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Factor Substitution in the Two-Sector Growth Model*

I INTRODUCTION

1.1. In a series of recent papers by Shinkai [3], Uzawa [6], [7], Solow [4], Inada [2], the familiar two-sector neoclassical production model has been systematically applied to the analysis of the growth process of capital accumulation.[1] In particular, the stability properties of balanced growth paths have been examined under various simplifying assumptions. It has been shown by Uzawa in [6], [7], that a sufficient condition for the stability of the balanced growth path is that the consumption-goods industry is more capital-intensive than the capital-goods industry. Shinkai [3] has shown that the above condition is a necessary and sufficient condition for stability in the case of fixed coefficients of production.

1.2. The main purpose of this paper is to provide a new and weaker set of sufficient conditions for the stability of the balanced growth path in the two-sector model. We find that if the elasticity of factor substitution of both production functions approaches zero, then the greater capital-intensity of the consumption-goods industry becomes a necessary and sufficient condition for stability. In any other case it remains a sufficient condition. As however we consider production functions with greater elasticities of substitution, this condition becomes an overly strong one. Finally, in our main stability theorem we establish that if the elasticity of factor substitution of either one of the production functions is greater than or equal to one, then the stability of the balanced growth path is obtained without regard to other conditions.

It can easily be explained why the balanced growth path, and the corresponding to it capital-labour ratio, are asymptotically stable whenever the elasticity of factor-substitution in either one of the industries is sufficiently high. The main point to be emphasized is that a relative " abundance "—relative to the balanced growth path—of any one of the factors at any time will be accommodated through two channels. The first is an inter-industry movement towards the industry which uses relatively more of that factor. The second, which will be operating in the case where alternative techniques for the production of one or of both of the goods exist, is an intra-industry movement. If e.g., the available quantity of one of the factors changes exogeneously than this will cause an alteration of the equilibrium factor prices; then with factor substitutability a change in the factor proportions used in the relevant industry will occur; if factor substitutability is high enough, this will insure the absorption of the relatively " abundant " factor with only small inter-industry movements.

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[1] We note that technical progress is not considered in these papers.

1.3. We also provide weaker sufficient conditions for the existence of a unique equilibrium of this economy in every time period.

The existence problem is a preliminary one to our main object of inquiry. However, its careful consideration is needed because when we examine the growth process we assume that the economy is continuously at equilibrium.

II THE MODEL

2.1. The model examined below is the familiar two-sector neoclassical model as it has been formulated by Uzawa [6]. Our notation also follows [6]. The capital-goods sector is denoted by the subscript 1, and the consumption-goods sector by 2. The workers and the capitalists offer in each market the services of the goods they own in exchange for consumption and capital goods which will become available in the next time period. Competitive equilibrium is achieved in each market and by means of the exogenous growth of the labour force and the utilization of the newly produced capital goods a process of successive equilibria through time is generated.

2.2. The main assumptions are the following:

(A_1) In each time period there exist only four distinct commodities: labour services, capital services, capital goods, and consumption goods. Consumption goods are not used for production. Labour and capital goods do not yield direct services for consumption. Each good is produced separately by the use of capital and labour services.

(A_2) The production structure is described by two neo-classical production functions, denoted by $F^i(K_i, L_i)$, $K_i, L_i \geqq 0$, $i = 1, 2$, which are twice continuously differentiable, homogeneous of degree one, concave, and have positive marginal rates of substitution.[1]

(A_3) A given percentage of the capital goods existing in each period is withdrawn from production on account of depreciation. Otherwise all capital goods are homogeneous and they are freely transferable between the two sectors.

(A_4) The marginal (and average) propensity to save out of the current income of workers and capitalists are given constants.[2]

2.3. *Competitive Equilibrium.* The constant-returns-to-scale feature of the model enables us to deal mainly with the ratios of inputs and of prices. We write s_w and s_r for the workers' and capitalists' propensity to save,[3] $k = \frac{K}{L}$ for the aggregate capital-labour ratio, $k_i = \frac{K_i}{L_i}$ for that in the ith sector, $l_i = \frac{L_i}{L}$ for the proportion of labour employed

[1] Namely, (a) $F^i(K_i, L_i) > 0$ for all $K_i, L_i > 0$, (b) $F^i(\lambda K_i, \lambda L_i) = \lambda F^i(K_i, L_i)$ for all $\lambda > 0$, and (c) $F^i_K(K_i, L_i) > 0$, $F^i_L(K_i, L_i) > 0$, $d^2F^i \leqq 0$ for all $K_i, L_i > 0$.

