A MODEL OF BANK PORTFOLIO SELECTION

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1. Introduction

Over the course of the last century, the implications of the assumption of profit maximization for the behavior of the firm have been tracked down in ever greater detail. Curiously, however, this firm has almost always been a seller of non-financial goods; banking has been studiously exempted from the application of such theory. The exemption is curious because the commercial bank seems in many respects more likely to fit the conditions of such static theory than the product manufacturer. The "method" of production and the "product" itself do not change, and hence the unpleasant necessity of neglecting some of the most interesting features of markets in order to devise marginal conditions does not arise. Even the proverbial conservatism of bankers is prop to such theory, for it may well make banking less prone to upsetting expectational factors than other markets.

The reason for this neglect of banking probably lies in the implication of straightforward profit maximization: that the bank should acquire a portfolio consisting entirely of the asset whose yield (less any costs of maintenance or acquisition) is greatest.¹ But this procedure misses the very essence of banking, which is to "borrow short and lend long." Thus, the "profit" which a bank derives from its portfolio must be interpreted in terms of not only the money return but also the liquidity and capital certainty which the portfolio offers. There is no reason why the concepts of profit maximization cannot be applied to bank operations, provided that "profit" is conceived in this broader sense.

The core of bank operations is uncertainty, and hence any reference to profits must be in a probabilistic sense. In this paper, it will be assumed that the bank considers the expected value of its profits (i.e., additions to surplus during the planning period) under various conditions of risk,² principally that of change in size of deposits. The problem is somewhat analogous to recent demand-risk-inventory theory for the selling firm,³ where cash (and other assets readily convertible into cash) represents inventories, the carrying cost of these "inventories" is the surrender of earning power, and various penalties are incurred for insufficient "inventories."

This approach to commercial bank operations is not new, having been first indicated by Edgeworth in 1888,⁴ although at that time bankers still considered loans to be liquid (in the sense of self-liquidating) and securities frozen, a view which lingered into the 1920's.⁵ Edgeworth indicated the im-

¹. Diversification can be explained only if the bank is a monopsonist in the market of the highest-yield asset or if it is required by law to carry reserves of low-yield assets.

². In deference to received literature, the word "risk" is used rather than "uncertainty" since the bank is assumed to know, with certainty, the probability distributions.


⁵. For a review of this revolution in bankers' ordering of relative liquidities, whereby securities became "secondary reserves" and loans frozen assets, cf.
portance of probability to banking through the device of a simple game:

I have imagined a new game of chance, which is played in this
manner: each player receives a disposable fund of 100 counters,
part of which he may invest in securities not immediately realizable,
bearing say 5 per cent per ten minutes; another portion of the 100
may be held at call, bearing interest at 2 per cent per ten minutes;
the remainder is kept in the hands of the player as a reserve against
certain liabilities. ... [22 digits are drawn at random every two
minutes, and the difference between their sum and their expected sum,
99, is calculated.] The special object of the reserve above mentioned
is to provide against demands which exceed that average. If the player
can meet the excess of demand with his funds in hand, well; but if not
he must call in part, or all, of the sum placed at call, incurring a
forfeit of 10 per cent on the amount called in. But if the demand is so
great that he cannot even thus meet it, then he incurs an enormous
forfeit, 100£ or 1000£. ¹

Unfortunately, Edgeworth then proceeded to solve for the optimum portfolio by
a kind of enlightened common sense, claiming that "the calculus cannot in-
deed, I think, by itself determine what chance of great disaster it might be
prudent to incur for the probability of a moderate gain."² If however, values
can be placed upon the various aspects of this "great disaster," the calculus
can do just that.

2. Assumptions
It is uncertainty, in its various guises, far more than anything else
which makes the banker's job a difficult one. The important areas of this un-
certainty arise because the bank cannot know exactly:
1. How large will be its deposit liabilities at any moment of the future.
2. The market value of the non-matured securities in its portfolio at
any moment of the future.
3. What proportion of its borrowers will default the loans which the
bank has extended to them. ³
4. The degree of "frozen-ness" of the loan portfolio at any moment
of the future, where this degree depends upon the ability (and, to
a certain extent, willingness) of customers to accept refusal of
loan renewals. ⁴

B. Suviranta, "The Shiftability Theory of Bank Liquidity," Economic Essays

1. Edgeworth, op. cit., p. 120.

2. Ibid., p. 121.

3. There is also the possibility of default on securities, but Government ob-
ligations comprise so large a part of banks' portfolios that this area of un-
certainty may be neglected.

4. The nominal maturity distribution of the bank's loan portfolio may have
little to do with the actual degree of "frozen-ness" of its loans. A study by
the Federal Reserve Bank of Cleveland indicated that continuous borrowing
through renewal of short-term loans was quite widespread. While only six
per cent of the loans of banks in that district matured in five years or more,
25 per cent of the total dollar amount of loans had been made to borrowers who
While the first element of uncertainty is particularly critical to the bank, the last three areas are clearly not unimportant. If bonds were always marketable at par and loans callable on demand, without possibility of default, the bank could never become illiquid no matter how erratic the behavior of its deposits. The greater the extent to which any or all of these latter three uncertainties exist, the greater becomes the bank's first concern for the future course of deposits. Thus, no one of these four aspects may be properly neglected in a model of bank operations.

The assets which the bank can hold may be divided into three general categories: cash assets, securities, and loans. Since the problem of diversification within each of these portfolios (that is, what types of securities and loans are held) will not be of concern in this paper, each of these categories will be assumed internally homogeneous. "Cash" assets in fact consist of Federal Reserve Bank reserves, vault cash, net balances with other banks, and bills of very near maturity; here no such distinctions will be made, all "cash" being assumed 1) to provide no earnings and 2) to be completely free of risk of capital value change. The category "securities" will be assumed to include a homogeneous group of securities 1) without default risk, 2) readily salable upon established markets, 3) with maturity date beyond the end of the bank's present planning horizon, and 4) with a fixed coupon per bond per planning period. The distinction between "cash" and "securities," is clearly one of degree and not of kind. The portfolio of an actual bank will invariably consist of a variety of assets in the range from cash to fairly long-term bonds; in this simplified representation, the choice of the bank is narrowed. "Loans" are assumed 1) to be not callable during the planning period, 2) to be not marketable, and 3) to be "shiftable" only to the extent that they are eligible as collateral for borrowing from the Federal Reserve Banks. Thus, the essential difference between "securities" and "loans" is that there is a market for the former so that securities may be readily converted into cash, although at an uncertain price, while loans can be so converted only through the Federal Reserve Bank. Since these are assumed to be the only assets which the bank can hold, 1 it must be true that cash plus securities plus loans equals deposits plus total capital accounts; the bank is assumed to have no liabilities other than deposits, and, of course, no part of the total capital accounts (which will be called simply "net worth") can be withdrawn from the bank.

What has been called the 'planning period' is that span of time upon which the bank concentrates all its attention and over which it sets, and does not plan to alter, its asset portfolio. This is obviously unrealistic, for every bank is always planning and re-planning its asset portfolio. Even if the fact of continually maturing securities—which forces the bank to re-plan by automatically replacing securities with cash assets—were removed, as it is in the model, actual banks would make continual changes in their portfolio plans. Nevertheless, it is equally true that portfolios are not planned with the in-

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1. Non-financial assets comprised less than one per cent of the assets of member banks of the Federal Reserve System in 1956.

2. Deposits are also assumed internally homogeneous, i.e., no distinctions are made between demand and time deposits. The question of the bank's optimal proportion of time to demand deposits is briefly treated in Appendix D.
tention of making frequent changes, and it seems more realistic to assume that the basic portfolio decisions with respect to the fundamental components, cash, securities, and loans, are made fairly seldom and with reference to a sizable span of time. Forcing this flexible procedure into a planning period of fixed length is very simplifying but, it is hoped, not badly distorting.

The choice of this portfolio is assumed to depend entirely upon anticipations concerning the circumstances of the ensuing period and in no way upon past events, past portfolio selections, or expectations of events occurring after the end of the ensuing period. To expose the importance of the assumptions of this last sentence some further amplification is required. First, it is assumed that the bank is not influenced by past events except insofar as these affect its estimates of the future. Thus, if the bank has had an unusual proportion of its loans defaulted in the previous period (the third element of uncertainty), this may induce it to re-value its estimates of such risk but it does not reduce its loans solely on the basis of a "once burnt, twice shy" code of behavior. More relevant to the real world is the connection to the "pin-in" effect. Any such effect is assumed away in the model; previous declines in the security price level (as a result of the second element of uncertainty) cannot cause the bank to carry a greater proportion of securities than it otherwise would desire. Second, past portfolio selections do not influence the current choice (except, of course, through their influence on the bank's appraisal of the ensuing period); this requires that there be leeway in the portfolio at the start of any period. Clearly, cash and securities can be converted into other assets at any time so this assumption really applies only to loans—if all the bank's borrowers "required" renewals of their loans (the fourth element of uncertainty) for the ensuing period, the bank would not be able to reduce its loans, whether it wished to or not. Thus, under this assumption the model can only consider those banks for which the proportion of "required" renewals, while perhaps large, is never so large that the bank cannot start the new period with the exact quantity of loans it wishes to make. Third, the bank knows, or estimates with complete confidence, all the parameters of its environment that are relevant to its portfolio choice for the ensuing period. And fourth, the portfolio of the current period is not affected by expectations of change in the parametric climate of the next period. In short, the assumptions about the "period" are those required to keep the model manageable and static—"once burnt, once shy," unitary elasticity of expectations (in the Hicksian sense), and limits to the degree of the fourth area of uncertainty.

