Hardy, Littlewood, and Polya [**4**, pp. 276–295].

Let y be a bounded measurable function on [0, 1], and let $S(y; v)$ be the set on which $y \geqslant v$:

$$(11) \qquad S(y; v) = \{t: y(t) \geqslant v\}\,.$$

Now define a function φ_y as follows:

$$(12) \qquad \varphi_y(v) = mS(y; v)\,,$$

where m denotes Lebesgue measure. The function φ_y is nonincreasing and left-continuous. Consider the right-continuous inverse of φ_y, to be denoted $\bar{y}$:

$$(13) \qquad \begin{aligned} &\bar{y}[\varphi_y(v)] = v \quad \text{for all} \quad v\,, \\ &\bar{y}(t+0) = \bar{y}(t) \quad \text{for all} \quad t \ \text{in} \ [0, 1)\,. \end{aligned}$$

$\bar{y}$ is well defined by (13) on the interval [0, 1). To define $\bar{y}(t)$ at $t = 1$, let us agree that

$$(14) \qquad \bar{y}(1) = \inf_{t \in [0,1)} \bar{y}(t)\,.$$

$\bar{y}$ will be referred to as *the nonincreasing rearrangement of* y. The following are three properties of nonincreasing rearrangement, stated without proof:

First, let f be a measurable function which is bounded on the range of y. Then

$$(15) \qquad \int_0^1 f(\bar{y}(t))dt = \int_0^1 f(y(t))\,dt\,.$$

Second, if furthermore, the function f is nondecreasing and if one defines $\psi(t) = f(y(t))$ for all t in [0, 1], then

$$(16) \qquad \bar{\psi} = f(\bar{y})\,.$$

Finally, for any two bounded and measurable functions x and y on [0, 1], the inequality

$$(17) \qquad \int_0^1 x(t)y(t)\,dt \leqslant \int_0^1 \bar{x}(t)\,\bar{y}(t)\,dt$$

is valid. With these observations we return to the main line of the discussion.

Existence of Solutions

Let $\bar{C}_\lambda$ be the set of all nonincreasing and right-continuous members of C_λ:

$$(18) \qquad \bar{C}_\lambda = \{c: \ c \in C_\lambda, \ c \ \text{nonincreasing and right-continuous}\}\,.$$

Consider any function $c \in C_\lambda$. Its nonincreasing rearrangement, $\bar{c}$, is in the set $\bar{C}_\lambda$. In fact, $\bar{c}$ is clearly bounded and measurable, and by (15),

$$\int_0^1 \bar{c}(t)\,dt = \lambda\,, \tag{19}$$

and hence $\bar{c} \in \bar{C}_\lambda$. As a matter of fact, the set $\bar{C}_\lambda$ coincides with the set of nonincreasing rearrangements of all functions $c \in C_\lambda$. Using the properties of monotone rearrangements, one can readily establish the following proposition:

LEMMA 3: *If α is a nonincreasing function, then for any $c \in C_\lambda$, the inequality $U(\bar{c}) \geqslant U(c)$ is valid.*

In other words, when α is nonincreasing, the search for an optimal plan may be restricted to the set $\bar{C}_\lambda$.

PROOF: Since g is nondecreasing, (16) says that $g(\bar{c})$ is the nonincreasing rearrangement of $g(c)$. Now, if α is nonincreasing, then $\bar{\alpha}$ and α coincide, and property (17) yields

$$\int_0^1 \alpha(t)g[\bar{c}(t)]\,dt \geqslant \int_0^1 \alpha(t)g[c(t)]\,dt \quad \text{for all} \quad c \in C_\lambda\,, \tag{20}$$

which completes the proof.

At this point it becomes possible to state a necessary and sufficient condition for the existence of a solution for the maximization problem:

THEOREM 1: *For the maximum of the function U to be attained in the set $\bar{C}_\lambda$, it is necessary and sufficient that there exists a real number K which has the property that for any $c \in \bar{C}_\lambda$ there exists a $c^* \in \bar{C}_\lambda$ such that $c^*(t) \leqslant K$ for all t and $U(c^*) \geqslant U(c)$.*

In other words, the maximum is attained in $\bar{C}_\lambda$ if and only if the search for the optimal plan can be restricted to a uniformly bounded subset of $\bar{C}_\lambda$.