[2] (A_4) specifies the saving patterns of the economy and completes the model by introducing the demand side of the economy. This assumption that workers and capitalists have formed rigid saving patterns, which are not influenced by the accumulation process itself, is a really strong assumption. (A_4) in effect separates each market from all successive ones. Price expectations and interest rates do not play any independent role in our model because of (A_4).

An alternative approach is that in which saving patterns are considered as derived from utility maximization over time; see e.g. Srinivasan [5]. However, in such a model perfect foresight for the whole future has to be assumed. The present model is exactly the opposite in this respect; the economic horizon extends over only one time period.

[3] We have $1 \geqq s_r, s_w \geqq 0$. We exclude the trivial cases $1 = s_r = s_w$, $0 = s_r = s_w$. s_r is the gross propensity to save out of the gross capitalists' income, i.e., $s_r rK$ represents the gross savings of the capitalists.

in the ith sector, and also $y_i = \frac{Y_i}{L_i}$, $\omega = \frac{w}{r}$, $p = \frac{p_1}{p_2}$. Then $Y_i = F^i(K_i, L_i) = L_i F^i(k_i, 1) = L_i f_i(k_i)$, or $y_i = f_i(k_i)$.[1]

We easily find (cf. Uzawa [6, pp. 41-42]) that the competitive equilibrium conditions are given by

(1) $y_i = f_i(k_i), \quad i = 1, 2,$

(2) $\omega = f_i(k_i)/f_i'(k_i) - k_i, \quad i = 1, 2,$

(3) $k_1 l_1 + k_2 l_2 = k,$

(4) $l_1 + l_2 = 1,$

(5) $s_r k + s_w \omega = l_1 f_1(k_1)/f_1'(k_1),$

(6) $k_i, l_i, y_i, \omega \geqq 0,$

and $k = \bar{k} \geqq 0.$

(1) and (2) are derived from the production equilibrium conditions; i.e., the wage rate w is equal to the value of the marginal product of labour, and the rental of the capital goods r is equal to the value of the marginal product of capital in both sectors. (3) and (4) express the equilibrium conditions in the markets for productive services; i.e., labour and capital services are *fully* employed and freely transferable between sectors. (5) expresses the equilibrium condition in the market of newly produced capital goods.[2] In the market at t the available quantities of capital and labour services are given.

We will show that (1)-(5) have a unique solution for y_i, k_i, l_i, ω, which is positive for $\bar{k} > 0$.

III EXISTENCE OF COMPETITIVE EQUILIBRIUM IN EACH TIME PERIOD[3]

3.1. In sections 3.3*-3.8* below we will prove two existence theorems. We have: *Existence Theorem* 1:[4] For any aggregate capital-labour ratio k satisfying (12), the equilibrium ω, k_i, y_i, w, r, p, are uniquely determined and they are positive, in any of the following cases:

1. If $k_1[\omega(k)] \neq k_2[\omega(k)]$, whenever (a) $s_r > s_w$, $k_2[\omega(k)] > k_1[\omega(k)]$, or (b) $s_w > s_r$, $k_1[\omega(k)] > k_2[\omega(k)]$, or (c) $s_r = s_w$; and

2. If $k_1[\omega(k)] = k_2[\omega(k)]$.

We note that in this theorem we completely ignore the possibilities of factor substitution in any of the two sectors. Instead we rely on the difference between the capital-labour ratios in the two sectors. Thus e.g., in the most important case, i.e., that where $s_r > s_w$,

[1] (A_2) is now given by (a) $f_i(k_i) > 0$ for $k_i > 0$, (b) f_i is twice continuously differentiable, (c) $f_i'(k_i) > 0$, $f_i(k_i) - k_i f_i'(k_i) > 0$, $f_i''(k_i) < 0$ for $k_i > 0$.

[2] The equilibrium condition in the market for the consumption goods follows from the rest.

[3] It can be easily seen that no essential differences are created whether we treat time as a continuous or a discrete variable. We will follow the first alternative, and thus the analysis will be carried out in terms of differential equations.

Proofs are placed in starred sections separate from the statements of the corresponding theorems, so that all proofs may be omitted without loss of continuity.