The first area of uncertainty, that of the future course of the level of deposits, implies that the bank must be prepared for the possibility that withdrawals exceed additions to deposits over a particular time-span. A net reduction of deposits will always occur over the moment of time during which one depositor makes a withdrawal. Not infrequently, a bank will find its deposit levels declining over a few days or weeks. And it is not impossible that seasonal, cyclical, or secular factors will cause a fall in deposits over longer periods. At the beginning of the planning period, the bank recognizes that its

1. The fact that transaction costs of change give the existing portfolio some inertia is assumed to be of little importance at the margin.

2. J. R. Hicks, Value and Capital, 2nd ed., Oxford, Oxford Press, 1946, p. 205. This is not strictly correct since expectations are there assumed single-valued. In this paper, any expectations that are not single-valued are assumed to have a probability distribution which, at every moment of time has an arithmetic mean equal to the current market value.
level of deposits during the period may follow a myriad of possible paths, of which continuous rises or continuous falls are but two. In reality, the complete shapes of the possible paths are implicitly considered in the specification of the bank's asset portfolio. But one aspect of the shape of each of these possible paths is of such great importance to the bank that it is here assumed the only aspect considered by the bank—namely, the lowest point to which deposits fall in each of the paths. For it is at this "deposit-low" of the period that the bank is forced to make the most radical adjustment of its asset portfolio in order to meet the demands of its depositors.  

This assumption of sole concern with "deposit-lows" is not in itself sufficient to permit complete neglect of the time-shape of deposit changes since the date of occurrence of any "deposit-low" may still be important to the bank. In the interest of simplicity, this problem of the time-path of deposits will be avoided in the following way. At some point toward the end of the period, deposits will reach their low, at which time the bank makes any asset adjustments required; this perhaps necessitates selling some of its securities and/or borrowing on the collateral of some of its securities or loans.

This is not the place for a full discussion of the complex manner (use of Federal Funds, security sales, Federal Reserve Bank discounts, etc.) in which banks in fact can and do meet the problem of insufficient reserves (i.e., insufficient cash assets). Basically, the process may be simplified into the following stages:

Stage 1. The bank meets net withdrawals from its cash assets as long as it can without drawing these assets down below their minimum required level.

Stage 2. Should the cash assets prove insufficient, the bank sells from its security portfolio, at the going market price, and continues to do so as long as it has securities to sell.

1. Of course, if deposits rise throughout the period, the "deposit-low" is zero and the bank need make no adjustments as far as meeting withdrawals is concerned. Thus, the "most radical" adjustment may well be no adjustment at all. This "most radical" adjustment may also be slight if the bank has access to short-term borrowings (such as Federal Funds or the discount window) and is not reticent to use them for short periods in hope of improvement in its deposit position. In this case, end-of-period deposits may be a more important variable than the "deposit-low" of the period. The conservative bias that the "deposit-low" assumption gives the portfolio is indicated in Appendix F.

2. If deposits, on the average, should rise during the period, the "deposit-low" will probably not be much, if at all, below zero and will probably occur toward the beginning of the period. Since little asset readjustment is required in this case and since we neglect the net rise in deposits that follows, the assumption that the "deposit-low" occurs toward the end of the period is innocuous.

3. For simplicity, it is assumed that the banks sell, rather than borrow from the Federal Reserve Banks on the collateral of securities. Given bankers' dislike of debt and the fact that interest charges would probably exceed transaction costs of selling and later repurchasing, sales rather than borrowing would occur in the world postulated by the model, that is, a world of no "pin-
in" effects and unitary elasticity of expectations of bond prices.

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Stage 3. Should the sale of all its securities also be inadequate to meet the deposit depletions, the bank borrows from the Federal Reserve Bank on the collateral of its outstanding loans. 1 This it continues to do as long as necessary or until its stock of such collateral is exhausted.

To these three stages, a fourth might be added: should all its assets be converted, to the greatest extent possible, into means of payment and still be insufficient to cover deposit withdrawals, Edgeworth's "great disaster" would occur par excellence—the bank would then be in the throes of a liquidity crisis beyond its ability to handle. At the very least, it would have to call for exceptional aid from the Federal Reserve System; it might be forced to close its doors, and it might find itself insolvent as well. However, it will be assumed that such a "Stage 4" is so costly to the bank that no bank's optimum portfolio permits any possibility of this occurrence. Such a result may (and, in fact, does, if seldom) occur, but it could only happen, by the assumptions here, through a misestimation by the bank of the parameters of its operations. 2

The method of treatment of the four areas of uncertainty may now be more accurately specified.

1. The size of deposits at any moment of the future. Although the bank expects (in the probability sense) deposits to stay at their start-of-period level, it recognizes that they may fall or rise. The relevant distribution function is that relating each of the various "deposit-lows" during the ensuing period to the probability of its occurrence. If the random variable, u, is defined to be the "deposit-low" as a fraction of initial deposits, the distribution of u may be defined only over the range, zero to unity. For simplicity, the frequency distribution of u, f(u), is assumed to be a linearly increasing function of the amount by which u exceeds s, where s is the smallest "deposit-low" (as a fraction of initial deposit) to which the bank assigns a non-zero probability (and clearly, 0 < s < 1). 3 Since the cumulative of f(u) must equal one, specification of s is sufficient to determine:

\[ f(u) = \frac{2(u - s)}{(1 - s)^2} \]

2. The market value of its securities at any moment in the future. A "unit" of securities is defined as a dollar's worth at the market prices prevailing at the start of the planning period; this "unit" carries a coupon paying g dollars per "unit" per period where it is assumed, without undue restriction, that 0 < g < 1. The market price at the end of the period may be written as (1 + w) where w, the change in security prices during the period (absolute

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1. Alternatively, one may think of this process as straightforward rediscouning in the traditional, if in fact little used, manner.

2. It is possible that some bank may be forced to accept the possibility of such a "Stage 4" if its net worth were low and its lowest possible "deposit-low" very near zero. Of course, the bank could meet this situation by holding very large cash assets, but this may be so unprofitable as to induce it to accept the possibility of "Stage 4." Such a bank is not considered in this paper.

3. This "triangular" distribution is assumed because it is believed to be the best simple approximation to the actual distribution of banks' "deposit-lows." For a theoretical derivation of the distribution, see Appendix A. Alternatively, a uniform distribution of "deposit-lows" is considered in Appendix B.
and percentage, is assumed to be uniformly distributed over the range, -a to a (0 < a < 1). Since the "deposit-low" occurs toward the end of the period, the price of securities sold at the moment of the "deposit-low" may also be considered to be (1 + w).

3. The proportion of loans defaulted. This aspect of bank uncertainty will be most summarily treated, not because it is felt to be unimportant to a complete theory of bank operations but because its basic effects upon the bank's portfolio can be seen in the present model without complex treatment. It is assumed that the bank charges a pure interest rate of e per dollar of its loans; it then adds to this rate some amount according to the default risk which just suffices to insure that the bank will not lose through defaults in the long run. The final "gross" rate is e' (where 0 < e < e' < 1), but we shall here concern ourselves only with the bank's earnings net of default. 

4. The degree of intra-period "frozen-ness" of the loan portfolio. The fear on the part of the bank that it may not be able in an emergency to reduce its loans sufficiently, even over several periods, means that any debt incurred to help meet deposit depletions may well be long-term debt, a position which bankers dislike. It is because of this that Stage 2, sales of securities, is assumed to precede Stage 3, borrowing on loan collateral, in the process of meeting deposit depletions. For the same reason, the cost of such borrowing in Stage 3 may be interpreted to include not only the charge

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1. Unfortunately, fixing the distribution of w so that its expected value is zero implies that the expected value of the equivalent distribution of interest rates is greater than zero, in contradiction of the assumption of static expectations with respect to interest rates. In the case of consols the expected value of the distribution of changes in the interest rate is:

\[ g \left[ \frac{a^2}{3} + \frac{a^4}{5} + \frac{a^6}{7} + \ldots \right] \]

which is sufficiently near zero for small values of g and a that this contradiction is not serious. This results from the property of the number system, whereby the average of the reciprocals does not equal the reciprocal of the average.

2. It must always be remembered, however, that to the extent that the bank is worried about the time-path of defaults or the default rate is positively related to the quantity of loans, the present model will overstate the amount of loans which the bank will desire to make.

3. Much nebulous writing has appeared on this subject, but the bankers' aversion seems real enough, probably basically deriving from their fear that heavy indebtedness will have adverse effects upon their relations with depositors, borrowers and correspondent banks. The view is not without its dissenters, however; for example, see A. Murad, "The Ineffectiveness of Monetary Policy," "Southern Economic Journal, Vol. 22, No. 3 (January 1956), pp. 339-351. Of bankers' supposed aversion to steady borrowing, Murad says (p. 346):

- It may be that bankers feel that way or say that they feel that way, but they certainly do not act that way. Whenever they have reserve deficiencies they borrow and if necessary remain for years in debt to the Federal Reserve banks.

As a general phenomenon, this last sentence is open to great doubt, for the fact of increasing or large aggregate indebtedness is not proof of a decreasing or small antipathy toward permanent indebtedness.
of the Federal Reserve Bank but also a subjective "cost" of being in what may prove long-term debt.

