THE ACCUMULATION OF RISKY CAPITAL: A SEQUENTIAL UTILITY ANALYSIS

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This paper presents a utility analysis of personal saving in which the only storable asset, capital, exposes its holder to the risk of capital gain or loss. The consequences of this risk and the effect of other parameters upon the optimal consumption policy is analyzed by means of dynamic programming.

This paper investigates the optimal lifetime consumption strategy of an individual whose wealth holding possibilities expose him to the risk of loss. The vehicle of analysis is a stochastic, discrete-time dynamic programming model that postulates an expected lifetime utility function to be maximized. All wealth consists of a single asset, called capital.

The problem described belongs mainly to the theory of personal saving. Models of saving behavior thus far have been entirely deterministic [4, 7, 8, 11, 12, 13], whereas, in fact, the saver is typically faced with the prospect of capital gain or loss. So it seems appropriate to determine whether the results of the conventional theory carry over or have to be qualified upon admitting capital risk into the theory. The question also arises as to the effect of capital risk itself upon the level of consumption. This neglected factor may play a role in the explanation of certain inter-group differences in saving behavior.

These questions are easier to raise than to answer, and this paper is frankly an exploratory effort. No generality or definitiveness is claimed for the results obtained. A brief outline of the paper and sketch of some of these results follow.

In the first two sections, a utility function and a stochastic capital growth process are postulated and discussed. Subsequently, the "structure" of the optimal consumption policy, that is, the way in which consumption depends upon the individual's age and capital, is established. One's expectations, based on existing "deterministic" theory, are confirmed: Optimal consumption is an increasing function of both age and capital. Little else appears deducible without further restrictions upon the utility function.

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1 For helpful discussions on this subject I am grateful to T. N. Srinivasan and S. G. Winter.

2 An exception is a Cowles Foundation Discussion Paper by Martin Beckmann [2]. That paper (which deals with wage rather than capital uncertainty) uses a technique similar to the one here.

3 The model below resembles Ramsey's more than contemporary models [7, 11] so that it is largely his results that are modified.
Thereafter attention is confined to certain monomial utility functions. These special cases cannot yield general theorems but they do have the function of providing counter-examples to conjectures and of serving to suggest other hypotheses for empirical test.

For example, it is shown that the classical phenomenon of "hump saving" [8, 12] need not occur, quite apart from reasons of time preference, if capital is risky. Instead a low-capital "trap" region is possible in which it is optimal to maintain or decumulate capital, no matter how distant the planning horizon.

These utility functions all make consumption linear homogeneous in capital and permanent nonwealth income, and linear in each of these variables. But the straight-line classroom consumption function is not really upheld: Consumption cannot be expressed as a function of aggregate expected income because expected wage income (treated as certain) and expected capital income have different variances, whence different impacts upon the level of consumption. The marginal propensity to consume out of risky income is smaller than out of sure income. This result may help to explain why households which depend primarily upon (risky) capital income (e.g., farmers, wealthy heirs) are comparatively thrifty.

Finally, we consider the effect upon the consumption level of variations in the riskiness and in the expected rate of return of capital (given capital and nonwage income). Not surprisingly, the direction of effect of both are unpredictable without knowledge of the type of utility function; the familiar conflict between substitution and income effects applies as much to risk as to the rate of return. Two closely related utility functions give opposite results. But it is interesting that risk always "opposes" return. Where increase of the rate of return raises (reduces) the propensity to consume, an increase in risk reduces (raises) it; and where return has no effect, neither does risk.

1. THE BEHAVIOR OF CAPITAL

Capital is treated as homogeneous in the sense that each unit of the asset experiences the same rate of return.⁴

The individual's consumption opportunities occur at discrete, equally spaced points in time. These points divide the lifetime of the consumer into \( N \) periods. The state of the system at the beginning of each period, \( n = 1, 2, \ldots, N \), is described by the variable \( x_n \), the amount of capital then on hand. At this time the individual chooses to consume some amount \( c_n \) of this capital.

