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On a Theorem of Scarf[1]

In [3], Herbert Scarf has given a remarkable solution for a classical problem of economics. In this note, I wish to suggest a simplification of his proof, and a slight weakening of his assumptions.

Let Ω denote the non-negative orthant of the commodity space R^l. The economy is made up of N infinite sequences of consumers. For each $j = 1, \ldots, N$, all the consumers of the jth sequence have the same resources I_j in the interior of Ω, and the same complete preference preordering $\underset{j}{\gtrsim}$ on Ω satisfying

(1) $\{x \in \Omega \mid x \underset{j}{\gtrsim} x'\}$ and $\{x \in \Omega \mid x' \underset{j}{\gtrsim} x\}$ are closed for every x' in Ω,

(2) for every x in Ω, there is x' in Ω such $x' \underset{j}{>} x$,

(3) $x' \underset{j}{>} x$ implies $t\,x' + (1 - t)\,x \underset{j}{>} x$ for every t such that $0 < t < 1$,

(4) $x \underset{j}{>} x'$ for some x' implies that x is interior to Ω.

An allocation is an N-tuple of infinite sequences $(x_1^i), \ldots, (x_N^i)$ of points of Ω, where x_j^i is the consumption of the ith consumer in the jth sequence, such that

$$(5) \qquad \lim_{n \to \infty} \left(\sum_{=1}^{n} \sum_{j=1}^{N} x_j^i - n \sum_{j=1}^{N} I_j \right) = 0.$$

A finite coalition S of consumers blocks an allocation $\big((x_1^i), \ldots, (x_N^i)\big)$ if, for every consumer (i, j) in S, there is a consumption y_j^i in Ω such that $\sum_{(i,j) \in S} y_j^i = \sum_{(i,j) \in S} I_j^i$, and $y_j^i \underset{j}{\gtrsim} x_j^i$ for every (i, j) in$S$, while $y_j^i \underset{j}{>} x_j^i$ for at least one (i, j) in S.[2] The core of the economy is the set of allocations that no finite coalition blocks.

An allocation $\big((x_1^i), \ldots, (x_N^i)\big)$ and a price system p form an equilibrium of the economy if, for every (i, j), the consumption x_j^i is a greatest element of the set $\{x \in \Omega \mid p \,.\, x \leqq p \,.\, I_j\}$ for $\underset{j}{\gtrsim}$.

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[2] It is convenient, here, to identify the resources of consumer (i, j) by I_j^i, although I_j^i is a constant with respect to i. Given the assumptions made on preferences, our definition of a blocking coalition is easily seen to be equivalent to H. Scarf's.

THEOREM : *Given an allocation* $\big((x_1^i), \ldots, (x_N^i)\big)$ *in the core, there is a price system p with which it forms an equilibrium.*

Proof: By (1), there is a continuous utility function u_j on Ω for every j ([1], p. 56). We denote $u_j\,(x_j^i)$ by v_j^i. Two cases have to be distinguished:

(*a*) *for every j,* $\mathit{Inf}_i\, v_j^i = \underline{\mathit{lim}}_i\, v_j^i$.

We introduce the notation

$$C_j^i = \{x \in \Omega \mid u_j\,(x) > v_j^i\}, \quad \Gamma_j^i = C_j^i - \{I_j\};$$

$$C_j = \{x \in \Omega \mid u_j\,(x) > \mathrm{Inf}_i\, v_j^i\}, \quad \Gamma_j = C_j - \{I_j\}.$$

All these sets are non-empty, by (2), and convex, by (3) and (1) ([1], p. 60). They also have non-empty interiors, for every C_j^i does. Indeed, let x be a point in C_j^i, i.e., such that $x \underset{j}{>} x_j^i$. By (1), x has a neighbourhood in Ω all of whose elements $\underset{j}{>} x_j^i$. But, in that neighbourhood, there are points interior to Ω. Any one of them is interior to C_j^i.

The basic property of the sets Γ_j is

(6) 0 is not interior to the convex hull of $\bigcup\limits_{j=1}^{N} \Gamma_j$.

To establish this, we denote the interior of a set S by Int S, its convex hull by $H(S)$, and its closure by $\bar{S}$, and we first prove that

$$(7) \quad \mathrm{Int}\, H\,(\bigcup_j \Gamma_j) \subset H(\bigcup_j \mathrm{Int}\ \Gamma_j).$$

$$\mathrm{Int}\quad H\,(\bigcup_j \Gamma_j) \subset \mathrm{Int}\, H\,(\bigcup_j \overline{\mathrm{Int}\ \Gamma_j}) \subset \quad \mathrm{Int}\, H\,(\overline{\bigcup_j \mathrm{Int}\ \Gamma_j}) \subset \mathrm{Int}\ \overline{H\,(\bigcup_j \mathrm{Int}\ \Gamma_j)}$$

$$= \mathrm{Int}\, H\,(\bigcup_j \mathrm{Int}\ \Gamma_j).$$

Assume now that (6) does not hold. According to (7), there are, for each j, a point y_j' in Int Γ_j, and a non-negative real number α_j, with $\sum\limits_{=1}^{N} \alpha_j = 1$, such that

$$\Sigma\ \alpha_j\, y_j' = 0.$$

Thus, one can find, for each j, a point y_j in Γ_j, and a non-negative *rational* number r_j, with $\sum\limits_{=1}^{N} r_j = 1$, such that