[4] This theorem is a generalization of the existence theorems in [6] and [7]. Also Inada [2] has independently proved case 1.a of the theorem along with the corresponding stability theorem in section 4.5 below.

we have to assume that the consumption-goods sector is always more capital intensive than the capital-goods sector. However, many objections can be raised concerning the reasonableness of this *capital intensity condition.*[1]

Actually, a closer consideration of the possibilities of factor substitution in both sectors makes apparent that such capital intensity conditions are, in most cases, unnecessarily strong sufficient conditions.

3.2. Let

$$\sigma_i(\omega) = \frac{\omega}{k_i}\frac{dk_i}{d\omega} \tag{7}$$

be the elasticity of substitution between capital and labour in sector $i = 1, 2$. We will also show

Existence Theorem 2: Let $\sigma_i(\omega)$, $i = 1, 2$, be such that $\sigma_1(\omega) + \sigma_2(\omega) \geqq 1$, (or satisfy (18) or (19)). Then, for any aggregate capital-labour ratio k satisfying (12), the equilibrium ω, k_i, y_i, w, r, p, are uniquely determined and they are positive.

3.3.* First, $\alpha_i(k_i) \equiv \dfrac{f_i(k_i)}{f_i'(k_i)} - k_i$ is a strictly increasing function of k_i, for $k_i > 0$.

Let $\underline{\omega}_i = \lim\limits_{k_i \to 0} \alpha_i(k_i)$, $\overline{\omega}_i = \lim\limits_{k_i \to \infty} \alpha_i(k_i)$, $\underline{\omega} = \max[\underline{\omega}_1, \underline{\omega}_2]$, and $\overline{\omega} = \min[\overline{\omega}_1, \overline{\omega}_2]$. Of course $0 \leqq \underline{\omega}_i \leqq \overline{\omega}_i \leqq +\infty$. We assume that f_i are such that $\underline{\omega} < \overline{\omega}$ holds. Then, *for any positive wage-rentals ratio* ω, (2) *can be solved for a unique positive capital-labour ratio* $k_i = k_i(\omega)$ *in each sector, provided*

$$\underline{\omega} < \omega < \overline{\omega} \tag{8}$$

Second, for any ω satisfying (8) and such that $k_1(\omega) \neq k_2(\omega)$, equations (3)-(4) give us

$$l_1(\omega) = \frac{k_2(\omega) - k}{k_2(\omega) - k_1(\omega)}, \text{ and } l_2(\omega) = \frac{k - k_1(\omega)}{k_2(\omega) - k_1(\omega)} \tag{9}$$

3.4.* Finally, (5) can be written, using (2) and (9), as

$$k = \psi(\omega) \equiv \frac{k_1(\omega)k_2(\omega) + [s_w k_1(\omega) + (1 - s_w)k_2(\omega)]\omega}{(1 - s_r)k_1(\omega) + s_r k_2(\omega) + \omega} = \frac{a(\omega)}{b(\omega)} \tag{10}$$

Now, $\psi(\omega) > 0$ for $\underline{\omega} < \omega < \overline{\omega}$. Thus $k = \psi(\omega)$ has a solution for any k such that $\inf\limits_{\omega\varepsilon(\underline{\omega}, \overline{\omega})} \psi(\omega) < k < \sup\limits_{\omega\varepsilon(\underline{\omega}, \overline{\omega})} \psi(\omega)$.

A unique solution for ω is insured if $\psi(\omega)$ is a strictly monotonic function of ω.

Let $c(\omega) = k_1(\omega)\,[\omega + k_2(\omega)]\,[s_r k_2(\omega) + s_w\omega]$, $d(\omega) = k_2(\omega)\,.\,[\omega + k_1(\omega)]\,[(1 - s_r)k_1(\omega) + (1 - s_w)\omega]$, and $e(\omega) = \omega[k_2(\omega) - k_1(\omega)]\,.\,[s_r(1 - s_w)k_2(\omega) - (1 - s_r)s_w k_1(\omega)]$.

We can show that

$$\frac{1}{\psi(\omega)}\,\psi'(\omega) = \frac{1}{\omega}\,\frac{c(\omega)\sigma_1(\omega) + d(\omega)\sigma_2(\omega) + e(\omega)}{a(\omega)b(\omega)} \equiv \frac{1}{\omega}\,\sigma(\omega) \tag{11}$$

[1] See e.g., Solow [4, p. 48].