⁴ Alternatively, capital might have been envisioned more like identical female rabbits. In any short time period, some units of the asset would multiply while others not. This might be termed subjective or \textit{ex ante} homogeneity.
The unconsumed capital is left to grow at a rate which is not then known. In addition to the capital growth, the individual receives an amount, $y$, of nonwealth income at the end of the period. This income is the same each period. Consequently the amount of capital available for consumption in the next period is given by the difference equation

\[(1.1) \quad x_{n+1} = \beta_n(x_n - c_n) + y, \quad x_1 = k,\]

where $\beta_n - 1$ is the rate of return earned on capital in the $n$th period.

We shall assume that the random variables $\beta_n$ are independent and drawn from the same probability distribution. There are $m$ possible rates of return, $0 \leq \beta_i, i = 1, 2, \ldots, m$. The probability of the $i$th rate of return will be denoted by $p_i$ (the same from period to period). In addition we shall assume that $\bar{\beta} = \sum_i^m p_i \beta_i > 1$ so that the consumer expects capital to be productive. However, $\sum_i^m p_i (\beta_i - \bar{\beta})^2 > 0$, and so the realized return may differ from the expected one.

2. THE UTILITY FUNCTION

This model postulates a consumer who obeys the axioms of the von Neumann-Morgenstern utility theory. His consumption strategy (or policy) can therefore be viewed as maximizing the expected value of utility, which is determined up to an increasing linear transformation.

Second, we suppose that the lifetime utility associated with any consumption history is a continuously differentiable function of the amount consumed at the beginning of each period.

The lifetime utility function is assumed to be of the independent and additive form

\[(2.1) \quad U = \sum_{i=1}^N \alpha^{x_i-1} u(c)_i, \quad 0 < \alpha \leq 1.\]

The implications of this functional form are several. Preferences for the consumption "chances" or distributions of any period are invariant to the consumption levels befalling the individual in other periods (separability). Preferences among consumption subhistories in the future are independent of the age of the individual (stationarity). Preference for a consumption strategy is independent of or unaffected by any serial correlation in the random consumption sequence associated with that strategy (independence).\(^5\)

\(^5\) However the necessary and sufficient conditions for independence of utilities when choice takes place under uncertainty have yet to be investigated. The independence of utilities when choice takes place in an environment of certainty has been axiomatized by Debreu [6]. The meaning of additivity with a variable utility discount factor and an infinite number of periods has also been investigated by Koopmans [9].
The same axioms which yield the von Neumann-Morgenstern utility indicators also imply that \( U(c_1, \ldots, c_N) \) is bounded from above and below. Consequently \( u(c_n) \) is also a bounded function. Let \( \hat{a} \) and \( \underline{a} \) denote the upper and lower bounds of \( u(c_n) \), respectively.

Finally, we postulate that the individual strictly prefers more consumption to less (monotonicity) and that he is strictly averse to risk (concavity). The latter means that for every pair of consumption histories \( (c_1, \ldots, c_N) \) and \( (c'_1, \ldots, c'_N) \) to which he is not indifferent, he will strictly prefer the certainty of the compromise history \( \theta c + (1 - \theta)c' \) to the mixed prospect offering him the history \( c \) with probability \( \theta \) and the history \( c' \) with probability \( 1 - \theta \), \( 0 < \theta < 1 \). It follows trivially that \( u(c_n) \) is a strictly increasing and strictly concave function.

3. DERIVATION OF THE FUNCTIONAL EQUATIONS

We seek the consumption strategy (or, equivalently, policy)—denoted by the sequence of functions \( \{c_n(x)\} \) for \( x \geq 0 \), \( n = 1, 2, \ldots, N \)—which maximizes expected lifetime utility:

\[
J_N(c) = \exp \frac{U}{\beta^n}
\]

subject to the relation (1.1). Notice that the optimal \( c_n \), \( n = 1, \ldots, N \), will be a stochastic rather than a predetermined function of \( n \).

To treat this variational problem we turn to the technique of dynamic programming [3]. Observing that the maximum expected value of lifetime utility depends only upon the number of stages in the process and the initial capital, \( k \), we define the function

\[
w_N(k) = \max J_N(c)
\]

where the maximum is taken over all admissible policies. The function defined may be interpreted as the utility-of-wealth function of the optimizing consumer having \( N \) periods of life remaining.

