ON A THEOREM OF HALMOS CONCERNING UNBIASED ESTIMATION OF MOMENTS

H. S. KONIJN

(received 11 August 1963)

1. Introduction

In [4] Halmos considers the following situation. Let $\mathcal{D}$ be a class of distribution functions over a given (Borel) subset $E$ of the real line, and $F$ a function over $\mathcal{D}$. He investigates which functions $F$ admit estimates that are unbiased over $\mathcal{D}$ and what are all possible such estimates for any given $F$. In particular he shows that on the basis of a sample (of size $n$) one can always obtain an estimate of the first moment which is unbiased in $\mathcal{D}$ and that the central moments $F_m$ of order $m \geq 2$ have estimates which are unbiased in $\mathcal{D}$ if and only if $n \geq m$, provided $\mathcal{D}$ satisfies the following properties: $F_m$ exists and is finite for all distributions in $\mathcal{D}$ and $\mathcal{D}$ includes all distributions which assign probability one to a finite number of points of $E$. Halmos also finds that symmetric estimates which are unbiased on $\mathcal{D}$ are unique and have smaller variances on $\mathcal{D}$ than unsymmetric unbiased estimates.

He recognizes that his assumptions are too restrictive for most applications and mentions in particular the case where $\mathcal{D}$ is the class of all normal distributions. The present paper addresses itself to that case.

2. Statement of results

If $\mathcal{D}$ is the class of all nondegenerate univariate normal distributions, then, on the basis of a sample (of size $n$), an estimate of the first moment which is unbiased over $\mathcal{D}$ exists (and is unique when $n = 1$); and a central moment of order $2r \geq 2$ has estimates which are unbiased over $\mathcal{D}$ if and only if $n \geq 2$, and has a unique symmetric unbiased estimate when $n = 2$, but not when $n > 2$.

Specifically, this means the following:

Let $z_1, \ldots, z_n$ be a sample from a normal distribution with mean $\mu$ and variance $\sigma^2 > 0$. Let $\bar{z} = \frac{1}{n} \sum z_i$, $S^2 = \sum (z_i - \bar{z})^2$. Recall that the even

\textit{It will be convenient to call a function on a $k$-dimensional Euclidean space the unique function satisfying a certain property if any other function on this space satisfying the property may differ from it only on a set of $k$-dimensional Lebesgue measure zero.}
central moments $\bar{F}_{2r}$ equal $\omega^{2r-2}(2r)!/r!$ and the odds vanish.

(a) If $n = 1$, $\bar{z}$ is the unique unbiased estimate of $\nu$, and no unbiased estimate of $\bar{F}_{2r}$ exists for $r = 1, 2, \ldots$. In [5] this seemingly uninteresting fact turns out to be the key to a quite practical question.

(b) If $n \geq 2$,

$$\bar{f}_{2r} = \frac{\{(n-3)/2\}!(2r)!}{\{(n+2r-3)/2\}!r!} (S/2)^{2r}$$

is an unbiased estimate of $\bar{F}_{2r}$ ($r = 1, 2, \ldots$), and is the unique symmetric unbiased estimate if $n = 2$, but not if $n > 2$. It then follows from [6] that $\bar{z}$ and $\bar{f}_{2r}$

(c) are the unique unbiased estimates of $\nu$ and $\bar{f}_{2r}$, respectively, which depend only on the sufficient statistic $(\bar{z}, S^2)$

(d) have the smallest variance among all unbiased estimates.

Note that $\bar{z}$ and $S^2$ are symmetric functions of the observations. The usual symmetric estimate $\bar{f}'_{2r}$ for $\bar{F}_{2r}$, which is unbiased for all distribution functions for which $\bar{F}_{2r}$ exists, is defined only when $n \geq 2r$. When $r = 1$ it coincides with $\bar{f}_2$, when $r = 2$ it equals [2, 27.6]

$$\bar{f}'_4 = (n!)^{-1}(n-4)! \{n(n^2-2n+3) \sum (z_i - \bar{z})^4 - 3(2n-3)S^4 \} \quad (n \geq 4).$$

For any family $\mathcal{D}$ as first mentioned in the introduction or mentioned in the final section $\bar{f}'_{2r}$ is the only symmetric estimate which is unbiased for all distributions of $\mathcal{D}$. But, if for $\mathcal{D}$ we take the class of nondegenerate univariate normal distributions, our results imply that the symmetric estimate $\bar{f}'_{2r}$ is also unbiased over this class and has a smaller variance than $\bar{f}_{2r}$ for $r > 1$.

In the next two sections we prove the parts of (a) and (b) which are not immediate.

3. Nonexistence of an unbiased estimate of $\bar{F}_{2r}$ in a sample of one

In this section denote $z_1$ by $z$. If $h(z)$ is an unbiased estimate of $\bar{F}_{2r}$, then

$$\int_{-\infty}^{\infty} \{h(z+\nu) - z^{2r}\} \exp \left(-\frac{1}{2}\omega^2 \nu^2\right) d\nu$$

should vanish for all $\nu$ and all $\omega > 0$. This integral can be written as an