We will examine below several cases for which $\sigma(\omega)$ is positive for all $\omega \varepsilon (\underline{\omega}, \overline{\omega})$.[1] In all these cases equation (10) either expresses k as a function of ω, $k = \psi(\omega)$ for all $\omega \varepsilon (\underline{\omega}, \overline{\omega})$ or ω as a function of k, namely,

(10′) $$\omega = \psi^{-1}(k), \text{ for } k \text{ satisfying}$$

(12) $$\psi(\underline{\omega}) < k < \psi(\overline{\omega})$$

Of course $$\omega'(k) = \frac{1}{\psi'(\omega)} = \frac{1}{\sigma[\omega(k)]} \frac{\omega(k)}{k}, \text{ and}$$

$$\sigma(\omega) = \frac{\omega}{\psi(\omega)} \psi'(\omega) \text{ is thus equal to the elasticity of } \psi(\omega)$$

Hence, *provided that $\sigma(\omega)$ is positive for all $\omega \varepsilon (\underline{\omega}, \overline{\omega})$,* (10) *uniquely relates the equilibrium wage-rental ratio and the equilibrium aggregate capital-labour ratio throughout the process of capital accumulation.*

3.5.* Since $a(\omega)$, $b(\omega)$, $c(\omega)$, $d(\omega)$ and $\sigma_i(\omega)$ are all positive for all $\omega \varepsilon (\underline{\omega}, \overline{\omega})$, we will first examine the cases in which $e(\omega)$ is also positive. $e(\omega)$ is positive if and only if

$$k_2(\omega) \begin{Bmatrix} > \\ < \end{Bmatrix} k_1(\omega) \text{ and } \frac{s_r(1 - s_w)}{s_w(1 - s_r)} \begin{Bmatrix} > \\ < \end{Bmatrix} \frac{k_1(\omega)}{k_2(\omega)}.$$

Hence $\psi'(\omega)$is positive for all $\omega \varepsilon (\underline{\omega}, \overline{\omega})$, if

(a) $s_r > s_w$, $k_2(\omega) > k_1(\omega)$; (b) $s_w > s_r$, $k_1(\omega) > k_2(\omega)$; and (c) $s_r = s_w$. *For these cases any given capital-labour ratio $\bar{k}$, such that $\psi(\underline{\omega}) < \bar{k} < \psi(\overline{\omega})$, uniquely determines the equilibrium wage-rental ratio $\omega(\bar{k}) > 0$.*

3.6.* Through $\omega(k)$ all other quantities and prices at t are determined. We can easily show that they are all positive for all k satisfying (12). From (10) we get

(13) $$k - k_1 = (k_2 - k_1) \frac{(1 - s_r)k_1 + (1 - s_w)\omega}{(1 - s_r)k_1 + s_r k_2 + \omega}$$

(14) $$k + k_2 = (k_1 - k_2) \frac{s_r k_2 + s_w \omega}{(1 - s_r)k_1 + s_r k_2 + \omega}$$

Thus we see that for any k satisfying (12)

(15) $$\max_{i=1,2} \; k_i[\omega(k)] > k > \min_{i=1,2} \; k_i[\omega(k)]$$

holds. This is naturally a consequence of the feasibility of factor substitution. For any k satisfying (12) the equilibrium wage-rental ratio $\omega(k)$ will be such that (15) is satisfied. From (15) and (9) we immediately see that $l_1[\omega(k)]$ are positive. Also, the equilibrium w, r, p, are positive.

[1] The existence of a unique solution of (10) for ω is of course also insured if $\sigma(\omega) < 0$, for all $\omega \varepsilon (\underline{\omega}, \omega)$, provided that $\psi(\overline{\omega}) < k < \psi(\underline{\omega})$. This case however will not be considered at length, since no simple sufficient conditions for $\sigma(\omega) < 0$ can be given.

3.7.* Till now we have assumed that for all relevant k, $k_1[\omega(k)] \neq k_2[\omega(k)]$ holds. However, the production functions may be such that there exist $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, for which $k_1(\omega) = k_2(\omega)$. Then (3)-(4) give us

$$k_1(\omega) = k_2(\omega) = k \tag{16}$$

Since both $k_i(\omega)$ are strictly increasing functions of $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, (16) is satisfied by a unique $\omega = \omega(k)$, for given $k > 0$. Then from (2) and (5) we get

$$l_1[\omega(k)] = \frac{s_r k + s_w \omega(k)}{k + \omega(k)} \tag{17}$$

Hence $0 < l_1[\omega(k)] < 1$, and finally $0 < l_2[\omega(k)] < 1$ is determined from (4).