While use of the "discount window" is a privilege and not a right, no Federal Reserve Bank would refuse to extend advances to a bank which found itself unable to cover exceptionally large deposit withdrawals without such aid. The only questions before the bank are, then, how much borrowing could they do on the basis of their total loan portfolio and how much would it cost. If the bank gets into Stage 3, it can take a typical dollar's worth of its loans to the Federal Reserve Bank and receive an advance of \((1 - m)\) dollars, where \(m\), which might be labelled the "excess-collateral rate," is, of course, between 0 and unity. On this advance, the borrowing bank is charged an interest cost which, it is here assumed, is different from the real "cost" because of bankers' dislike of such debt. There are many ways in which such a "cost" might be handled, but the one to be assumed in this paper is that the real "cost," \(q\), of such borrowing is to some extent greater than the interest charge, where \(0 < q < 1\).

Cash assets are of two types, those required to be held as reserves in the Federal Reserve Bank and those which the bank holds (in various forms) in excess of these requirements. The amount of the former at any moment of time must be a specified fraction of the bank's deposit liabilities. While, in fact, a rise in reserve requirements usually results in the lowering of the amount of other cash assets which the bank feels it requires, it is here assumed that the amount of cash assets other than required reserves is also a specified fraction of its current deposit liabilities. Thus, the "required" amount of cash assets can be written as a fraction, \(k\), of the bank's deposit liabilities, where \(k\) is somewhat larger than the reserve requirement ratio. Most banks would meet these requirements almost continually, but as the model is set up, it need only be prepared to meet them at the moment of the "deposit-low" to be sure of having a sufficient amount at every other moment of the period. Assuming a fixed fraction of cash assets in this fashion means that Stage I is not possible. But this assumption is not as restrictive as it might seem at first since the excess of cash assets over \(k\) would probably be very small unless \(g\) were near zero and/or a extremely large.

Each of the balance sheet items will be written as a fraction of start-of-period deposits—the fraction of cash assets being \(k\), of securities, \(B\), of loans, \(L\), and of net worth, \(N\). \(B\) and \(L\) are variables under the control of the bank, while \(N\) is assumed previously determined and unalterable at least over the ensuing period.

It remains only to fix the criterion by which the bank balances its portfolio between possible gains and losses. One often stated by bankers themselves is that they minimize the probability of losses (or, in reverse, maximize the probability of some gain); but this implies that the portfolio be prepared to meet any possible deposit reduction out of cash assets, while in fact

1. If there were no addition of a subjective "cost" and the bank knew that it could repay its debt in exactly one period, \(q\) would be equal to the Federal Reserve Bank discount rate (per period). To the extent that there is a subjective element or such last-resort borrowing is felt to be of longer duration, \(q\) may be well above the discount rate.

2. The real-world analog to the parameter \(g\) is the difference between long-term security rates and the bill rate. Not infrequently, this difference is very slight, but such times have little relevance here for the assumption of static expectations concerning future interest rates is then almost certainly violated.

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banks do incur an unnecessary, if small and profitable, risk of losses. A variant of the above is the minimization of the probability of incurring losses within the constraint of a reasonable expected profit. Such a criterion is rejected on the grounds that setting the definition of "reasonable" is more important than the minimization process that follows. A criterion advanced by recent portfolio theory\(^1\) is that some point is chosen, according to the selector's preferences, on the frontier (or locus) of the maximum expected return for every possible variance of return. This criterion was introduced because it was useful in explaining diversification; in the present model, the variable, variance of return, is not needed to explain diversification and so, for simplicity, will not be included in the text.\(^2\) Here, the bank is assumed to choose its asset portfolio so as to maximize its expected additions to net worth during the period. Thus, it maximizes its expected additions to net worth function with respect to one of the asset variables, \(B\) and \(L\), the other being then determined by the accounting identity:

\[
1 + N = k + B + L
\]

In summary, the symbols to be used in the paper are:

**Variables:**
1. \(B\), securities as a fraction of initial deposits.
2. \(L\), loans as a fraction of initial deposits.

**Random elements:**
3. \(u\), the "deposit-low" of the period, as a fraction of initial deposits. \(u\) is defined over the range, \(0 < s < u < 1\), by the distribution,
   \[
f(u) = \frac{2(u - s)}{(1 - s)^2}.
\]
4. \(w\), the change between the start of the period and the "deposit-low" (and the end) of the period in the market price of securities. \(w\) is defined over the range, \(-a < w < a(0 < a < 1)\), by a uniform distribution,
   \[
f(w) = \frac{1}{2a}.
\]

**Parameters:**
5. \(N\), net worth, unchanging, as a fraction of initial deposits.
6. \(g\), the coupon per dollar's worth of securities (at initial market prices); \(0 < g < 1\).
7. \(e\), the earning rate on loans (net of default risk); \(0 < e < 1\).
8. \(k\), the amount of cash assets which the bank holds, as a fraction of current deposit liabilities; \(0 < k < 1\).
9. \(q\), the "cost," both actual and subjective, of borrowing a dollar during the period from the Federal Reserve Bank; \(0 < q < 1\).
10. \(m\), the "excess-collateral rate." A dollar of loans as collateral enables the bank to borrow \((1 - m)\) dollars from the Federal Reserve Bank (the latter acting in its capacity of "lender of last resort"); \(0 < m < 1\).


\(^2\) The implications of a bank preference map which is a function of both expected gain and variance of gain are drawn, for a simplified version of the model, in Appendix B. Also, some inference about the change in results which this would cause is made there.
3. Structure of the Model

The amount of profit which the bank makes during the period will clearly depend upon the "stage" into which the "deposit-low" forces it, and, for Stage 3, upon how far into that stage it goes. The expected addition to net worth \( \Delta N \) for each stage is:

\[
\Delta N = gB + wB + eL
\]

(2)

The profit is composed of: \( gB \), earnings on securities; \( wB \), capital gains or losses on securities; and \( eL \), earnings (net) on loans.

Stage 3. (Borrowing on loan collateral is required.)

\[
\Delta N = gB + wB + eL - q(1-m)x
\]

(3)

where \( x \) is the amount of loans put up as collateral (and \( (1-m)x \) the amount borrowed) and

\[
(1-m)x = (1-k)(1-u) - (1+w)B
\]

(4)

The profit is composed of: \( gB \), securities earnings; \( wB \), capital gains or losses on the (completely sold) securities; \( eL \), earnings on loans; and \( q(1-m)x \), the cost of the bank's borrowings.

According to the values assumed by the random variables, \( u \) and \( w \), the bank finds itself in one of these two stages. The ranges of \( u \) and \( w \) which bring about each stage are:

Stage 2. This may occur in either of two ways:

i) \( s < u < 1; \ 1 + \frac{(1-s)(1-k)}{B} < w < a \). It may be possible that the most extreme (conceivable) deposit depletion can be met through bond sales alone, provided that the price of securities rises sufficiently (or falls sufficiently little) during the period.

ii) \( 1 - \frac{(1+w)B}{1-k} < u < 1; \ -a < w < 1 + \frac{(1-s)(1-k)}{B} \). If securities prices do not rise enough, only a certain degree of possible deposit withdrawals can be handled by means of bond sales alone.

Stage 3. \( s < u < 1 - \frac{(1+w)B}{1-k}; \ -a < w < 1 + \frac{(1-s)(1-k)}{B} \). Any deposit depletion too extreme to be met by securities sales alone can be met by bond sales plus borrowings on loan collateral.

These three cases are exhaustive since we have already excluded the possibility of a Stage 1, the running down of cash assets to meet withdrawals, and of a Stage 4, where even borrowings on all the bank's loan collateral are insufficient to cope with the deposit losses. But the fact that the three possibilities listed above cover the entire of the ranges of \( w \) and \( u \) does not automatically imply that every stage is relevant for every bank--for example, the net worth of a bank may be so high that it is able to cover any conceivable deposit depletion by bond sales alone, even if the price of bonds drops to their lowest conceivable level. In technical terms, for a stage to be possible of occurrence, the lower limits of both \( u \) and \( w \) (for that stage) must indeed be lower than the upper limits. If all three of the cases cited above--Stage 2(i), Stage 2(ii), and Stage 3--are considered relevant to the bank, then the following assumption about the size of the bank's securities holdings is being made implicitly:
(5) \( \frac{(1-s)(1-k)}{1+a} < B < \frac{(1-s)(1-k)}{1 - a} \)

for if the left-hand inequality does not hold, Stage 2(i) does not exist; and if the right-hand inequality does not hold, Stage 2(ii) and Stage 3 do not exist.

The bank for which Stage 3 is not even a remote possibility is not only rare but uninteresting, so there should be few qualms about assuming the right-hand inequality. But the left-hand inequality is not so easily handled; a taxonomic approach would construct the model for both directions of the inequality sign, but here only that one which is felt most closely to describe reality will be extensively treated. 1 In a world where banks do not expect extremely large bond price fluctuations and do make a significant amount of loans relative to their net worth and lowest possible "deposit-lows," most banks are probably not able to cover their lowest conceivable "deposit-lows" by means of securities sales alone, even if bond prices rise to as high a level as is considered possible. Thus, in what follows, it is assumed that Stage 2(i) is not a possibility, or, in other words, that inequalities (5) may be replaced by:

(6) \( 0 < B < \frac{(1-s)(1-k)}{1+a} \)

4. Determination of the Optimum Portfolio

The stage in which the bank finds itself is determined, as follows, by the value assumed by the random variable, \( u \):

Stage 2(ii). \( 1 - \frac{(1+w)B}{1-k} < u < 1 \), \( -a < w < a \).