PROOF: Necessity is immediate, in view of the fact that all the functions in the set $\bar{C}_\lambda$ are bounded. In order to show sufficiency, assume that a real number K, satisfying the conditions of the theorem, exists. Let $\{c_n\}$ be a sequence of members of $\bar{C}_\lambda$ having the following properties:

$$c_n(t) \leqslant K \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad n = 1, 2, \ldots, \tag{21}$$

and

$$\lim_{n\to\infty} U(c_n) = \sup_{c \in \bar{C}_\lambda} U(c)\,.^6 \tag{22}$$

[6] It is easily seen that the supremum in the right-hand side of equation (22) is always finite.

Selecting such a sequence is clearly possible. Now, since the functions of the sequence $\{c_n\}$ are monotone and uniformly bounded, one can apply Helly's Theorem,[7] which states that there exists a subsequence of $\{c_n\}$ which converges at every point of the interval [0, 1]. More precisely, Helly's Theorem states that there exists a subsequence $\{c_k\}$ of $\{c_n\}$ and a bounded nonincreasing function c^* such that

$$(23) \qquad \lim_{k\to\infty} c_k(t) = c^*(t) \text{ for all } t \text{ in } [0, 1] .$$

To show that $c^* \in \bar{C}_\lambda$ we need only prove that $\int c^*(t)\,dt = \lambda$. But by a Theorem of Lebesgue[8] we have

$$(24) \qquad \int_0^1 c^*(t)\,dt = \lim_{k\to\infty} \int_0^1 c_k(t)\,dt = \lambda ,$$

so c^* is in $\bar{C}_\lambda$. Finally, by applying the same theorem of Lebesgue to the functional U, one obtains

$$(25) \qquad \sup_{c\in \bar{C}_\lambda} U(c) = \lim_{k\to\infty} U(c_k) = U(c^*) .$$

Hence, the maximum of U is attained in $\bar{C}_\lambda$. This completes the proof.

An immediate consequence of Theorem 1 is that if the class of admissible plans is uniformly bounded to begin with, then the existence of a solution is no longer a problem. In other words, if one adds to the definition of the set C of admissible plans the requirement that $c \in C$ only if $c(t) \leqslant K$ for all t, then attainment of the maximum in C is automatically assured. It is indeed quite reasonable in many contexts to impose on the function c the constraint that it be less than some K at all points. However, it is important to note that imposing this additional constraint, while eliminating the existence problem, increases (sometimes considerably) the difficulty of characterizing the solution, because it makes it necessary to determine the set in the interval [0, 1] on which this additional constraint is effective (i.e., the set on which $c(t) = K$).

Before going on to make further use of Theorem 1, it is necessary to recall the differentiability properties of the function g. It follows from concavity that g possesses a derivative at all points of its domain, except possibly for a denumerable number of them.[9] The symbol $g'(x)$ will be the derivative of g at x, and if a derivative does not exist at x, then $g'(x)$ will be taken to mean the right derivative at x, which always exists. g' is a nonnegative and nonincreasing function on $[0, \infty)$. Hence $\lim_{x\to\infty} g'(x)$ must exist. Call it simply $g'(\infty)$.

[7] See Natanson [7, Vol. I, pp. 220–223].

[8] Lebesgue's Theorem says that the integral of the limit equals the limit of the integrals, given convergence in measure. In the present case, pointwise convergence implies convergence in measure.

[9] For a discussion and a proof, see Fenchel [3].

THEOREM 2: *Under the hypothesis that α is nonincreasing, a sufficient condition for the maximum of U to be attained in $\bar{C}_\lambda$ is*

(26) $$\alpha(0)g'(\infty)<\alpha(t^*)g'(\lambda/t^*) \text{ for some } t^* \text{ in } [0, 1].$$

In other words, a sufficient condition for the existence of a solution is that the quantity $\alpha(t)g'(\lambda/t)$ not be maximum at $t=0$.

PROOF: Suppose that condition (26) holds. It is then possible to find a real number K and a point t^* in [0, 1] such that

(27) $$\alpha(0)g'(K)<\alpha(t^*)g'(\lambda/t^*).$$

We shall show that for every $c\in\bar{C}_\lambda$ there exists some $c^*\in\bar{C}_\lambda$ such that $c^*(t)\leqslant K$ for all t in [0, 1] and such that $U(c^*)\geqslant U(c)$. Theorem 1 will do the rest.