Next one reduces the problem with \( N \) decision variables to a sequence of \( N \) problems, each involving only one policy variable, the decision which must be taken at the current moment. This approach leads to the following functional equations:

6 A proof of boundedness may be found in [1] and [5]. The proof uses the "continuity axiom" and a generalization of the St. Petersburg game, the idea for which Arrow [1] credits to K. Menger.

7 The argument starts with the observation that with the elapse of each period the individual is confronted with another multistage decision problem which differs only in having one less stage and, in general, a different initial capital. By the "principle of
\( w_N(x) = \max_{0 \leq c \leq x} [u(c) + \alpha \sum_{i=1}^{m} \beta_i w_{N-1}(\beta_i(x - c) + y)], \quad N \geq 2, \)

and

\( w_1(x) = \max_{0 \leq c \leq x} u(c) \)

which defines the utility of wealth in the single stage process. Without a subscript, the symbol \( c \) shall always denote the value of consumption in the first period of the (not necessarily original) multistage process. Similarly \( x \) shall denote capital at the start of whatever process is being considered.

4. PROPERTIES OF THE OPTIMAL CONSUMPTION POLICY

A number of standard results follow from this model: First, the optimal consumption strategy is unique; the optimum value of \( c_n \) is a unique function of \( x_n \) for every \( n \).

The proof consists of showing that the utility of wealth function is strictly concave if the utility of consumption function is strictly concave; therefore the maximand in each period is a strictly concave function of current consumption, whence the maximizing consumption level is unique.\(^8\)

Second, consumption is an increasing function of capital and age. The latter result depends upon the further assumption made now that \( \alpha \beta > 1 \). It will become clear in the next section that this inequality is also a necessary condition for positive accumulation of capital.

The proof is rather involved and is omitted here. It can be shown that if \( \alpha \beta > u'(0)/u'(y) \) then, with \( N \geq 2 \) periods remaining, consumption is the following function of capital:

\[ c = \begin{cases} 0, & 0 \leq x \leq \bar{x}_N, \\ c_N(x), & x \geq \bar{x}_N, \end{cases} \tag{4.1} \]

where \( c_N(x) = 0 \) at \( x = \bar{x}_N \), \( c_N(x) > 0 \), and \( d_N(x) < x \). The function \( c_N(x) \)

\(^8\) Readers who are unfamiliar with this type of proof may wish to consult [3]. Proofs of the result above and of the other results stated but not proved in this section can be found in an earlier version of this paper (same title) by the author, published as Cowles Foundation Discussion Paper No. 109, which is available on request to the Cowles Foundation.
represents the interior portion of the solution where consumption is not constrained by the nonnegativity requirement.

It can be further shown that the marginal utility of wealth declines with age and capital and that the "consumption function" in (4.1) shifts leftward and upward as age increases:

\[
\omega_1(x) < \omega_2(x) < \ldots < \omega_N(x) < \ldots,
\]

\[
(4.2)
\]

\[
\alpha(x) > \ldots > \alpha_N(x) > \ldots,
\]

\[
0 < \bar{\alpha}_2 < \ldots < \bar{\alpha}_N < \ldots.
\]

Of course, when \( N = 1 \), \( c = x \).

In the other case, where \( \kappa \beta \leq \mu'(0)/\mu'(y) \), the constraint that consumption cannot exceed capital becomes binding for \( N = 2 \) and possibly for larger \( N \)—when capital is sufficiently small. If there is a value of \( x \geq 0 \) for which \( \alpha_N(x) = x \) then, denoting this value by \( \bar{\alpha}_N \), we obtain

\[
c = \begin{cases} 
\alpha, & 0 \leq x \leq \bar{\alpha}_N, \\
\alpha_N(x), & x \geq \bar{\alpha}_N.
\end{cases}
\]

Again, as age increases, \( N \) decreases, the marginal utility of wealth function decreases and the consumption function shifts upward. Consequently the intersection where \( c = x \) shifts rightward:

\[
\bar{\alpha}_2 > \ldots > \bar{\alpha}_N \geq 0.
\]

A typical possibility is graphed in Figure 1. This consumption function is of the second type. As \( N \) becomes small, the consumption schedule shifts upward. When \( N = 2 \), the function intersects the \( c = x \) line. When \( N = 1 \), \( c = x \) at all \( x \).