Stage 3. \( s < u < 1 - \frac{(1+w)B}{1-k} \), \( -a < w < a \).

The expression for the expected addition to net worth, \( E[\Delta N] \), is:

(7) \[
E[\Delta N] = \int_{w=-a}^{a} \int_{u=s}^{1} \left[ gB + wB + eL \right] \left[ \frac{u - s}{a(1-s)^2} \right] du \, dw \\
+ \int_{w=-a}^{a} \int_{u=s}^{1} \left[ -q(1-k) + q(1+w)B + q(1-k)u \right] \left[ \frac{u - s}{a(1-s)^2} \right] du \, dw
\]

which becomes, after integration,

(8) \[
E[\Delta N] = gB + eL - \frac{q}{3(1-s)^2(1-k)^2} \left[ (1-s)(1-k)-B \right]^3 \\
- \frac{a^2 q B^2}{3(1-s)^2(1-k)^2} \left[ (1-s)(1-k)-B \right]
\]

L may be eliminated as a variable by means of the accounting identity (1); then the derivative of \( E[\Delta N] \) with respect to \( B \) is:

1. Enough has been worked through for the other case to indicate that the results are similar. See Appendix C for a brief treatment of the bank which is able, for some conceivable level of bond prices, to meet the worst conceivable "deposit-low" without recourse to borrowing.
\[ (9) \quad \frac{dE[\Delta N]}{dB} = g - e + \frac{q}{(1-s)(1-k) - B} \left[ \frac{(1-s)(1-k) - B}{(1-s)^2(1-k)^2} \right]^2 \]

and the second derivative:
\[ (10) \quad \frac{d^2E[\Delta N]}{dB^2} = \frac{2q}{(1-s)(1-k)^2} \left[ \frac{(1-s)(1-k)(1 + \frac{a^2}{5})}{5} + (1+a^2)B \right] \]

If \( B \) is at its lowest permissible level (i.e., zero)\(^1\), the first derivative, \( dE[\Delta N]/dB \), is positive, and, if \( B \) is at its highest value (i.e., \( \frac{(1-s)(1-k)}{1 + a} \)), the derivative is negative, provided that the following inequalities hold:\(^2\)
\[ (11) \quad \frac{2a^2(2 - a)}{3(1+a)^2} < \frac{e - g}{q} < 1 \]

If inequalities (11) hold, there exists a regular maximum and the optimum fraction of assets in securities can be found by setting the derivative (9) equal to zero. Solving this quadratic equation in \( B \) yields:
\[ (12) \quad B = \frac{(1-s)(1-k)(1 + \frac{a^2}{3})}{1 + a^2} \left\{ 1 - \sqrt{1 - \frac{(1 + a^2)(1 - \frac{e-g}{q})}{(1 + a^2/3)^2}} \right\} \]

Much of the complexity (or rather simplicity!) of the form of equation (12) results from the particular functional representation of the distribution of the random variables, \( u \) and \( w \); but there are three interesting properties of (12) which are not dependent upon the choice of distribution functions:

1. The fact that \( (1-s)(1-k) \) enters in linear fashion. In words, this quantity is the fraction which does not need to be held in cash assets \( (1-k) \) of the largest conceivable loss in deposits \( (1-s) \). The rest of equation (12), involving the parameters, \( a \), \( e \), \( g \), and \( q \), serves to fix the quantity of securities as a fraction of this term, \( (1-s)(1-k) \). Thus, the determination of the optimum portfolio can be divided into two problems, first, the calculation of the maximum amount of securities which the bank would ever need to sell to meet deposit losses (on the assumption that bond prices do not change), and second, the decision as to what fraction of this amount the bank will actually hold (which will depend upon the various earning and borrowing cost rates as well as the expected fluctuation in security values).

2. The manner in which the earning and borrowing cost rates enter the equation. By means of the single expression, \( (e-g)/q \), the bank measures the relative advantage of loans vis-a-vis securities, the advantage being greater the larger is the difference in earning rates and/or the lower is the

---

1. See inequalities (6).

2. See Appendix C. If the left-hand inequality does not hold, the present model is inapplicable and that treated in Appendix C becomes the relevant one. Unless very high values of \( a \) are considered, however, the expression on the left will be very close to zero. If the right-hand inequality does not hold, there is a corner maximum, with no securities entering into the portfolio.
cost of borrowing on loan collateral. It is interesting to note that, if the bank
is to include any securities at all in its optimal portfolio, it is not necessary
that the borrowing cost be greater than the loan earning rate (i.e., a "penalty"
rate is not essential), but only that it be larger than the difference between
the earning rate on loans and that on securities.

3. The fact that \( N \), the net worth of the bank, plays no part in the
determination of the optimum quantity of securities. This independence be-
tween optimum \( B \) and \( N \) implies that any change in \( N \) induces a change in
the bank’s loans of the same amount and direction. This conclusion is not
surprising if one recognizes that net worth, from the viewpoint of the bank's
liquidity problems, can be treated as a deposit liability with no possibility of
withdrawal.

What is really interesting about equation (12) is not the level of \( B \),
and hence of \( L \), but rather the way in which the optimum holdings of these
assets vary as a result of changes in the different parameters. The first
and second partial derivatives of \( B \) with respect to the various parameters
are given in the table below.

<table>
<thead>
<tr>
<th>( \partial B/\partial x )</th>
<th>( \partial^2 B/\partial x \partial y )</th>
<th>where</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a )</td>
<td>( y = q )</td>
<td>( y = e )</td>
</tr>
<tr>
<td>(- )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = s )</td>
<td>( + ^{1} )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = k )</td>
<td>( + ^{1} )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = e )</td>
<td>( + ^{2} )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = g )</td>
<td>( + ^{2} )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = q )</td>
<td>( + ^{2} )</td>
<td>( + )</td>
</tr>
<tr>
<td>( x = a )</td>
<td>( + ^{3} )</td>
<td>( + )</td>
</tr>
</tbody>
</table>

1. upper or lower sign holds according as:
\[
\frac{e - g}{q} > \frac{1}{9}
\]

2. upper or lower sign holds according as:
\[
\frac{e - g}{q} < \frac{3 - 2a^2}{9}
\]

3. evaluated at \( a = 0 \); upper or lower sign holds according as:
\[
\frac{e - g}{q} > \frac{1}{9}
\]

(see Appendix E for the signs when a uniform distribution of "deposit-
lows" is used.)

A similar table of partial derivatives could be constructed for the
changes in optimum holdings of loans, but this is not necessary; since, by
accounting identity (1), \( L = 1 + N - k - B \), the first and second partial of
L with respect to any parameter (except $k$) is simply the negative of the relevant partial of $B$. It can be shown that an increase in $k$ decreases $L$ as well as $B$, and that the second derivative ($\partial^2 L/\partial k^2$) is zero.

One could find quantitative estimates of these derivatives by assuming particular parameter values, but more generally we can plot the value assumed by $B^{1/(1-\delta)}(1-k) \over (1-\delta)(1-k)$ (written hereafter as $B'$) for all possible values of $\delta$ and of the composite parameter, $(e - g)/q$. This is done, for various fixed levels of $(e - g)/q$, in Figure 1. The dotted line is the border above and to the right of which Stage 2(i) becomes a possibility.  

---

1. Although that region is neglected in Figure 1, numerical examples based on Appendix C indicate that the curves could be smoothly extrapolated into the State 2(i) area without much, if any, error.
The most obvious lesson of Figure 1 is that the effect of changes in the anticipated fluctuation in security prices is uncertain, with respect to both direction and magnitude. When \((e - g)/q\) is in the neighborhood of 1/9 or of unity (i.e., when \(B'\) is in the neighborhood of .67 or zero), changes in the parameter, \(a\), have almost no effect upon the composition of the optimum portfolio. The farther \((e - g)/q\) is from these critical values, the greater will be the effect on the portfolio of \(a\). If \((e - g)/q\) is less than 1/9, greater certainty about the course of future bond prices will induce the bank to hold less securities; while if \((e - g)/q\) is between 1/9 and unity, greater bond-price certainty will induce larger holdings of securities.

Lest Figure 1 give the impression that the value of \(a\) is a critical determinant of the portfolio composition, another diagram, Figure 2, is included which relates \(B'\) to \((e - g)/q\) for two very different values of \(a\). The solid line shows the relation at \(a = 0\) and the dotted line at \(a = 1/2\).

---

1. The dotted line is not continued to the point where \((e - g)/q\) is zero because for the very low values, Stage 2(i) becomes possible. For some values of \(a\), as Figure 1 shows, the dotted line will cross the solid one for low values of \((e - g)/q\).
It will be seen from Figure 2 that \( a \) has no more than a very marginal effect upon the portfolio, \(^1\) and that the important parameters for the division of the portfolio between loans and securities are, not surprisingly, the difference in earning rate between loans and securities, \((e - g)\), and the cost of borrowing, \(q\). It is the influence of these latter parameters (as well, of course, as \(s\) and \(k\)) with which the rest of this paper is concerned; considerable simplification will henceforth be achieved by the assumption that \(a\) is zero. The first step will be to drop the unrealistic assumption that \(e\) is a constant, unaffected by the quantity of loans which the bank makes.

5. Imperfect Competition in the Loan Market

In order to conceive of the bank as, to some degree, a monopolist in its loan market, the meaning of the loan demand curve must be analyzed. As long as banks are a homogeneous group, each of which have available the same information concerning the credit-worthiness of every potential borrower, the rate of interest charged a customer for a loan is simply the going market rate on riskless lending (the "pure" or "prime" rate, \(e\)) plus a certain risk premium. \(^2\) It would be a matter of indifference to both borrowers and banks to which bank a particular borrower went; each bank would get no business if it charged more than the going rate and more business than it could handle if less.