In the first place, select K in such a way that $K>\lambda/t^*$. This is clearly possible. Let c be a function in the set $\bar{C}_\lambda$ such that $c(t)>K$ for some t in [0, 1]. By the monotonicity of c, there exists a point t_0 in [0, 1] such that

(28) $$\begin{aligned} c(t) &> K \quad \text{for} \quad 0\leqslant t<t_0, \\ &\leqslant K \quad \text{for} \quad t_0\leqslant t\leqslant 1. \end{aligned}$$

Furthermore, these also exists in [0, 1] a point t_1 such that

(29) $$\begin{aligned} c(t) &> \lambda/t^* \quad \text{for} \quad 0\leqslant t<t_1, \\ &\leqslant \lambda/t^* \quad \text{for} \quad t_1\leqslant t\leqslant 1. \end{aligned}$$

Since $K>\lambda/t^*$, we have $t_0\leqslant t_1$.

Now define the function c^* as follows

(30) $$\begin{aligned} c^*(t) &= K \quad \text{for} \quad 0\leqslant t<t_0, \\ &= c(t) \quad \text{for} \quad t_0\leqslant t<t_1, \\ &= \lambda/t^* \quad \text{for} \quad t_1\leqslant t<t_2, \\ &= c(t) \quad \text{for} \quad t_2\leqslant t\leqslant 1, \end{aligned}$$

where t_2 is determined so as to have $\int c^*(t)\,dt=\lambda$. In other words, t_2 is determined from the equation

(31) $$\int_0^{t_0} (c(t)-K)\,dt = \int_{t_1}^{t_2} ((\lambda/t^*)-c(t))\,dt.$$

Note that $t_2\leqslant t^*$. This is true because, by definition, $c^*(t)\geqslant\lambda/t^*$ for $0\leqslant t<t_2$, and if $t_2>t^*$ were true, then one would have to have $\int c^*(t)\,dt>\lambda$, which is impossible (see Figure 1).

To show that $U(c^*)\geqslant U(c)$ one proceeds as follows:

(32) $$U(c^*)-U(c) = \int_{t_1}^{t_2} \alpha(t)\{g(\lambda/t^*)-g[c(t)]\}\,dt - \int_0^{t_0} \alpha(t)\{g[c(t)]-g(K)\}\,dt.$$

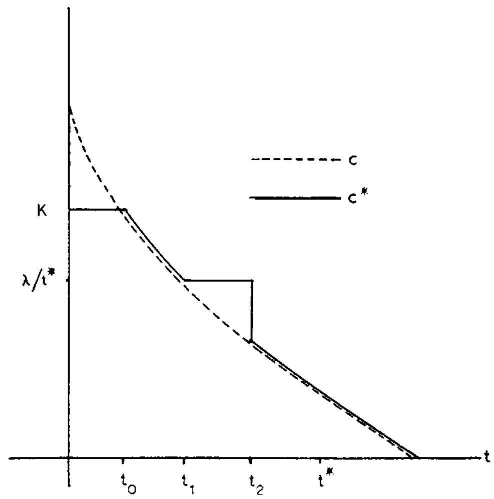

FIGURE 1

Consider each of these integrals separately: For t in $[t_1, t_2]$ monotonicity of α implies $\alpha(t) \geqslant \alpha(t^*)$ and concavity of g implies that $g(\lambda/t^*) - g[c(t)] \geqslant g'(\lambda/t^*)[(\lambda/t^*) - c(t)]$. Hence

$$(33) \qquad \int_{t_1}^{t_2} \alpha(t)\{g(\lambda/t^*) - g[c(t)]\}\,dt \geqslant \alpha(t^*)g'(\lambda/t^*)\int_{t_1}^{t_2} ((\lambda/t^*) - c(t))\,dt\,.$$

Next, for t in $[0, t_0]$, monotonicity of α implies $\alpha(t) \leqslant \alpha(0)$ and concavity of g implies $g[c(t)] - g(K) \leqslant g'(K)(c(t) - K)$. Hence

$$(34) \qquad \int_0^{t_0} \alpha(t)\{g[c(t)] - g(K)\}\,dt \leqslant \alpha(0)g'(K)\int_0^{t_0} (c(t) - K)\,dt\,.$$

Combining these inequalities and making use of equation (31) leads to

$$(35) \qquad U(c^*) - U(c) \geqslant [\alpha(t^*)g'(\lambda/t^*) - \alpha(0)g'(K)]\int_0^{t_0} (c(t) - K)\,dt\,,$$

and the latter quantity is always positive if condition (26) holds. This completes the proof.