The \( I(x) \) function is defined in the next section.
5. CONDITIONS FOR EXPECTED ACCUMULATION

The preceding theorems confirm our expectations about the qualitative behavior of optimal consumption. They do not go far enough to permit inferences about the behavior of capital as a function of age and initial capital. One might ask if the model generates “hump saving” \([8, 12]\), so important in the theory of aggregate capital formation. The “hump saver” saves when he is young and dissaves as he grows older. Therefore we ask: Can one find a value of \(N\) sufficiently large to induce the individual to save —more precisely, to cause the expected value of his subsequent capital to exceed the value of his present capital?\(^9\)

Let us define “expected income,” \(I(x)\), to be the amount of consumption such that the expected value of capital in the next period equals present capital. Now \(\exp x_{n+1} = y + \beta(x_n - c_n)\). Expected stationarity, \(x_{n+1} = x_n\), implies \(c_n = (y/\beta) + [(\beta - 1)/\beta] x_n = I(x)\). Expected income is displayed as a function of capital in Figure 1. Our question is then whether, in the limit, as \(N\) approaches infinity, \(c_N(x) < I(x)\) for all \(x \geq y\).

The answer is clear cut when capital is riskless. Then \(\beta_i = \beta\) for all \(i\) and we obtain the following recurrence relation in the limiting utility of wealth function:

\[
(5.1) \quad \omega(x) = \max_{c} \{u(c) + \alpha \omega(\beta(x - c) + y)\}.
\]

The maximum is an interior one for \(x \geq y\) so that \(c(x)\) defined by

\[
(5.2) \quad u'(c) - \alpha \beta \omega'(\beta(x - c) + y) = 0
\]

determines \(c\) as a function of \(x\).

Differentiating totally with respect to \(x\) gives

\[
(5.3) \quad \omega'(x) = \alpha \beta \omega'(\beta(x - c) + y) + c'(x) [u'(c) - \alpha \beta \omega'(\beta(x - c) + y)]
\]

\[
= \alpha \beta \omega'(\beta(x - c) + y) \quad \text{[by (5.2)].}
\]

Since \(\omega'(x)\) is monotone decreasing, (5.3) implies that \(x_{n+1} > x_n\) if and only if \(\alpha \beta > 1\). Therefore, denoting the limiting consumption function by \(c(x)\),

\[
(5.4) \quad c(x) < I(x) \quad \text{for all } x \geq y.
\]

This simple result fails to extend to risky capital. When \(\beta_i \neq \beta\) for some \(i\), (5.3) becomes

\[
(5.4) \quad \omega'(x) = \alpha \sum p_i \beta_i \omega'(\beta_i(x - c) + y).
\]

From (5.4) no general conclusions concerning the conditions for expected capital growth can be drawn. Of course capital cannot be expected to grow

\(^9\) Of course, an affirmative answer would not be very interesting if the necessary value of \(N\) exceeds human life expectancy!
very long unless $\beta > 1$. But $x \beta > 1$ is insufficient to guarantee "expected" capital growth.\footnote{Several plausible cases are the following. First, there may be no capital level at which the expected returns to saving repays the risks. Or it may be that the individual can "afford" the risks of net expected saving only when capital exceeds a critical value at which $c(x)$ intersects $I(x)$ from above. In the opposite case, additional wealth is worth the risks only as long as capital falls short of the level where $c(x)$ intersects $I(x)$ from below.}

It is clear that the critical value which $x \beta$ must exceed if capital growth is to be expected will depend upon the distribution of $\beta_i$ and the shape of the marginal utility function $w'(x)$. The only practical procedure here is to investigate the implications for capital growth of particular classes of utility functions.

6. IMPLICATIONS OF SELECTED MONOMIAL UTILITY FUNCTIONS

In this section we investigate the implications of certain types of monomial utility functions for the consumption function and for the expected path of capital.