3.8.* If we take into consideration any possibilities of factor substitution in both sectors, we can give another set of sufficient conditions for having $\sigma(\omega) > 0$, for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$. Let us examine the sign of $g(\omega) \equiv \alpha c(\omega) + \beta d(\omega) + e(\omega)$, where $\alpha \geqq 0$, $\beta \geqq 0$. We find that

$$g(\omega) = [\alpha(s_r + s_w) + \beta(2 - s_r - s_w) - (s_r(1 - s_w) + (1 - s_r)s_w])k_1 k_2 \omega$$

$$+ [\beta k_2 + s_w\omega](1 - s_r)k_1^2 + [\alpha k_1 + (1 - s_w)\omega]s_r k_2^2 + [\alpha s_w k_1 + \beta(1 - s_w)k_2]\omega^2$$

Hence $g(\omega)$ is positive if $\alpha(s_r + s_w) + \beta(2 - s_r - s_w) - (s_r(1 - s_w) + (1 - s_r)s_w) = (s_r + s_w)(\alpha - \beta - 1) + 2\beta + 2s_r s_w > 0$.

Considering the range of values of s_r, s_w, we easily see that the last expression is positive, for $1 \geqq s_r$, $s_w \geqq 0$, if and only if $\alpha + \beta \geqq 1$. Therefore, if $\sigma_1(\omega) + \sigma_2(\omega) \geqq 1$ holds for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, then $\sigma(\omega) > 0$, for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, is satisfied. If e.g., either one of $\sigma_1(\omega)$ is greater than or equal to one for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, $\sigma(\omega) > 0$ holds.

In general, if we ignore the possibilities of factor substitution in the capital-goods sector, or in the consumption-goods sector, respectively, then [1]

$$\sigma_1(\omega) > \max\left\{-\frac{e(\omega)}{c(\omega)}, 0\right\}, \text{ or} \tag{18}$$

$$\sigma_2(\omega) > \max\left\{-\frac{e(\omega)}{d(\omega)}, 0\right\}, \text{ respectively,} \tag{19}$$

must hold, for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, in order to have $\sigma(\omega) > 0$ and thus $\psi'(\omega) > 0$ for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$.

IV EXISTENCE AND STABILITY OF THE BALANCED GROWTH PATH

4.1. We now examine the behaviour through time of the aggregate capital-labour ratio, and by means of it that of all other variables, given that the economy is continuously at equilibrium.[2] We assume

[1] Of course $d(\omega) + e(\omega) > 0$, $c(\omega) + e(\omega) > 0$, for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$.

[2] We assume that a unique equilibrium is reached in every period. As we have seen in section 3, this is accomplished if $\sigma(\omega) > 0$ for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$, namely if $\psi(\omega)$ is a strictly increasing function of $\omega\varepsilon(\underline{\omega}, \bar{\omega})$. In the following, $\sigma(\omega) > 0$ for all $\omega\varepsilon(\underline{\omega}, \bar{\omega})$ is assumed to hold.

(A_5) The rate of growth of the labour force is exogenously determined and remains constant throughout the accumulation process.

Let λ be this rate. Then $\hat{L} = \lambda$.[1]

4.2. The process of capital accumulation is given by

$$\dot{K} = Y_1 - \mu K, \text{ or} \tag{20}$$

$$\hat{K} = \frac{Y_1}{K} - \mu = \frac{y_1 l_1}{k} - \mu \tag{21}$$

Then

$$\hat{k} = \hat{K} - \hat{L} = \frac{y_1 l_1}{k} - (\lambda + \mu) \tag{22}$$

Since the economy is always at equilibrium, $\frac{y_1 l_1}{k}$ must satisfy (5). Thus

$$\hat{k} = \frac{s_r k + s_w \omega(k)}{k} f_1'[k_1[\omega(k)]] - (\lambda + \mu)\text{ [2]} \tag{23}$$

k^* is a *balanced capital-labour ratio* if

$$\frac{s_r k^* + s_w \omega^*}{k^*} f_1'(k_1^*) = \lambda + \mu \tag{24}$$

where ω^*, k_1^* are the equilibrium wage-rental ratio and capital-labour ratio in sector 1, respectively, which correspond to k^*.