The actual banking mechanism differs, fundamentally, in two ways from this hypothetical competitive system. \(^3\) First, banks do not all have the same knowledge concerning the credit-worthiness of a potential borrower, and as a result, different banks do not add onto the pure rate the same risk premium for the same borrower. Other things being equal, the typical businessman is able to borrow at a lower gross rate in his own locale than elsewhere and at a still lower rate at his customary bank than at a new one. The more strange the borrower, the less sure is the bank of his ability and reliability, \(^4\) and hence the higher will be the risk premium that is added to the prime rate. Second, the potential borrower knows all this and therefore tends not to shop around each time he seeks a loan—he will accept the rate quoted by his traditional bank unless he is convinced that it is far out of line with the market situation.

Consequently, the bank is not faced with a horizontal demand curve for loans (in terms of the net rate) but has two degrees of freedom concerning the rate it charges. It can demand a rate higher than the prime rate plus its

---

1. A rise in \( a \) from 0 to .50 never decreases the optimum security holdings by more than \(0.022(1-s)(1-k)\) (i.e., by more than \(1/40\) of deposits even if both \(s\) and \(k\) are zero).

2. This "certain risk premium" is here, it will be recalled, such as to insure the bank against default losses in the long run.

3. It differs as well in a third way, in that two banks (or the same bank at two different moments of time) may differ in their attitudes toward making risky loans. Certainly the assumption that all banks merely mark up the pure rate, \(e\), so as to avoid default losses in the long run is no better than a very crude first approximation; but it is sufficient for present purposes.

4. And the more expensive it is to ascertain. If the bank has been dealing with the borrower for a long time, it does not need to incur the costs of careful credit investigation.
proper estimate of the risk premium and not lose all its customers because even this gross rate will be lower than many of its borrowers could get elsewhere. Moreover, even those of its borrowers who could do better by taking a higher risk premium but a lower gross rate at another bank are not likely to realize this immediately.\footnote{1}

Just as the bank does not lose all its loan business by raising its pure rate above the going market rate, so also does it not gain an infinite amount of new loan demand by undercutting the going rate. For it would need to undervalue its risk premiums and overcome other bank-borrower inertias in order to gain the new business.

All the above assumes that the same pure rate must be charged to all borrowers. To the extent to which the bank can discriminate between borrowers,\footnote{2} it improves its situation (at least until its customers find out). At the extreme of perfect discrimination, the demand curve for the bank’s loans in terms of the pure rate represents not its average but its marginal earning rate (net of default) per dollar of loans. While the following argument applies, with the requisite adjustments, equally well for this case, it will not be explicitly treated.

The marginal earning rate, or marginal revenue, of loans (net of default), \( e \), should in the general case be written as a function of the amount of loans which the bank makes; however, since the fraction of initial deposits which the bank lends (\( L \)) is a linear transformation of the dollar amount of loans it makes, \( e \) may equally well be considered a function of \( L \). It is far beyond the scope of this paper to predicate the details of this functional relationship; the only property it is safe to assume is that \( e \) declines as \( L \) increases.

One possible procedure from this point would be to hypothesize a specific form for the function; for example, that \( e \) is linear in \( L \):

\[
(13) \quad e = h - jL
\]

where \( h \) and \( j \) are both positive; the average earning rate on loans would then be \( (h - \frac{1}{2} jL) \). The technique of Section 4 can again be used to derive an explicit expression for the fraction of initial deposits which the bank optimally carries in securities. The monopolistic analog of equation (12), written for simplicity on the assumption that \( a = 0 \), is:

\[
(14) \quad B = \frac{1}{q} \left(1 - s\right)^2(1 - k)^2 + (1 - s)(1 - k) \left\{ \frac{1}{q^2} \left[ \frac{h - \delta}{q} + \frac{1}{q^2} \left[ (1 - s)(1 - k) - 2(1 + N - k) \right] + \frac{1}{q^2} (1 - s)^2(1 - k)^2 \right] \right\}
\]

which reduces to equation (12) with \( a = 0 \) whenever \( j = 0 \) and \( e = h \). It

1. And having realized it, many may not wish and/or be able to take advantage of it immediately.

2. The probability that this occurs is augmented by the fact that the gross rate will differ between borrowers anyway, and differential risk premiums help to disguise the existence of differential pure rates as well. Furthermore, the fraction (if any) of the loan which the bank insists (or strongly suggests) be retained in the borrower’s deposit may vary among borrowers— and this practice is essentially nothing more than a rate increase.
is interesting to note three important differences between equations (12) and (14): first, the term \((1 - s)(1 - k)\) no longer enters as a mere proportioning factor in the determination of optimum \(B\); second, \(j\), slope of the marginal revenue from loans function, is an essential element in the determination of \(B\); and third, \(N\) now has an effect upon the size of \(B\) as well as \(L\).

One could now proceed, as in the previous section, to derive the various properties of the derivatives of equation (14), but this will not be done partly because of the mathematical complication, partly because the results would have validity only for the special case of a linear demand for loans function,\(^1\) but mostly because it is unnecessary. Traditional economic theory suggests that, if we know the marginal revenue from loans function, we can deduce the relevant implications of the market if only we can discover the marginal cost function. And such a function we have already found--implicitly--in the derivation of equation (12).

6. The Marginal Cost of Loans

Equation (12) is the expected profit maximizing relation between the optimum quantity of bonds and the parameters of the model. By means of the accounting identity (1), the following equation for the optimum amount of loans may also be found:\(^2\)

\[
L = N + s(1 - k) + (1 - s)(1 - k) \sqrt{\frac{e - q}{q}}
\]

So long as all the parameters on the right-side of equation (15) are considered unalterable constants, \(L\) is simply a function of these parameters, and there is no need to go further. But in Section 4, the possibility was introduced that the marginal earning rate of loans depends upon the quantity of loans made. When this is so, it is better to view equation (15) as a relationship between two variables, the optimum proportion of loans, \(L\), and the marginal loan earning rate, \(e\).\(^3\) Then equation (15) may be put into a form more readily seen as a marginal cost function merely by algebraic manipulation of \(e\) to the left-side of the equation:

\[
e = q \left[ \frac{L - N - s(1 - k)}{(1 - s)(1 - k)} \right]^2 + g
\]

In effect, equation (16) is the bank's "marginal cost of loans" schedule in that it shows what the marginal revenue (or marginal earning rate) of loans must be if the bank is to make any given amount of loans.

---

1. If \(e\) were made a function of powers of \(L\) higher than the first, the maximization equation would involve third (or higher) powers of \(B\), and hence \(B\) could be written as an explicit function of the parameters. Implicit differentiation would, of course, still be possible.

2. Throughout this section, too, the parameter, \(a\), is assumed zero for simplicity.

3. The \(e\) which enters in equations (12) and (15) is the marginal and not the average earning rate on loans. This can be seen by looking at equations (8) and (9): The \(e\) appears as a result of the differentiation of the term \((EL)\), and this derivative of the total revenue of loans is the marginal revenue of loans.
The range of $L$ over which equation (16) is relevant is, however, limited. First, $L$ must be less than (or equal to) the total of the non-cash asset portfolio, $(1 + N - k)$, since $B$ has then taken its smallest possible value, zero. Once $L$ has attained this limit, the marginal cost of loans, $e$, will equal $(g + q)$, and no further increases in $e$ can induce the bank to augment its loan portfolio. Hence, "capacity" limitations imply that the marginal cost of loans curve becomes vertical (i.e., perfectly inelastic with respect to $e$) at a value of $L$ equal to $(1 + N - k)$. The second limitation is less obvious. Equation (12) and hence equation (16), has been derived on the assumption that there is a possibility of Stage 3 (where borrowing on the collateral of loans is required if the bank is to meet the withdrawals of its worst possible "deposit-low"). But this assumption is violated if $L$ becomes less than $[N + s(1 - k)]$. Thus equation (16) is the relevant marginal cost function only over the range, 

\[
[ N + s(1 - k) ] < L < [ 1 + N - k ]
\]

No elaborate theory is needed to understand the bank's actions in the range, $0 < L \leq [N + s(1 - k)]$, for then the bank faces no potential liquidity problems—sales of securities will always suffice to cover any conceivable amount of withdrawals. The bank's only concern is its earnings, and it will maximize these by making loans as long as the marginal earnings rate on loans exceeds the earning rate on securities; once $e$ equals $g$, the bank will expand loans no further, holding the remainder of its non-cash asset portfolio in securities. The marginal cost of loans function is, therefore, horizontal (i.e., infinitely elastic with respect to $e$) at the level of $g$ over the range of $L$ less than $[N + s(1 - k)]$.

This marginal cost of loans curve is shown in Figure 3, where $L_0 = [N + s(1 - k)]$, $L_1 = [1 + N - k]$, and $e^* = g + q$. The function has three principal regions: one of infinite elasticity for low values of $L$; one of zero elasticity once all possible loans have been made; and one of intermediate values of $L$ where the elasticity declines continuously from infinity to zero as $L$ increases. It is true that the resemblance of this to the traditional manufacturing cost curve is slight—both are equated to marginal revenue to deter-
mine the optimum "output"--but this follows from the fact that manufacturing cost curves are only indirectly aligned to opportunity costs. As long as \( L \) is less than \( [N + s(l - k)] \), the banks loan costs are simply the opportunity costs of an alternative "output," i.e., holding securities. If \( L \) is large enough to make Stage 3 possible, to these opportunity costs is added an illiquidity-incurred cost of increasingly greater size as \( L \) rises.