We consider first the utility function\footnote{The function (6.1) fails to have the boundedness property assumed up to this point and thus it contradicts the "continuity axiom" mentioned in Section 2. Whatever the merits of that axiom, the function has received sufficient study in the context of deterministic models [4, 12, 13] to deserve our attention here.}

\begin{equation}
(6.1) \quad u(c_n) = \tilde{u} - \lambda c_n^\gamma, \quad \tilde{u}, \gamma > 0, \lambda > 1.
\end{equation}

Solving successively for the sequence of unknown functions $\{w_n(x)\}$, $N = 1, 2, \ldots$, yields

\begin{equation}
(6.2) \quad w_N(x) = \lambda (1 + x + \ldots + x^{N-1}) - \lambda (x b^{-\gamma})^{N-1} \left[ 1 + (x b^{-\gamma})^{N-1} \right]^{-1} + \frac{x}{(x b^{-\gamma})^{N-1} \gamma + 1}
\end{equation}

and

\begin{equation}
(6.3) \quad c_N(x) = \frac{(x b^{-\gamma})^{N-1}}{1 + (x b^{-\gamma})^{N-1} \gamma + 1}
\end{equation}

where

\[ b = \left( \sum x \beta_i x^{-\gamma} \right)^{-1} \]
If the reader applies (6.3) to \( c_{N+1}(x) \) and uses (6.2) he will obtain an expression for \( w_{N+1}(x) \) having the same form as (6.2). Note also that if \( \alpha = \beta_t = 1 \) for all \( i \), formula (6.3) calls for consuming a fraction \( 1/N \) of the individual's net worth, \( x + (N - 1)y \).

Provided that \( \alpha b^{-\gamma} < 1 \) (for which \( \alpha < 1, \beta > 1, \gamma > 0 \) is sufficient in the certainty case), the expressions in (6.2) and (6.3) converge as \( N \) approaches infinity, giving the solutions to the "infinite stage" process:

\[
\begin{align*}
(6.4) \quad w(x) &= \frac{\bar{a}}{1 - \alpha} - \lambda \left[ \frac{\frac{1}{\alpha b^{-\gamma} x^{\gamma+1} - 1}}{\alpha b^{-\gamma} x^{\gamma+1} - 1} \right]^{\gamma} + 1 \quad \left( x + \frac{\gamma}{b - 1} \right)^{-\gamma} \\
\text{and} \quad c(x) &= (1 - \alpha b^{-\gamma} x^{\gamma+1}) \left( x + \frac{\gamma}{b - 1} \right).
\end{align*}
\]

This limiting consumption function is useful as an approximation to \( c_N(x) \) for large \( N \).

(i) Properties of the consumption function.

A number of properties of the consumption functions (6.3) and (6.5) can be observed immediately. First, the consumption function is linear homogeneous in capital and nonwealth income. Of two households, both having identical utility functions like (6.1), if one household enjoys twice the capital and nonwealth income of the other, it will also consume twice as much.

Second, consumption is linear in capital and nonwealth income. The coefficient of wealth, \( \partial c/\partial x \), may be called the marginal propensity to consume (MPC) out of wealth.

The convergence condition \( \alpha b^{-\gamma} < 1 \) insures that \( \partial c/\partial x > 0 \). And \( \partial c/\partial x < 1 \) for all finite \( \alpha, b > 0 \).

The coefficient \( \partial c/\partial y \) may be called the MPC out of "permanent," sure, (nonwealth) income. Clearly \( \partial c/\partial y > 0 \) if and only \( b > 1 \) (given the convergence condition). What can be said concerning this condition?

When capital is risky (that is, when \( \beta_t \neq \beta \) for some \( i \)), then \( b < \bar{\beta} \). Therefore the postulate \( \bar{\beta} > 1 \) does not imply \( b > 1 \). We see thus that Keynes' "psychological law" stating that \( MPC > 0 \) applies only if capital has a positive net expected productivity and only if capital is sufficiently productive at that. However, we do observe positive MPC and if we were to

\[12\] To see this, draw a diagram showing \( \beta_i^{-\gamma} \) as a function of \( \beta_t \). Since \( \beta^{-\gamma} \) is a convex function of \( \beta \), \( \sum \rho_i \beta_i^{-\gamma} > \bar{\beta}^{-\gamma} \) whence \( \sum \rho_i \beta_i^{-\gamma} < 1/\bar{\beta}. \)
fit this model to data we should presumably find that \( b > 1 \). At any rate, we shall assume \( b > 1 \) unless we indicate the contrary.

Is the MPC also less than one, as Keynes had it? Of course, with \( b > 1 \), the MPC out of an income stream beginning sufficiently far in the future is bound to be less than one. Usually one considers the effect on (immediate) consumption of immediate income. To do that in the present model—where the paycheck is received at the end of the period—suppose capital increases by the same amount as \( y \), as if last period’s paycheck were increased too. Is this MPC out of “immediate,” nonwealth income smaller than one?