4.3.* Let

$$\varphi(k) \equiv \frac{s_r k + s_w \omega(k)}{k} f_1'[k_1[\omega(k)]]\,. \tag{25}$$

We can show that

$$\frac{1}{\varphi(k)} \varphi'(k) = - \frac{s_w \omega(k)}{[s_r k + s_w \omega(k)]k} + \frac{s_w k_1[\omega(k)] - s_r k}{[s_r k + s_w \omega(k)][\omega(k) + k_1[\omega(k)]]} \cdot \omega'(k) \tag{26}$$

We will examine below several cases in which $\varphi'(k) < 0$ holds. Therefore, in all these cases a unique k^* exists, provided that

$$\lim_{k \to \psi(\underline{\omega})} \varphi(k) > \lambda + \mu > \lim_{k \to \psi(\overline{\omega})} \varphi(k). \tag{27}$$

4.4.* First, let us suppose that the *capital-intensity condition* holds, i.e., that the consumption-goods industry is always at least as capital intensive as the capital-goods industry. Thus, $k_2[\omega(k)] \geqq k_1[\omega(k)]$, for all k satisfying (12). We also assume that $s_r \geqq s_w$.

[1] . and ^ denote the time rate of change and the relative time rate of change of a variable.

[2] (23) generalizes the corresponding equation appearing in [6] and [7]. Solow [4] derives a relation which can be shown to be exactly the same as (23). (23) is also examined by Inada [2].

Under the present conditions, $s_r k \geqq s_w k_1[\omega(k)]$, and $\omega'(k) = 1/\psi'(\omega) > 0$, are seen to hold. Therefore, $\varphi'(k) < 0$ for all k satisfying (12), and there exists a unique k^* such that $\varphi(k^*) = \lambda + \mu$. Further, as $\varphi(k)$ is a strictly decreasing function of k, we easily see from (23) that this unique k^* is globally stable.

4.5. We have proved the following:

Stability Theorem 1: Under the capital intensity condition and if $s_r \geqq s_w$ holds, then along any path of intertemporal equilibria, for which (27) holds, the capital-labour ratio k approaches asymptotically the unique balanced capital-labour ratio k^*.

4.6.* Let us examine again (25) and (26). (26) can be written as

$$(26') \qquad \frac{1}{\varphi(k)}\,\varphi'(k) = \frac{1}{\sigma[\omega(k)]}\,\frac{\omega(k)}{k[\omega(k)] + k_1[\omega(k)]]}\,\frac{1}{[s_r k + s_w(k)]}$$

$$\cdot \left\{ s_w k_1[\omega(k)] - s_r k - s_w \sigma[\omega(k)] \,.\, \Big(\omega(k) + k_1[\omega(k)]\Big)\right\}$$

since $\omega'(k) = 1/\psi'(\omega) = \dfrac{1}{\sigma[\omega(k)]}\,\dfrac{\omega(k)}{k}$.

This equation gives us the necessary and sufficient conditions for stability of the balanced capital-labour ratio k^*. Global stability requires that $\varphi'(k) < 0$ holds for all k satisfying (12). Hence it requires that if for such k, $\sigma[\omega(k)] > 0$, then

$\left\{ s_w k_1[\omega(k)] - s_r k - s_w \sigma[\omega(k)] \Big((\omega(k) + k_1[\omega(k)]\Big)\right\} < 0$ holds, or that

$$(28) \qquad \sigma[\omega(k)] > \max \left\{ \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])}, 0 \right\},$$ [1] holds.

This is as far as we can go with respect to the necessary and sufficient conditions for stability. We may remark that if $s_w = 0$, then direct consideration of (26) shows that stability is insured if and only if $\sigma[\omega(k)] > 0$, i.e., if and only if $\psi(\omega)$ is an increasing function of ω. Finally, if $s_r = s_w$, then we know that $\sigma[\omega(k)]$ is positive and thus only (28) is relevant for stability.

4.7.* With respect to sufficient conditions for stability several results are now readily available.

First, $\sigma_1[\omega(k)]$, $\sigma_2[\omega(k)] \geqq 1$ is a sufficient condition for the stability of the balanced capital-labour ratio k^*. For then

$$\sigma[\omega(k)] \geqq 1, \text{ and } \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} < 1$$

holds for all k satisfying (12). Thus (28) is satisfied.

[1] Similarly, if $\sigma[\omega(k)]$ is negative for all k satisfying $\psi(\bar{\omega}) < k < \psi(\underline{\omega})$ (cf. footnote 1 above), then

$$(28') \qquad \sigma[\omega(k)] < \min \left\{ \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])}, 0 \right\}$$

must also hold for the stability of the balanced capital-labour ratio k^*.

Second, let us ignore any possibilities for factor substitution in the capital-goods industry, and also assume that $\sigma_1[\omega(k)] = 1$, for all k satisfying (12).