Condition (26) is, of course, a marginal utility condition. It may seem somewhat unusual for a marginal utility condition to appear as a sufficient condition for the existence of an optimal plan, when it would normally be expected to appear as a necessary condition for the optimality of a given plan. However, condition (26) compares the marginal utilities at selected points on different plans, rather than at different points on the same plan. For each t in the interval [0, 1], let δ_t be given by

(36) $$\delta_t(\tau)=\lambda/t \quad \text{for} \quad 0\leqslant\tau\leqslant t\,, \qquad =0 \quad \text{for} \quad t<\tau\leqslant 1\,.$$

δ_t is an admissible plan for all t, except for $t=0$, in which case δ_t is the (inadmissible) Dirac delta function. What condition (26) says is that one should compare the marginal utilities at t on the plans δ_t. If, for some $t>0$, this marginal utility is greater than it is for $t=0$, then an optimal plan exists not only in this restricted class, but also in the class of all admissible plans.

It may be of some interest to compare condition (26) with the requirement that g be a bounded function. It is easily seen that, except for trivial cases, whenever g is bounded, condition (26) must always be met. More precisely, let t_α be defined by

(37) $$\alpha(t_\alpha)=0\,, \qquad \alpha(t)>0 \quad \text{for all } t \text{ in } [0, t_\alpha)\,.$$

Thus, t_α is the smallest t for which $\alpha(t)=0$. Now suppose that g is bounded above. Then either $g'(\lambda/t_\alpha)=0$ or condition (26) is met. The case where $g'(\lambda/t_\alpha)=0$ is trivial, because then the plan which is constant up to t_α and zero thereafter is optimal. Otherwise, $g'(\lambda/t_\alpha)>0$ and condition (26) is always met because $\alpha(0)g'(\infty)=0$. However, condition (26) is valid in a great many cases where g is not bounded. For example, $g'(\infty)$ may be zero even though g is unbounded. Thus, condition (26) represents a fairly substantial weakening of the requirement that g be bounded.

As the number λ becomes small, condition (26) becomes both necessary and sufficient. More precisely:

THEOREM 3: *If the maximum of U is attained, then there exists a $\lambda^*>0$ (depending on g and λ) such that condition* (26) *of Theorem* 2 *is satisfied for every* λ, $0<\lambda\leqslant\lambda^*$.

PROOF: Suppose no such λ^* exists. This means that condition (26) is violated for all $\lambda>0$. In symbols,

(38) $$\alpha(0)g'(\infty)\geqslant\alpha(t)g'(\lambda/t) \quad \text{for all} \quad \lambda>0 \quad \text{and} \quad t \text{ in } [0, 1]\,.$$

Let $\{\lambda_n\}$ and $\{t_n\}$ be two sequences such that $\lambda_n>0$, $0\leqslant t_n\leqslant 1$, and

(39) $$\lim \lambda_n=\lim t_n=\lim (\lambda_n/t_n)=0\,.$$

Finding such sequences is always possible. Using (38) together with the properties of g and α, one obtains

(40) $$\alpha(0)g'(0)=\lim \alpha(t_n)g'(\lambda_n/t_n)\leqslant\alpha(0)g'(\infty)\,.$$

Hence, $g'(0)=g'(\infty)$ and g must be linear. But when g is linear we know in advance that the maximum of U is not attained for any $\lambda>0$. This completes the proof.

COROLLARY: *The only case in which the maximum is not attained for any $\lambda>0$ is the case in which g is a linear function.*

4. AN EXAMPLE

At this point, it may be worth while to illustrate the use of condition (26) with a numerical example: Let the functions α and g be given by

$$\text{(41)} \qquad \alpha(t)=1-t \qquad 0\leqslant t\leqslant 1\,, \qquad g(x)=x-e^{-x} \qquad x\geqslant 0\,.$$

In other words, the utility function is assumed to be of the form

$$\text{(42)} \qquad U(c)=\int_0^1 (1-t)(c(t)-e^{-c(t)})\,dt\,,$$

which is quite ordinary-looking. Our aim is to state conditions under which an optimal plan exists when U is maximized subject to $c(t)\geqslant 0$ for all t, and $\int c(t)\,dt=\lambda$.

According to condition (26), an optimal plan exists if

$$\text{(43)} \qquad (1-t)(1+e^{-\lambda/t})>1 \quad \text{for some } t \text{ in } [0,1]\,,$$

which reduces to

$$\text{(44)} \qquad \lambda<t\log\frac{1-t}{t} \quad \text{for some } t \text{ in } [0,1]\,.$$

By consulting a table of natural logarithms, one finds that the greatest value of $t\log[(1-t)/t]$ for t in the unit interval is approximately .278. Thus, condition (26) tells us that an optimal plan exists if λ falls below .278.