This MPC is

\[
1 - (\alpha b^{-\gamma})^{\frac{1}{\gamma+1}} \frac{b}{b - 1}
\]

and is smaller than one if and only if \( \alpha b > 1 \).

This is an interesting condition. This same condition, we show now, is necessary and sufficient for positive capital accumulation at all possible values of income and capital.

Note first that \( c(x) < I(x) \) for all \( x \geq y \)—causing the expected growth of capital—if and only if \( c(y) < y \) and \( c'(x) \leq I'(x) \). Now \( c(y)/y \) equals the MPC just analyzed so that \( \alpha b > 1 \) means \( c(y) < y \). The condition that \( c'(x) < I'(x) \) is

\[
1 - (\alpha b^{-\gamma})^{\frac{1}{\gamma+1}} < \frac{\beta}{\beta - 1}
\]

for which \( \alpha b > 1 \) is sufficient (although unnecessary).\(^{13}\)

The significance of this exercise lies in the possibility that \( 1 < b \leq 1/\alpha \), in which case capital will be expected to grow only if it exceeds a certain threshold. Suppose \( \alpha b = 1 \). Then all nonwealth income is consumed and there is “net expected saving”—that is, \( c(x) < I(x) \)—only if \( x > y \), i.e., only if the individual starts the period with some capital over and above his just-received wage of the previous period. Otherwise there will be no “hump saving” (in this case), even though \( \beta > 1/\alpha \).

A comparison of the MPC’s leads to an interesting finding: The greater nonwealth income, \( y \), as a proportion of total expected income, \( I(x) \), the larger is the ratio of consumption to expected income. This is because the MPC out of (sure, immediate) nonwealth income, \( c'(x) b/(b - 1) \), is greater than the consumption effect of that increase in current capital which is required to raise expected income by one dollar. Writing

\[
x = \frac{\beta}{\beta - 1} \left[ I(x) - \frac{x}{\beta} \right],
\]

\(^{13}\) Note that all these conditions reduce to \( b > 1 \) if \( \alpha = 1 \).
we see that the latter consumption effect is $c'(x)(\bar{\beta}/(\bar{\beta} - 1))$. Recalling that $b < \bar{\beta}$, we find that "sure" income has the stronger effect. This implies that, among households who have like utility functions and who face the same capital growth process, those whose expected income depends relatively heavily on risky capital will be observed to be relatively thrifty. This may help to explain why wealthy heirs, farmers, and certain other groups save a comparatively large proportion of their incomes. Further, the result suggests that capital income and labor income ought not to be aggregated in econometric analyses of consumption.

(ii) Variations of risk and return.

The last question taken up here relates to the effect upon consumption of variations in the riskiness and expected return from capital. Since the consumption function is linear homogeneous we can write

$$c = \frac{\partial c}{\partial x} x + \frac{\partial c}{\partial y} y,$$

whence these variations influence consumption through the marginal propensities, which are a function of $b$ (and independent of $x$ and $y$).

Let us consider first the effect of variations in risk and return on the value of $b$.

An increase in the expected return on capital is defined here as a uniform shift in the probability distribution of $\beta$, which leaves all its moments the same except the mean, $\bar{\beta}$. Such a shift increases $\bar{\beta}$ and $b$.

What effect has risk on the value of $b$? When capital is risky, $b < \bar{\beta}$. Thus the presence of risk (as distinct from marginal increases therein) decreases $b$.

Hence, capital's (net) productivity and its riskiness affect consumption in the opposite direction.

A second kind of risk effect results from a change in the degree of risk, somehow measured.

A probability distribution which offers a simple measure of risk is the uniform or rectangular distribution. This is a two-parameter distribution with mean $\bar{\beta}$ and range $2h$. The variance is $h^2/3$ so that $h$ is the measure of risk.

We show now that increases in $h$ reduce $b$ so that the "structural" and "marginal" effect of risk on $b$ are in the same direction. Noting that $\frac{\partial b}{\partial h} < 0$ means $\frac{\partial b^{-\gamma}}{\partial h} > 0$, we examine $b^{-\gamma}$.