Throughout this paper, we have neglected the possibility of a Stage 4 (where sale of all securities and borrowing on the collateral of all loans are insufficient to cover the withdrawals of the worst conceivable "deposit-low"). This neglect was justified on the grounds that no bank would attempt to carry out the portfolio that permitted any possibility of so fearful an occurrence. But it is possible that Figure 3, as drawn, violates this assumption. Stage 4 emerges as a possibility if \( L \) is greater than \( [N + s(l - k)]/m \), where \( m \) is the "excess-collateral rate" required by the Federal Reserve Bank for the bank's borrowing on the collateral of its loans. For very small values of \( m \), it is clear that this consideration will be irrelevant, but there are also values of \( m \) large enough to induce the Stage 4-avoiding bank to cease its loan expansion, regardless of the marginal earning rate, before loans comprise the entire non-cash asset portfolio. This Stage 4 constraint becomes potentially operative if

\[
m > \frac{N + s(l - k)}{1 + N - k}
\]

The bankers talk of being "loaned-up" while their portfolios still carry some securities is perhaps partial evidence that some such Stage 4 restriction generally does occur. If so, the marginal cost curve becomes vertical not at \( L_1 \) but at \( L_2 \) (where loans equal \( [N + s(l - k)]/m \)); this is shown in Figure 4, where the dotted curve indicates the part of the Figure 3 marginal cost curve which becomes irrelevant as a result of Stage 4 considerations.

Perhaps more important than the shape of the bank's marginal cost of loans function is the way in which this function changes as a result of changes in the different parameters. Given the demand curve for the bank's loans, any factor that causes the marginal cost function to rise (and/or shift to the left) will tend to bring about a reduction in the amount of loans which the bank wishes to make (and vice versa). The changes in the marginal cost function,

---

1. In the extreme case, where \( m \) equals zero, Stage 4 can never occur, even if \( N \) and/or \( s \) are zero.

2. However, as long as \( m \) is not one, there are always some (large) values of \( N \) and \( s \) that make Stage 4 impossible.

3. If there is uncertainty about the value of \( m \) (i.e., the bank is unsure to what extent the Federal Reserve System will support it in a liquidity crisis), the "loaned-up" limit will probably occur at a value of \( L \) less than that calculated by using the expected value of \( m \). How much less we are not equipped to say, on the basis of our too simple assumption that the bank never incurs any possibility of Stage 4. Scott suggests the possibility that each bank's "loaned-up" limit of \( L \) is based upon other banks', on the grounds that it is certain that the Federal Reserve System will not permit a general liquidity crisis to occur under any conditions. See p. 219 (especially footnote 15) of L. O. Scott, "The Changing Significance of Treasury Obligations in Commercial Bank Portfolios," Journal of Finance, Vol. XII, No. 2 (May 1957), pp. 213-222.
for the five parameters, $s$, $k$, $q$, $g$, and $N$, are illustrated in Figures 5 - 9 respectively; the solid line is the marginal cost for a lower value of the parameter and the dotted one for a higher, and $L_0^1$ and $L_0^2$ represent the values of $L_0$ and $L_0$ pertinent to the higher parameter value.

A fall in $s$ and $N$ and a rise in $k$, $q$, and $g$ still have the same general effect of raising the marginal cost function, but the details of these shifts differ. Only $s$, $k$ and $N$ are capable of affecting the "loaned-up" limit (imposed by Stage 4 possibilities); and only $g$ can affect the curve in its perfectly elastic range, although changes in $s$ and $k$ can alter the point at which the elasticity becomes finite. Although Figures 5 - 9 are illustrative and not empirically derived, they help to show how Federal Reserve System actions may achieve an impact upon bank portfolios; the traditional central bank policy weapons of reserve requirements, discount rate, and open-market operations are seen to operate primarily, in the world of the model, through the parameters, $k$, $q$, and $g$, respectively. And, if these (or any other) central bank measures should alter the bank's uncertainty about its deposits future, then there is an effect through $s$. A rise in $g$ will generally inflict capital losses on the bank and, to the extent that these reduce $N$, will further raise the marginal cost of loans schedule of the bank.

The preceding discussion of the bank's supply of loans little resembles the current literature on credit "availability." Although a rise in bond interest rates is effective in shifting the schedule upward and/or to the left (through $g$, $N$, and possibly $q$ or $s$), there is in the model nothing to indicate that small changes in interest rates will have large effects, a basic
tenet of the "new theory of credit control." Notice of this incompatibility is worthwhile, for it implies that availability theories rest upon assumptions different from, or supplementary to, those of the model. Some of the more obvious are: 1) explicit consideration of multi-period profit maximization, which might frequently mean a failure to maximize in the short-run; 2) oligopolistic pricing policies (through prime rate conventions, usury laws or market leadership) which distort the individual bank's effective loan demand schedule; 3) expectations that are not so simple as here assumed (i.e., unitary elasticity); 4) an inter-relationship of loan demand with the level of security interest rates (and possible lags in the adjustments); and 5) greater concern with uncertainty of profits, credit-worthiness of borrowers and the non-price factors of loans than has here been included. This list, certainly not exhaustive, indicates some of the ways that the model would have to be extended if the availability implications were to be more closely approached. It is perhaps at least as important to know what cannot be derived from a particular set of assumptions as to know what can.

To derive implications for monetary policy directly from the bank's marginal cost of loans function is a great temptation, but a dangerous one. For the model here presented is no more than a theory of the "firm;" no theoretical structure has been developed about the adjustment mechanism of the banking "industry" nor of the other sectors of the economy, from whence comes the banks' demand for loans. Knowledge about macro-economic behavior requires, ultimately, macro-economic analysis. The hypotheses about the individual bank's loan behavior here developed are only of value, from the viewpoint of aggregative analysis, if they help to place it upon a more sound micro-economic foundation.

7. Conclusion

Not too long ago, it would have been considered presumptuous to claim that knowledge of the bank's portfolio of earning assets could be useful to analysis of monetary theory or policy. Once economists had become convinced that commercial banks really could "create" money, they became enamored with the fact that the amount of money thus created was limited by the quantity of currency and reserves which the central bank issued. To give precision to the formula relating currency and reserves to the money supply, all that was needed was knowledge of the public's and the banks' propensities to hold currency and the circumstances in which the banks keep excess reserves. Behind such total concern for the money supply always lies the assumption, explicit or implicit, that the velocity of money (or, in more acceptable modern terminology, the relationship of aggregate demand to the money supply) was constant, or at least fairly predictable.

In a world where money was used primarily for transactions purposes, and where only a small and relatively unchanging fraction of the total was used as a way of holding wealth, the quantity theory, at least in its more

1. This phrase is the title of an excellent summary of the innovations to monetary theory of the 1950s by A. Lindbeck, (Uppsala: 1959).

2. And this was not so long ago as we would like to think. As late as 1921, Professor Conant wrote: "If cloak-room attendants managed to lend out three-quarters of the bags entrusted to them...we should certainly not accuse the cloak-room attendants of having 'created' the number of bags indicated by the excess of bags on deposit over bags in the cloak-rooms." Page 31 of "The Meaning of Bank Deposits," Economics, No. 1, January 1921, pp. 28-36.
sophisticated presentations, would be a good approximation of reality. And neglect of bank portfolios, beyond the problem of changes in currency and reserve holdings (either required or desired) would be thoroughly consistent with the theory. What the bank's earning assets were technically labelled--securities, advances, call loans, etc.--might, and obviously would, matter to the banker, but to the monetary theorist they would be just different ways of placing active money balances in the hands of the public.¹ Should the bank decide to somewhat alter its portfolio from securities to loans, the final result would merely reduce the spending potential of those individuals who increased their holdings of securities while increasing, to the same extent, the spending potential of the recipients of the new loans.

Recognition that "idle" balances are neither an insignificant nor an unchanging fraction of the total money supply and the fact that contemporary governments provide a wide range of default-free forms of wealth-holding, from currency to consols (varying each from the other only slightly, in liquidity and yield), forces upon monetary theorists an entirely different mode of analysis. While the theory of the determination of the velocity of money was dramatically revised by Keynes's General Theory, the relevance of this to monetary policy has only gradually become apparent. It is only during the current decade that monetary authorities have finally become as concerned with the manner in which the banking system makes money available to the public as with the total quantity. A movement by banks out of securities into loans cannot be uninteresting to monetary policy: the bond-buying public may be merely transferring a part of its wealth to a less liquid form, while the recipients of new loans are almost certainly increasing their spending.

Thus the division of the bank's earning assets between securities and loans is of relevance not only to the bank itself (concerned as it is with liquidity and profits) but also to analysis of the inflationary impact of different allocations of a given supply of money. Coincidentally--though perhaps providently so--the bank's choice between greater liquidity and greater earnings is also society's choice between lesser and greater inflationary forces. Most of the weapons of contemporary monetary policy can be understood, and are in fact proposed, as an effort by the central bank to "encourage" (where that word covers a spectrum of meanings from "suggest" to "force") the commercial banks to hold assets which are relatively more liquid (though the means of achieving this often involves making such assets less liquid). It is toward an improved understanding of the ways in which various aspects of monetary policy affect the bank's choice between different earning assets, and hence aggregate demand, that this paper is aimed.