In order to find all the values of λ for which an optimal plan exists, it is necessary to solve the maximization problem. Let c^* be the optimal plan. If c^* exists, it satisfies the following first-order condition at all points where the constraint $c(t)\geqslant 0$ is ineffective:

$$\text{(45)} \qquad (1-t)(1+e^{-c^*(t)})=k'\,,$$

where k' is a constant satisfying $k'>1$. This equation reduces to

$$\text{(46)} \qquad c^*(t)=\log\frac{1-t}{k+t}\,,$$

where $k=k'-1$. It is easy to see from equation (46) that k must lie in the open unit interval. Moreover, it is clear that $c^*(t)\geqslant 0$ in equation (46) only for $0\leqslant t\leqslant (1-k)/2$. Hence, the optimal plan c^* (if it exists) is given by

$$\text{(47)} \qquad c^*(t)=\log\frac{1-t}{k+t} \quad \text{for} \quad 0\leqslant t\leqslant\frac{1-k}{2}\,,$$
$$\qquad\qquad =0 \qquad \text{for} \quad \frac{1-k}{2}\leqslant t\leqslant 1\,.$$

To evaluate k, one uses the constraint

$$\text{(48)} \qquad \int_0^1 c^*(t)\,dt=\int_0^{(1-k)/2}\log\frac{1-t}{k+t}\,dt=\lambda\,,$$

which reduces to

$$k \log k-(1+k) \log \frac{1+k}{2} = \lambda . \tag{49}$$

Now for k in the unit interval, the quantity $k \log k-(1+k) \log [(1+k)/2]$ is easily seen to be decreasing in k. For the optimal plan c^* to exist it is therefore necessary (and also sufficient) that λ fall strictly below the level which the quantity $k \log k-(1+k) \log [(1+k)/2]$ attains when $k=0$. This level is precisely log 2. Hence, an optimal plan exists if $\lambda < \log 2$ and does not exist if $\lambda \geqslant \log 2$.[10] Comparing log 2 (which is approximately .693) with the number .278 which was obtained by using condition (26), we see that the latter is still a fairly strong condition.

5. A NON-MONOTONE DISCOUNT FUNCTION

In this section we drop the assumption that α is a nonincreasing function. The symbol C_λ will be used once again to denote the set of all nonnegative measurable functions c on the unit interval which are bounded and satisfy $\int c(t)dt=\lambda$. Let $\bar{\alpha}$ be the nonincreasing rearrangement of α. Define a functional $\bar{U}$ as follows:

$$\bar{U}(c) = \int_0^1 \bar{\alpha}(t) g[c(t)]\, dt, \qquad c \in C_\lambda . \tag{50}$$

As might be expected, there is a close relationship between the attainment of the maximum of U and the attainment of the maximum of $\bar{U}$:

THEOREM 4: *The maximum of U is attained in C_λ if and only if the maximum of $\bar{U}$ is attained in C_λ. Moreover, the two maxima (if attained) are equal.*

PROOF: We shall show that

$$\sup_{c \in C_\lambda} U(c) = \sup_{c \in C_\lambda} \bar{U}(c) . \tag{51}$$

Let c be an arbitrary member of C_λ and let $\bar{c}$ be the nonincreasing rearrangement of c. Recall that $\bar{c}$ is also in the set C_λ. We already know that

$$\bar{U}(\bar{c}) = \int_0^1 \bar{\alpha}(t) g[\bar{c}(t)]\, dt \geqslant \int_0^1 \alpha(t) g[c(t)]\, dt = U(c) . \tag{52}$$

Hence,

$$\sup_{c \in C_\lambda} U(c) \leqslant \sup_{c \in C_\lambda} \bar{U}(c) . \tag{53}$$

[10] It may seem here that by measuring λ in large units, we can ensure the existence of an optimal plan. This, of course, is not the case, because the unit in which λ is measured is prescribed implicitly in the utility function U.

Next, let f be a transformation on the closed unit interval onto itself such that

(54) $\quad \bar{\alpha}(f(t)) = \alpha(t) \quad$ for all t in $[0, 1]$.