Then, $$\sigma[\omega(k)] = \frac{d[\omega(k)] + e[\omega(k)]}{a[\omega(k)]b[\omega(k)]} > 0.$$ We wish to show that

$$\sigma[\omega(k)] > \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} = \frac{s_w b[\omega(k)]k_1[\omega(k)] - s_r a[\omega(k)]}{s_w b[\omega(k)](\omega(k) + k_1[\omega(k)])}$$

or $$s_w \frac{d[\omega(k)] + e[\omega(k)]}{a[\omega(k)]} > s_w(1 - s_r)k_1[\omega(k)] - s_r(1 - s_w)k_2[\omega(k)]$$

This last inequality is easily seen to be true. Thus even when the capital-goods industry operates under (almost) fixed technical coefficients, we may still have stability of the balanced growth path if factor substitutability in the consumption-goods industry is high enough, *without regard to the factor intensities in the two industries.*[1]

We may also examine the opposite case, i.e., the case where $\sigma_1[\omega(k)] = 1$ for all k satisfying (12), while we ignore any possibilities of factor substitution in the consumption-goods industry.[2] Again

$$\sigma[\omega(k)] = \frac{c[\omega(k)] + e[\omega(k)]}{a[\omega(k)]b[\omega(k)]} > 0,$$

and we finally come to the inequality

$$s_w c[\omega(k)] > k_2[\omega(k)](\omega(k) + k[\omega(k)])(s_w(1 - s_r)k_1[\omega(k)] - s_r(1 - s_w)k_2[\omega(k)])$$

However this inequality cannot be unequivocally established.[3] Nevertheless, the global stability of the balanced capital-labour ratio k^* can be shown for some particular values of s_w and s_r. Thus if $s_w = 0$, then we have seen in page 15 that global stability is obtained since $\sigma[\omega(k)] > 0$. Also if $s_r = 1$, the above inequality holds. Since $\varphi'(k)$ is a continuous function of the parameters s_r, s_w, global stability is obtained for s_r near one and s_w near zero.

4.8. We have proved the following:

Stability Theorem 2: Let (a) $\sigma_2[\omega(k)] \geqq 1$ hold for all k satisfying (12); or (b) $\sigma_1[\omega(k)] \geqq 1$ hold for all k satisfying (12) and s_r be sufficiently near one or s_w be sufficiently near zero. Then along any path of intertemporal equilibria, the capital-labour ratio k approaches asymptotically the unique balanced capital-labour ratio k^*.

V FINAL REMARKS

5.1. The uniqueness of a competitive equilibrium in each time period, as well as the existence of a unique balanced capital-labour ratio, was shown in sections 3 and 4 on the

[1] This result can be more easily shown if we consider directly a Cobb-Douglas production function in the consumption-goods industry and fixed technical coefficients in the capital-goods industry.

[2] This may throw some light on the impact of the asymmetricity exhibited in our model by the fact that only capital goods are directly fed back into the process of capital accumulation. Otherwise, the case is of little practical interest.

[3] The inequality of course holds if $s_r \geqq s_w > 0$, $k_2[\omega(k)[> k_1[\omega(k)[$, which was shown in Section 4.4.*

basis of conditions (8), (12) and (27). These conditions indicate what the ranges of the functions $\alpha_i(k_i)$, $i = 1, 2$, $\psi(\omega)$, and $\varphi(k)$, respectively, must be for existence of solutions of the equations concerned.

It is possible to specify the ranges of the above functions by strengthening the conditions imposed on the production functions $f_i(k_i)$, $i = 1, 2$. E.g., in [7], and [2], the following conditions are imposed:

(29) $$f_i(0) = 0,\ f_i(\infty) = \infty$$
$$f_i'(0) = \infty,\ f_i'(\infty) = 0$$

Then, it can be shown that

(a) $\lim_{k_i \to 0} \alpha_i(k_i) = 0,\ \lim_{k_i \to \infty} \alpha_i(k_i) = \infty$

thus in (8) $\underline{\omega} = 0$, $\overline{\omega} = +\infty$. Further,

(b) $\lim_{\omega \to 0} \psi(\omega) = 0$, and $\lim_{\omega \to \infty} \psi(\omega) = \infty$.