Appendix A: The "Deposit-Low"

Most banks have many depositors and the typical deposit transaction involves but a small fraction of the owner's account. For this reason, any attempt to derive theoretically the shape of the distribution of "deposit-lows" for a bank must recognize that any particular "deposit-low" is the result of a long series of individual deposit withdrawals and additions. Avoiding the question of what causes a depositor to alter the size of his account, we will deduce the "deposit-low" distribution on various assumptions about the prob-

¹. Rather curiously, through most of the nineteenth century, a great many bankers and economists did believe that certain classes of bank assets were inherently less inflationary than others. But the "real bills" doctrine was founded not on a belief that the money thereby created was any less "active," but on a mistaken notion concerning aggregate supply.
ability of each deposit account transaction being an addition to or depletion of total deposits.

In line with the static nature of expectations in the text, let us assume that the bank "expects" no change in its total deposits during the ensuing period; this may be interpreted to mean that it believes that every dollar of transactions in its deposit accounts has a 50-50 chance of being a withdrawal of a dollar or a deposit of a dollar. It is certainly true, then, that the expected change of deposits, no matter how many transactions occur, will then be zero. But there will still be finite probabilities attached to "deposit-lows" less than zero. In general, if \((2N)\) transactions occur, the probabilities of the "deposit-lows" are given by:

\[
\begin{align*}
\Pr[0] &= \frac{(2N)!}{2^N \cdot (N!)^2} \\
\Pr[-2x] &= \Pr[-2x + 1] = \frac{(2N)!}{2^N \cdot (N-x)! \cdot (N+x)!}
\end{align*}
\]

where \(x\) is a positive integer. \(\Pr[-2x]\) means the probability of a "deposit-low" of \((2x)\) dollars less than initial deposits, and \(1 \leq x \leq N\). \(^1\) A smoothed picture of such a discrete probability function is given in Figure A-1, \(^2\), where

---

1. Cf. J. L. Doob, Stochastic Processes, (New York: 1953), pp. 106-108. Let \(L\) be the "deposit-low" and \(A\) the level of deposits after the completion of all \((2N)\) transactions. Doob proves, with the "reflection principle," that

\[
\text{Prob}[L < -2x] = 2\text{Prob}[A \leq -2x] - \text{Prob}[A = -2x]
\]

Since

\[
\text{Prob}[L = -2x] = \text{Prob}[L < -2x + 1] - \text{Prob}[L < -2x]
\]

it follows that

\[
\text{Prob}[L = -2x] = \text{Prob}[A = -2x + 1] + \text{Prob}[A = -2x]
\]

But \(\text{Prob}[A = -2x + 1] = 0\) (since an even number of \(+1\) or \(-1\) transactions can never result in an odd number of dollars withdrawn). Thus,

\[
\text{Prob}[L = -2x] = \text{Prob}[A = -2x]
\]

It can easily be shown also that

\[
\text{Prob}[L = -2x] = \text{Prob}[L = -2x + 1]
\]

Thus the distribution of "deposit-lows" is the same (with twice the density) as the distribution of end-of-period deposits over the range of net decreases of deposits. Clearly, as \(N\) becomes larger, the "deposit-low" distribution, being binomial, approaches a truncated (at zero) normal distribution.

2. It should perhaps be noted that this distribution differs from Patinkin's receipts-expenditures distribution for individuals because there is here no assumption that withdrawals must equal accretions (i.e., both equal to \(N\)). Cf. D. Patinkin, Money, Interest and Prices, Evanston: 1956, Chapter 7

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the solid curve represents the distribution for \((2N)\) transactions and the
dotted curve for a larger number of total transactions.\(^1\)

![Figure A-1](image)

It is unlikely that any bank is in a position, however, to consider each
transaction as totally independent of all or any previous transactions in the de-
posit accounts. At one extreme, if there is only one bank in the economy and
there is no change in the public's desire to hold cash (as opposed to deposits),
then any withdrawal must appear later as an addition; there is definitely an
inverse relation between the probability that a given transaction will be an ad-
dition and the proportion of previous transactions which were additions. The
more usual case, however, is that of a bank which experiences or expects, or
feels, positive correlation between the probability that a given transaction will
be a deposit and the relative number of previous transactions which were de-
posits.

As an example of this, suppose that, whenever a majority of the pre-
vious transactions have been deposits, the probability that the next transaction
will also be a deposit is somewhere between .5 and unity. In order to main-
tain the assumption of static expectations, the probability of a withdrawal
when a majority of the previous transactions were withdrawals must be equal
to the probability of a deposit when a majority were deposits. The earlier

\(^1\) The slope of the frequency distribution is positive over its entire range,
from \((-2n)\) to zero. It increases first at an increasing rate, but at a de-
formulae are now seen as the special case where these probabilities are .5. A rough picture of the effect of raising them above .5 is shown in Figure A-2, where the solid curve represents the same distribution as in Figure A-1, and the dotted curve the distribution when autocorrelation is introduced in the above manner. It is doubtful if the autocorrelation of deposit transactions is sufficiently large, for most banks, to cause a negative slope in the frequency distribution of "deposit-lows," but it may suffice to keep the probability of relatively low "deposit-lows" from being so small as to be negligible.

The "triangular" distribution has been chosen in the text as the closest simple approximation to this distribution. If a straight line were fitted through the distributions of Figure A-2, it could be seen that the "triangular" distribution understates the probability of occurrence of extremely low and zero-neighborhood "deposit-lows" and overstates the probability of the middle range of "deposit-lows."

Figure A-2

Appendix B: Variance of Profits

Because the consideration of variance of earnings involves great com-
plication of the model, only a very simplified version of it will be discussed here. In addition to the assumptions of the text, it is assumed that the bank has no cash requirements (k = 0), it has no net worth (N = 0), there is no "excess-collateral rate" (m = 0), securities have no earnings (g = 0) and no possibility of price change (a = 0), and it is considered possible that all deposits be withdrawn during the period (s = 0). Thus the accounting identity (1) of the text becomes:

(B-1) \[ 1 = B + L \]

creasing rate in the area to the immediate left of zero. The inflexion point occurs at \[ x^2 = \frac{1}{2} (N + 1) \].
The distribution of "deposit-lows" is: \( f(u) = 2u \); and the expected additions to net worth function is simple:

\[
(B-2) \quad E[\Delta N] = 2 \int_{u=L}^{1} (eL)udu + 2 \int_{u=0}^{L} (eL - qL + qu)udu = eL - \frac{qL^3}{3}
\]

Maximization of expected profit implies that:

\[
(B-3) \quad L = \sqrt{e/q}
\]

A regular maximum will occur, with \( L \) between zero and one, as long as \( e < q \) (both are positive).

The variance of profit, written \( S^2(\Delta N) \), is:

\[
(B-4) \quad S^2(\Delta N) = 2 \int_{u=L}^{1} (eL)^2udu + 2 \int_{u=0}^{L} (eL - qL + qu)^2udu - (E[\Delta N])^2
\]

\[
= qL^2 \left[ \frac{1}{6} - \frac{L^2}{q} \right]
\]

Minimization of \( S^2 \) (in the relevant range, \( 0 < L < 1 \)) clearly requires \( L = 0 \), at which point both \( E \) and \( S^2 \) are zero. No maximization process is needed to find the frontier of maximum \( E \) for each given \( S \) since, once one is specified, the other is uniquely determined. Although the equation of this frontier is complex, its slope at any point (determined by the value of \( L \)) is:

\[
(B-5) \quad \frac{dE}{dS} = [\frac{e}{q} - L^2] \frac{\sqrt{\frac{3}{2} - L^2}}{L(1 - L^2)}
\]

which is positive in the range \( 0 < L < \sqrt{e/q} \), and negative beyond. Thus, the expected gain (\( E \)) can be increased only at the expense of increased variance of gain (\( S^2 \)) up to the point of maximum \( E \). These opportunity loci are plotted in Figure B-1 for three sets of values of \( e \) and \( q \).

On the assumption that bankers are "risk-aversers" and "diversifiers," 1 it can be easily seen from Figure B-1 that they will choose \( L \) at least somewhat smaller than that value which maximizes \( E \). However, the change in the portfolio induced by a change in \( e \) or \( q \) can only be guessed unless the bank's complete preference function between \( E \) and \( S \) is specified, a task which will not be attempted.

Three conclusions may be drawn from this discussion:

1. The assumption that the bank maximizes \( E \) and neglects \( S \) entirely is equivalent in the Tobin sense, to assuming that the bank is on the border between risk-averting and risk-loving. 2 This is not as serious as the

1. Cf. Tobin, op. cit., pp. 16-22. The labels "risk-averter" and "diversifier" are applied to those investors whose indifference curves between \( E \) and \( S \) are concave upward. For opportunity loci such as those of Figure B-1, "plungers" and "risk-lovers" might also diversify (i.e., choose \( L \) not equal to zero or one), but "diversifiers" necessarily will diversify.