By definition of $b$,

$$b^{-\gamma} = \int_{-h}^{\bar{\beta} + h} \beta^{-\gamma} \left(\frac{1}{2h}\right) d\beta.$$
Evaluating the integral we find
\[ b^{-\gamma} = \frac{1}{(1 - \gamma)^2 h} [(\hat{\beta} + h)^{1 - \gamma} - (\hat{\beta} - h)^{1 - \gamma}] \, . \]

Differentiating with respect to \( h \) yields
\[ \frac{db^{-\gamma}}{dh} = \frac{1}{2(1 - \gamma)h^2} [(\hat{\beta} - h)^{-\gamma}(\hat{\beta} - \gamma h) - (\hat{\beta} + h)^{-\gamma}(\hat{\beta} + \gamma h)] \, . \]

Assuming \( \gamma > 1 \), \( db^{-\gamma}/dh > 0 \) if and only if
\[ \frac{\hat{\beta} - \gamma h}{\hat{\beta} + \gamma h} < \left( \frac{\hat{\beta} - h}{\hat{\beta} + h} \right)^\gamma \, . \]

\( \beta \) equal to zero is excluded, for otherwise \( b \) is not defined. Consequently \( h < \hat{\beta} \) and the right hand side of the inequality must be positive. But so may be the left hand side (if \( \gamma < \hat{\beta}/h \)). The following shows the inequality is satisfied for all \( \gamma > 1 \).

Dividing both sides of the inequality by \( \hat{\beta} \), and defining \( z = h/\hat{\beta} \), we obtain
\[ \frac{1 - \gamma z}{1 + \gamma z} < \left( \frac{1 - z}{1 + z} \right)^\gamma \, . \]

which, taking the logarithm of both sides, we find to be satisfied if and only if
\[ \log (1 - \gamma z) - \log (1 + \gamma z) < \gamma [\log (1 - z) - \log (1 + z)] \, . \]

Expansion of the logarithmic functions into Taylor's series yields
\[ - \gamma z - \frac{(\gamma z)^2}{2} - \frac{(\gamma z)^3}{3} - \ldots - \left( \gamma z - \frac{(\gamma z)^2}{2} + \frac{(\gamma z)^3}{3} - \ldots \right) \]
\[ < \gamma \left[ \left( - z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots \right) - \left( z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \right) \right] \]

whence
\[ \left( \gamma z + \frac{(\gamma z)^3}{3} + \frac{(\gamma z)^5}{5} + \ldots \right) > \left( \gamma z + \frac{\gamma z^3}{3} + \frac{\gamma z^5}{5} + \ldots \right) \, . \]

This inequality can be seen to hold for all \( \gamma > 1 \). Therefore a margina increase in risk reduces the value of \( b \). Recalling that an increase in the expected return increases \( b \), we note that changes in risk and return have opposite effects on consumption.

We consider now the effect of a change in \( b \) upon consumption. Does the substitution effect dominate here—so that a rise in \( b \) encourages
saving and reduces consumption? Or does the income effect dominate?

Turning first to \( \partial c/\partial x \), we see from (6.5) that an increase in \( b \) raises \( \partial c/\partial x \).

Turning next to \( \partial c/\partial y \), we note from (6.5) that \( \partial c/\partial y = 1/(b - 1) \cdot \partial c/\partial x \). It would appear that a rise in \( b \) might reduce \( \partial c/\partial y \), because of the downward recapitalization (using \( 1/(b - 1) \)) of the \( y \) stream, if \( b \) were sufficiently small (\( b > 1 \)). It can be shown that \( d(\partial c/\partial y)/db \geq 0 \) if and only if \( (s \text{e}^{-r})^{-1/(\gamma + 1)} \leq (1 + \beta \gamma)/(1 + \gamma) \). If \( s = 1 \) this is satisfied for all \( b > 1 \); otherwise it is satisfied only for values of \( b \) above some value \( \hat{b} > 1 \).

Thus, if there is no utility discount, the income effect dominates here; then a rise in the expected return on capital weakens the incentive to save and an increase in risk compels more saving in order to reduce the insecurity of the future. But if the future is discounted, the individual feels "poorer"; then a rise in the expected return may encourage saving up to a point, after which the income effect dominates; this point comes sooner the smaller is \( y \). In either case, risk and return variations have opposing qualitative effects upon consumption.