Finally,

(c) $\lim_{k \to 0} \varphi(k) = \infty,\ \lim_{k \to \infty} \varphi(k) = 0.$

However, conditions (29), suggested by production functions of the Cobb-Douglas type, are not met for production functions of other familiar types, as e.g., for constant-elasticity-of-substitution production functions.[1]

In the present paper we prefer to impose directly (8), (12), and (27), instead of (29), mainly because we consider various cases in which the elasticities of factor substitution exhibited by the production functions are within certain limits. It is not apparent that for such production functions the values of $\alpha_i(k_i)$, $\psi(\omega)$, and $\varphi(k)$, at the end-points of their domain of definitions, are those indicated by (29).

5.2. We have deliberately refrained from introducing an interest rate into the model. In the present model, in which our assumptions in effect separate one market from all successive ones, an interest rate does not play an independent role. It can simply be defined by an equation referring to the yield of the capital goods throughout their life span. Thus, if for simplicity we assume static expectations, we have:

(30) $$p = \int_0^\infty r(s) e^{-(\rho+\mu)s}\, ds = \frac{r}{\rho + \mu}$$

which serves to define ρ. Equivalently, we may think of p in (30) as being the " demand price " of capital goods, determined when r and ρ are given. On the other hand the maximizing behaviour of the entrepreneurs provides us with the " supply price " of the capital goods. Equality of the two is needed for an equilibrium position.

5.3. The assumptions of Section 2.2. can be relaxed in some respects. E.g., a part of (A_1) can be substantially relaxed. Namely, in each time period the total existing quantity of labour services, $\bar{L}$, may not be necessarily offered to the market at any price. On the contrary we assume that labour services yield directly consumable services. We assume

[1] See Arrow, e.a. [1[.
One can however show that for C-E-S production functions the limits indicated under (a) and (b) still hold.

however that the supply of labour services for production purposes is an increasing function of the real wage w. Since $w'(\omega) > 0$ can easily be shown to hold, we denote this function by $L(\omega)$. $L(\omega)$ may be of the form:

$$L(o) = 0,\ L(\omega) \leqq \bar{L}, \text{ for all } \omega \text{ satisfying (8), and } L'(\omega) \geqq 0$$

This case arises if e.g, the workers decide on the basis of their real wage what proportion of their labour services will be offered to the market and what will be withheld for direct consumption by themselves.

Further, we may assume that the rate of growth of the labour force λ is an increasing function of the real wage, or $\lambda = \lambda(\omega)$, within some exogenously given (physiological) limits. This is a reasonable assumption in the present context.

We can easily show that none of our conclusions change by this generalization.

The equation referring to the labour market now becomes

(31) $$L_1 + L_2 = L(\omega),\ L(\omega) \leqq \bar{L}$$

The capital-labour ratio at any time period is now a decreasing function of ω, $k(\omega)$. Thus (5) again provides us with a unique solution for ω under exactly the same conditions as before. The only case which is still excluded is that of a " backward-rising " supply function of labour services, in which case a unique solution ω is of course not necessarily insured.

With respect to intertemporal equilibria again our present assumptions do not affect our conclusions. We have:

(32) $$\hat{L} = \lambda[\omega(k)].$$

Then,

(23′) $$\hat{k} = \frac{s_r k + s_w \omega(k)}{k} - \lambda[\omega(k)] - \mu.$$

We again obtain the global stability of k^* under the same conditions as in Stability Theorems 1 and 2. As a matter of fact, global stability of k^* must be more prevalent here than before. This is of course natural for an instability of k^* would mean a continuous increase or decrease of k and would in due time generate corrective counteractions on the part of the labour force.

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REFERENCES

[1] Arrow, K. J., et al. " Capital-Labour Substitution and Economic Efficiency ", *The Review of Economics and Statistics*, 43 (1961), pp. 225-250.

[2] Inada, K. " On a Two-sector Model of Economic Growth: Comments and a Generalization ", Technical Report No. 110, Department of Economics, Stanford University, 1962.

[3] Shinkai, Y. " On Equilibrium Growth of Capital and Labour ", *International Economic Review*, 1 (1960), pp. 107-111.

[4] Solow, R. M. " Note on Uzawa's Two-sector Model of Economic Growth ", *The Review of Economic Studies*, 29 (1961), pp. 48-50.

[5] Srinivasan, T. N. " On a Two-sector Model of Growth ", Cowles Foundation Discussion Paper No. 139 (Revised), 1962.

[6] Uzawa, H. " On a Two-sector Model of Economic Growth ", *The Review of Economic Studies*, 29 (1961), pp. 40-47.

[7] ——————. " On a Two-sector Model of Economic Growth, II ", manuscript, 1961.