If $\bar{\alpha}$ is strictly decreasing, then f is given directly by $f(t) = \bar{\alpha}^{-1}(\alpha(t))$. However, if $\bar{\alpha}$ is not strictly decreasing, then caution must be exercised in choosing the transformation f (which is not unique in this case) so that it is *onto*. In any case, such a transformation f always exists and it can be shown to be measurable and *measure preserving*.[11] This implies that if y is any measurable function on the unit interval, then

(55) $$\int_0^1 y(f(t))\,dt = \int_0^1 y(t)\,dt\,.$$

In particular, for any $c \in C_\lambda$,

(56) $$\int_0^1 \bar{\alpha}(f(t))\,g[c(f(t))]\,dt = \int_0^1 \bar{\alpha}(t)\,g[c(t)]\,dt\,.$$

In other words,

(57) $$\bar{U}(c) = U(c(f))\,.$$

But by (55), $c(f)$ is in the set C_λ, so that one obtains

(58) $$\sup_{c \in C_\lambda} U(c) \geqslant \sup_{c \in C_\lambda} \bar{U}(c)\,.$$

Thus, the two suprema must be equal. The assertion of the theorem now follows immediately: If sup U is attained at c, then sup $\bar{U}$ is attained at $\bar{c}$, and if sup $\bar{U}$ is attained at c, then sup U is attained at $c(f)$. This completes the proof.

COROLLARY: *Theorem* 1 *is valid for any bounded and continuous discount function* α.

Theorem 4 makes it possible to reduce the case of a general discount function to one in which the discount function is nonincreasing. In practice, however, finding the monotone rearrangement $\bar{\alpha}$ of the discount function α is not likely to be easy. For this reason, it may be worth while to state an analogue of Theorem 2 for the case of a general discount function, one which will not require the rearrangement of this function in nonincreasing order.

To state this analogue, let t_0 be a point at which α is maximum on $[0, 1]$. Then:

THEOREM 2′: *A sufficient condition for the attainment of the maximum of U in the set* C_λ *is that there exist in* $[0, 1]$ *a subinterval* $[t_1, t_1+h]$ *and a point* t^* *in it, such that* $\alpha(t^*) \leqslant \alpha(t)$ *for all t in* $[t_1, t_1+h]$ *and*

(59) $$\alpha(t_0)\,g'(\infty) < \alpha(t^*)\,g'(\lambda/h)\,.$$

[11] The transformation f is said to be measure preserving if, for any measurable set E in the unit interval, the set $f^{-1}(E)$ is measurable and its measure is equal to that of E.

The proof of this theorem is very similar to that of Theorem 2, and it uses the fact (Corollary to Theorem 4) that Theorem 1 is valid for any discount function.

Theorem 2′ is more cumbersome in applications than Theorem 2, because the choice of h affects the strength of condition (59). The larger h, the weaker condition (59) becomes, so there are returns to picking h as large as possible.

It is an immediate consequence of Theorem 4 that, once again, if g is not a linear function, then for sufficiently small λ the maximum of U is attained.

6. ON GENERALIZING THE LINEAR CONSTRAINT

It may be well to conclude the discussion with one or two remarks on what happens when the constraint $\int c(t)dt=\lambda$ is replaced by a more general linear constraint, namely by

$$\int_0^1 r(t)c(t)\,dt=\lambda\,, \tag{60}$$

where r is a continuous function on [0, 1] satisfying $r(t)>0$ everywhere. In the consumer's choice interpretation which we have given to the maximization problem, this modification amounts to the introduction of a nonzero rate of interest.

As can be shown, Theorem 1 stands unaffected, while Theorem 2 must now be modified as follows: Let t_0 be a point at which the quotient α/r is maximum on [0, 1]. Then:

THEOREM 2″: *A sufficient condition for the maximum of U to be attained in the set of admissible plans is that there exist in* [0, 1] *a subinterval* $[t_1, t_2]$ *with a point* t^* *in it such that* $\alpha(t^*)\leqslant\alpha(t)$ *for all t in* $[t_1, t_2]$ *and*

$$\alpha(t_0)g'(\infty)<\alpha(t^*)g'\Big[\lambda/\int_{t_1}^{t_2} r(t)\,dt\Big]\,. \tag{61}$$

It is easily seen that when $r(t)\equiv 1$ on the unit interval, then (61) reduces to condition (59) of Theorem 2′. If the function α/r happens to be nonincreasing, then condition (61) is equivalent to the statement that the quantity

$$\alpha(t)g'\Big[\lambda/\int_0^t r(u)\,du\Big] \tag{62}$$

not be maximum at $t=0$.

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