(iii) Other utility functions.

To see that the implications of the utility function (6.1) for the effects of variations in risk and return are not general, one has only to modify the utility function thus:

\[
(6.6) \quad u(c) = \lambda c^\gamma, \quad \lambda > 0, \; 0 < \gamma < 1.
\]

All the equations (6.2)–(6.5) continue to hold with the difference that \( \lambda \) and \( \gamma \) are then replaced by \( -\lambda \) and \(-\gamma \), respectively. Hence the limiting consumption function is

\[
(6.7) \quad c(x) = \left[ 1 - (sbr^{1-\beta})^{-\frac{1}{\gamma - 1}} \right] \left( x + \frac{s \gamma}{b - 1} \right)
\]

where \( b^\gamma = \sum p_i \beta^\gamma \).

An increase in \( \beta \), other moments of the distribution unchanged, will increase \( b \).

Once again the effect of risk is easy to ascertain. Since \( \beta^\gamma \) is a concave function of \( \beta \), \( \sum p_i \beta_i^\gamma \leq \beta^\gamma \) whence \( b = (\sum p_i \beta_i^\gamma) < \beta \).

Turning finally to the effect of a marginal increase in risk upon \( b \), we find that the "natural" result \( d\beta^\gamma/db \leq 0 \) (meaning that global and marginal risk effects have like signs) depends upon the condition \( (\beta - \gamma h)/(\beta + \gamma h) > [(\beta - h)/(\beta + h)]^\gamma \), which is satisfied for all \( \gamma < 1 \).

Once again, risk and return work in opposite directions.

Consider now the effect of an increase in \( b \) upon consumption. Unlike the
previous example, \( \partial c/\partial x \) decreases with increasing \( b \), as can be seen from (6.7); the substitution effect dominates the income effect. And, as (6.7) clearly shows, \( \partial c/\partial y \) is also a decreasing function of \( b \) for all values of \( b > 1 \); the downward recapitalization of future income merely reinforces the substitution effect against the weaker income effect.

Thus an increase in expected return encourages saving while an increase of the riskiness of capital discourages saving. The implications of the utility function (6.6) are essentially opposite to those of the utility function (6.1).

To what can this contrast of results be attributed? The utility function is determined only up to a linear transformation, meaning that we can set \( \bar{u} = 0 \) in (6.1) without effect. Doing this reveals that both (6.1) and (6.6) are constant-elasticity utility functions with elasticity parameter \( \gamma \). The income effect dominates (unless \( b \) is small and \( y \) large) in the elastic case and the substitution effect dominates in the inelastic case.

Finally we examine a utility function that can produce some odd results, the logarithmic function in (6.8):

\[
(6.8) \quad u(c_N) = \log c_N.
\]

It appears to be impossible to solve for \( c_N(x) \) explicitly in terms of \( x \) and \( y \) except in the case \( y = 0 \). Then we easily find

\[
(6.9) \quad w_N(x) = (1 + \alpha + \ldots + \alpha^{N-1}) \log x + v(\theta, \alpha, N)
\]

where \( v(\theta, \alpha, N) \) depends only upon the parameters, denoted by \( \theta \), of the probability distribution of \( \beta \), \( x \) and \( N \), and not upon \( x \).

Also

\[
(6.10) \quad c_N(x) = \frac{x}{1 + \alpha + \ldots + \alpha^{N-1}}.
\]

When the utility function is logarithmic, the optimum consumption rate is independent both of the expected return and riskiness of capital. Consumption is linear homogeneous in capital. As \( N \) is increased, the consumption function flattens asymptotically until, in the limit,

\[
(6.11) \quad c(x) = (1 - \alpha)x.
\]

A limiting function exists only if \( \alpha < 1 \).\(^{14}\)

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\(^{14}\) For certain utility functions the existence of a limiting solution does not require \( \alpha < 1 \). Ramsey [12] argued that boundedness was sufficient but a condition on the elasticity or rate of approach to the upper bound is also necessary, at least in models not containing risk. Samuelson and Solow [14] assume that the upper utility bound is attained at a finite consumption rate, which is not a necessary condition.
REFERENCES