$$\Sigma\ r_j\, y_j = 0.$$

Multiplying by a common denominator of the r_j, we obtain

$$\Sigma\ k_j\ y_j = 0$$

for an N-tuple (k_j) of non-negative integers, not all zero. Since $y_j \in \Gamma_j$, one has $u_j(y_j + I_j) > \mathrm{Inf}_i\ v_j^i$. Therefore, according to (*a*), we can select, in the jth sequence, k_j consumers whose v_j^i are less than $u_j(y_j + I_j)$. This means that y_j belongs to the set Γ_j^i of each-one of these k_j consumers. Consequently, 0 belongs to the sum of the sets Γ_j^i of the $k_1 + \ldots + k_N$ consumers we have selected. And the coalition of these consumers would block the given allocation.

Having established (6), we apply Minkowski's theorem to the situation it describes, and we obtain a hyperplane through 0, with normal p, bounding for $\bigcup\limits_{j=1}^{N} \Gamma_j$, hence for every Γ_j. We write this as $p\ .\ \Gamma_j \geqq 0$, or $p\ .\ C_j \geqq p\ .\ I_j$. However, C_j^i is contained in C_j for every (i, j). In addition, by (3), every x such that $x \underset{j}{\gtrsim} x_j^i$ is adherent to C_j^i. Therefore

(8) for every (i, j), $x \underset{j}{\gtrsim} x_j^i$ implies $p\ .\ x \geqq p\ .\ I_j$.

In particular, $p\ .\ x_j^i \geqq p\ .\ I_j$ for every (i, j). If any of these inequalities were strict, the inner product of p and the vector in the parenthesis of (5) would not tend to zero when $n \longrightarrow \infty$.

Hence

$$p\ .\ x_j^i = p\ .\ I_j \text{ for every } (i, j).$$

Finally, since I_j is interior to Ω, it follows readily from (8) ([1], p. 69), that x_j^i is a greatest element of the set $\{x \in \Omega \mid p\ .\ x \leqq p\ .\ I_j\}$ for $\underset{j}{\gtrsim}$.

(*b*) *for some* j', $\mathit{Inf}_i\ v_{j'}^i < \underline{\mathit{lim}}_i\ v_{j'}^i$.

We will show that this case cannot occur. Notice first that, according to (5),

$$\lim_{n \to \infty} \left(\sum_{j=1}^{N} x_j^n - \sum_{j=1}^{N} I_j \right) = 0.$$

Therefore the sequence of N-tuples (x_j^n) is bounded, and we can extract a subsequence converging to the N-tuple (x_j^o). Clearly

$$(9) \qquad \sum_{j=1}^{N} x_j^o = \sum_{=1}^{N} I_j.$$

Moreover

$$u_j(x_j^o) \geqq \mathrm{Inf}_i\ v_j^i \text{ for every } j, \text{ and } u_{j'}(x_{j'}^o) > \mathrm{Inf}_i\ v_{j'}^i.$$

The last inequality, which follows from (*b*), implies $x_{j'}^o \underset{j}{>} x_{j'}^i$ for some i, hence, by (4),

$$x^o_{j'} \text{ is interior to } \Omega.$$

Let $s(x,r)$ denote the open sphere with center x and radius $r > 0$. We can choose r small enough for $s(x^o_{j'}, r)$ to be contained in Ω, and for the utility of every consumption in $s(x^o_{j'}, r)$ to be greater than $\mathrm{Inf}_i\, v^i_{j'}$. By (2) and (3), there is, for every $j \neq j'$, a consumption x^*_j in $s(x^o_j, \frac{r}{N})$ such that

$$u_j(x^*_j) > u_j(x^o_j) \quad (j \neq j').$$

We define $x^*_{j'}$ as equal to $\sum_{j=1}^{N} x^o_j - \sum_{j \neq j'} x^*_j$. Thus $| x^*_{j'} - x^o_{j'} | < r$. Consequently $x^*_{j'}$ is in Ω and

$$u_{j'}(x^*_{j'}) > \mathrm{Inf}_i\, v^i_{j'}.$$

Also, by (9),

$$\sum_{j=1}^{N} x^*_j = \sum_{j=1}^{N} I_j.$$

To conclude, select for each j, a consumer (i, j) such that $x^i_j \underset{j}{\lesssim} x^*_j$. The coalition of these N consumers blocks the given allocation.

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The theorem can be generalized without modification of the proof. For instance, the common consumption set X_j of the consumers of the jth sequence may be any unbounded closed, convex set with a non-empty interior (instead of being Ω), provided that the asymptotic cone of $X = \sum_{j=1}^{N} X_j$ satisfies $AX \cap (-AX) = \{0\}$ (to insure that the sequence (x^n_j), at the beginning of (*b*), is bounded). Assumptions (1), (2), (3), and (4) are made on the preferences $\underset{j}{\lesssim}$ on X_j. Then, given an allocation in the core, there is a price system with which it forms a quasi-equilibrium (a definition of this concept, and a discussion of its relation to the concept of equilibrium will be found in [2]).

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[3] Scarf, H., " An Analysis of Markets with a Large Number of Participants," *Recent Advances in Game Theory*, the Princeton University Conference, 127-155, 1962.