---

8 It has been remarked that it is obvious that from a sample of one it is not possible to obtain an unbiased estimate of two independent parameters (that is, two functions $F_1$ and $F_2$ on a class of distributions such that there exists no function $g$ in the plane with $g(F_1(D), F_2(D)) = 0$ for all distributions $D$ in the class). That this is not so is easily shown by an example. Let $\theta = r + \omega^2$, where $r$ and $\omega^2$, the mean and variance, are independent parameters when, e.g., the class is the normal class. Then $r$ and $\theta$ are also independent parameters over that class with unbiased estimates $z_1$ and $z_1^2$.‌
integral over the positive axis and then we can make the substitution \( u = z^r \) and obtain, setting \( \omega' = (2\omega)^{-1} \), that

\[
\int_0^\infty \{h(-u^r + v) + h(u^r + v) - 2u^r\}u^{-\frac{1}{2}} \exp\{-u\omega'\} du
\]

is zero for all \( v \) and all \( \omega' > 0 \). This being a Laplace transform of \( u^{-\frac{1}{2}} \times \) the expression in brackets, it follows that

\[
h(-z^r + v) + h(z^r + v) - 2z^{2r} = 0
\]

for all \( v \) and almost all positive \( z \). For all \( v \) there is a set \( S_\nu \) on the positive \( z \) axis such that the Lebesgue measure \( l \) of the positive points \( z \) not in \( S_\nu \) is zero and such that the above equality holds on \( S_\nu \). Denote \( \bigcap_{k=1,2,4,5} S_{(k+2)/8} \) by \( T \).

It is easily shown\(^5\) that there exists a pair of points \( a \) and \( \frac{3a}{2} \) in \( T \). Choosing \( v = a \) and \( 2a \) respectively gives for \( z = a \)

\[
h(0) + h(2a) = 2a^{2r}, \quad h(a) + h(3a) = 2a^{2r},
\]

so that

\[
h(0) + h(a) + h(2a) + h(3a) = 4a^{2r}.
\]

Choosing \( v = \frac{3a}{2} \) and \( 2\frac{3a}{2} \) respectively gives for \( z = \frac{1}{2}a \)

\[
h(0) + h(\frac{3a}{2}) = a^{2r}/2^{2r-1}, \quad h(2a) + h(3a) = a^{2r}/2^{2r-1},
\]

so that

\[
h(0) + h(a) + h(2a) + h(3a) = a^{2r}/2^{2r-2}.
\]

Since \( a \neq 0 \), this is a contradiction.

4. Uniqueness of the unbiased symmetric estimate of \( \hat{F}_{2r} \) in a sample of two and nonuniqueness in a larger sample

For \( n \geq 2 \) (so that \( S^2 \) is not identically zero) the sufficiency of the statistic \((\mathcal{I}, S^2)\) and the completeness of its distribution imply that \( \hat{F}_{2r} \) is the unique unbiased estimate of its expectation \( \hat{F}_{2r} \), among unbiased estimates depending on \((\mathcal{I}, S^2)\) only \([5]\). Now if \( n = 2 \), \((\mathcal{I}, S^2)\) determines the set \( \{z_1, z_2\} \) of observations, but not their order. Therefore \( \hat{F}_{2r} \) is also the unique unbiased estimate of \( \hat{F}_{2r} \), among unbiased estimates which are symmetric in the observations.

In general, when \( n > 2 \), for any \( a \neq 0 \),

---

\(^5\) Let \( a' \) be in \( T \) and let \( 0 < b < a' \). Define the disjoint intervals \( I_i \) from \( ia' + b \) to \( i(a' + b) \) for \( i = 1, 2, \) which have \( l(I_i) = ib \). Denote by \( p_i(I, T) \) the set of points \( x \) in \( I_i \) \( T \) such that \( i(x) \) is in \( I_i, T \); \( l(p_i(I, T)) = ib \). Now let

\[
T_x = T_{p_x}(I, T), \quad T_1 = p_1(I_1 T_1);
\]

then, since the \( T_i \) are subsets of \( T \) of measure \( ib \), there exists \( a > 0 \) such that \( \frac{1}{ib} a \) is in \( T_i \) for \( i = 1 \) and 2. In fact, there exist \( c \) such that, for almost all \( a \) in \( T \), \( \frac{1}{ib} a \) is in \( T \). For brevity use \( c = 0 \).
\[ \tilde{F}_r + a(n+1) \sum (x_i - \bar{x})^4 - 3(n-1)S^4 \]

will be an unbiased symmetric estimate of \( F_r \), different from \( \tilde{F}_r \), since the mean of \( \sum (x_i - \bar{x})^4 \) is \( 3n^{-1}(n-1)^2\sigma^4 \) and the mean of \( S^4 \) is \( (n-1)(n+1)\sigma^4 \), and since for \( n > 2 \) the bracket is not identically equal to zero. For example, if \( n = 3 \), \( 1 \frac{1}{2} \sum (x_i - \bar{x})^4 \) has mean \( \tilde{F}_4 + 3\tilde{F}_2^2 \) and, in the normal case, \( S^4 \) has mean \( 8\tilde{F}_2^2 \), so that \( 1 \frac{1}{2} \sum (x_i - \bar{x})^4 - S^4/4 \) and \( \frac{3}{4} \sum (x_i - \bar{x})^4 \) are unbiased estimates of \( \tilde{F}_4 \) different from \( \tilde{F}_4 = 3S^4/8 \).

5. Remarks

One could similarly discuss unbiased estimation of other functions over the class of normal distributions.

Fraser [3] adapts Halmos' argument to cases where \( \mathcal{D} \) is a certain class of distributions that have a density. Some cases of this kind have been found by Lehmann and Scheffé; see [1].

The writer is much indebted to T. C. Koopmans and T. N. Srinivasan for helpful suggestions.

References


Department of Economics,
The City College,
New York,
and
Cowles Foundation.