2. Ibid., p. 19. In Tobin's equation (3, 7) maximizing expected return im-

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The numbers beside the points are the fraction of total assets held in loans at that point; the curves end at the point, \( L = 1 \), except for the curve, \( e = 0.05, q = 0.20 \), which ends at \( L = 0.86 \) where \( E \) becomes negative.
words imply since much of the bank's risk is reflected in the model of the
text in the expected profit function itself.
2. The size of the larger earning (and less liquid) portion of the
bank's portfolio (i.e., L) would generally be somewhat smaller than the
values derived in the text if variance of return were explicitly considered
and the bank assumed a "diversifier." While this might be more realistic
than to place all the risk elements in the expected return function, the com-
plications of such a procedure can be seen from the simple version of the
model presented here.
3. The effects of changes in parameters upon the various assets be-
come less determinate when variance is introduced. Even if the indifference
curves between E and S are assumed to be concave upward (i.e., risk-
averting), the effect of an increase in e on the bank's willingness to make
loans depends upon the relative slopes of successively higher indifference
curves. If the slopes are believed to increase rapidly, then it is possible
that higher values of e will reduce the amount of loans the bank makes. 1
But inspection of Figure B-1 indicates that the direction of changes is prob-
ably not altered for plausible shapes of the indifference loci.
Risk aversion on the part of the bank undoubtedly plays a critical
role in the determination of the composition within its loan and security port-
folios. As between two loans with the same expected earning rate, the one
whose returns have lower variance and/or lower correlation with the returns
of those loans already in the portfolio will certainly be preferred; similarly,
the bank will often accept a low-yield security into its bond portfolio because
its potential capital variation is small. While not denying the importance of
these considerations in the bank's choice between different loans and between
different securities, the model of the text maintains that they are not crucial
in the bank's prior choice as to the basic division of the portfolio between
loans and securities. The determinancy gained by assuming the bank an ex-
pected profit maximizer is felt to be worth the perhaps slight loss of realism.

Appendix C: Stage 2(i)
If the left-hand inequality of equation (5) of the text is assumed, then
Stage 2(i) becomes a possibility, as well as Stage 2(ii) and Stage 3. Then the
expected addition to net worth E[\Delta N] is as follows:

\[
E[\Delta N] = \int_{w=-1}^{a} \left[ \int_{u=s}^{1} \frac{[gB + wB + eL] \left( \frac{u - s}{a(1-s)} \right)}{u-(1+s)(1-k)} dudw \right] \frac{(1-s)(1-k)}{B}
\]

(C-1)

\[
+ \int_{w=-a}^{1} \left[ \int_{u=1}^{(1+w)B} \frac{[gB + wB + eL] \left( \frac{u - s}{a(1-s)} \right)}{u-(1+s)(1-k)} dudw \right] \frac{(1-s)(1-k)}{B}
\]

implies a marginal utility of return \(U'(R)\) which is constant with respect to
changes in R.

1. This possibility has been called an "income" effect by J. Aschheim in
"Open-market Operations versus Reserve-requirement Variation," Economic
Journal, December 1959, pp. 697-704, where it is suggested that greater
bank profits, ceteris paribus, will reduce the quantity of loans.
\[ \begin{align*}
&+ \frac{1}{B} \int_{w=a}^{1+k} \int_{u=s}^{1+k} \left[ gB + wB + eL - q(l - k) \right] \\
&+ q(l + w)B + q(l - k)u \left[ \frac{1 - s}{a(l - s)^2} \right] du dw \\
\end{align*} \]

This becomes, once the integrations are performed,

\[ \text{(C-2)} \quad E(\Delta N) = gB + eL - \frac{q(l - a)^4}{24Ba(l - s)^2(l - k)} \left[ \frac{(l - s)(l - k)}{1 - a} \right] B^4 \]

After elimination of \( L \) by means of the accounting identity (1), the derivative of the expected profit with respect to \( B \) is:

\[ \text{(C-3)} \quad \frac{dE(\Delta N)}{dB} = -(e - g) \frac{q(l - a)^4}{24B^2a(l - s)^2(l - k)} \left[ \frac{(l - s)(l - k)}{1 - a} \right] B^3 \left[ \frac{(l - s)(l - k) + 3B}{l - a} \right] \]

The second derivative \( \left( \frac{d^2E}{dB^2} \right) \) is negative throughout the range of \( B \), the first derivative (equation (C-3)) is negative when \( B \) assumes its largest value (i.e., \( B = \frac{(l - s)(l - k)}{1 - a} \)), and the first derivative is positive when \( B \) assumes its smallest possible value (i.e., \( B = \frac{(l - s)(l - k)}{l + a} \)), provided that the following inequality holds:

\[ \text{(C-4)} \quad 0 < \frac{e - g}{q} < \frac{2a^2(2 - a)}{3(l + a)^2} \]

Thus, if (C-4) holds, there is a value of \( B \) in the permissible range for which equation (C-3) is equal to zero; and it is that value of \( B \) which maximizes expected profits.

If the right-hand inequality of equation (C-4) does not hold, expected profits decline continually as \( B \) rises from its smallest to its largest permissible value. In that case, the maximizing value of \( B \) is smaller than \( \frac{(l - s)(l - k)}{1 + a} \), and the model of the text is the appropriate one (i.e., Stage 2(i) is impossible). The common sense argument for choosing the assumption of the text that there is no Stage 2(i) is greatly strengthened by a consideration of the parameter values required if the right-hand inequality of (C-4) is to hold: the table below gives the highest possible values of \( (e - g) \), consistent with the existence of a Stage 2(i), for several combinations of values of \( a \) and \( q \):

<table>
<thead>
<tr>
<th>TABLE C-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>q = 0.05</td>
</tr>
<tr>
<td>a = 0</td>
</tr>
<tr>
<td>a = 0.1</td>
</tr>
<tr>
<td>a = 0.25</td>
</tr>
<tr>
<td>a = 0.50</td>
</tr>
</tbody>
</table>

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Inasmuch as e generally could be expected to exceed g (on any definition of loans and securities) by at least one or two per cent, surprisingly—if not implausibly—large values of a and q are required if the bank is to choose to hold enough securities to be able to meet its worst deposit depletions by securities sales alone—even if bond prices rise to their highest conceivable value.

Nevertheless, the implications for the bank’s optimum asset portfolio of changes in various parameters can be determined by taking partial derivatives of (C-3) while holding that equation equal to zero. The changes in optimum B are as follows:1

\[ \frac{\partial B}{\partial e} < 0 \]
\[ \frac{\partial B}{\partial g} > 0 \]
\[ \frac{\partial B}{\partial q} > 0 \]
\[ \frac{\partial B}{\partial s} < 0 \]
\[ \frac{\partial B}{\partial k} < 0 \]

The sign of \( \frac{\partial B}{\partial a} \) cannot be determined generally. That these signs are the same as those derived in the text, on the assumption that Stage 2(i) does not exist, is partial evidence that the influence of parameter changes upon optimum portfolios does not depend critically upon this choice of assumption.

Appendix D: Time Deposits

No distinction has been made, anywhere in the text, between different types of deposits, a neglect which will be here repaired by introducing time, as well as demand, deposits. The most obvious difference between the two classes of deposits lies in their cost, for time deposits are still permitted to, and do, earn interest for the depositor. On the other hand, the “earning power” of time deposits may exceed that of demand deposits in either or both of two ways. First, it is an institutional fact in most countries that the cash, or low-yield, reserve requirements on time deposits are lower than those on demand deposits. And second, to the extent that the lowest conceivable “deposit-low” is raised by the addition of less volatile time deposits, the bank may hold a larger portion of its portfolio in less liquid, but higher earning, assets.2

Banks attract new time deposits by offering to pay higher interest rates on them. Such increases in time deposits may “come” from several places: by a transfer from 1) the bank’s own demand deposits; 2) the public’s deposits with other banks; and 3) deposits and shares of the non-bank financial intermediaries.3 From the point of view of the individual bank, the relevant division is between the first case and the second and third cases.4

---

1. The arithmetic is sufficiently messy that only the first derivatives have been calculated.

2. Throughout this section, we neglect the possibility that greater time deposits may imply lesser private (or other financial intermediary) lending, and hence higher lending rates for the bank. Cf. J. Tobin, "Financial Intermediaries and the Effectiveness of Monetary Controls," Cowles Foundation Discussion Paper No. 63 (January 1959).

3. It is assumed that the public’s currency needs are fixed.

4. From the point of view of the banking system, the most interesting di-
Here it will be assumed that the bank cannot, through its own volition, alter the total quantity of its deposits, but only, by changing its time deposit interest rate, induce some of its depositors to hold time instead of demand deposits. The fraction of total deposits held as time deposits is assumed to grow from zero, as the time deposit interest rate \( i \) increases from zero, to some maximum value at which point further increases in \( i \) cannot induce further shifts from demand to time deposits. Thus, time deposits as a fraction of total deposits \( T \) will lie between zero and some maximum level \( T^* \) depending upon \( i \), where \( T^* \) is less than one.

This transfer of funds from demand to time deposits will affect the values of \( k \) and \( s \). The overall cash requirements, as a fraction of total deposits, will be:

\[
(D-1) \quad k = k_d - (k_d - k_t)T
\]

where requirements on demand deposits \( (k_d) \) are assumed to exceed those on time deposits \( (k_t) \). The value of \( s \), and the frequency distribution of the possible "deposit-lows," is not so easily calculated. It will depend upon the lowest possible "deposit-low" of time deposits (as a fraction of initial time deposits), \( s_{t_l} \), and that of demand deposits (as a fraction of initial demand deposits), \( s_{d_l} \); but it will also depend upon the form of the distribution of each of these "deposit-lows" and the covariance of the changes of each of these two types of deposits. If the frequency distribution of each type's "deposit-low" is assumed "triangular" (with a lowest possible "deposit-low" of \( s_{t_l} \) and \( s_{d_l} \) for time and demand deposits, respectively) and the two changes are uncorrelated, the resulting distribution of "deposit-lows" for the total deposits will not be "triangular" but will have a convex (from below) segment over very low "deposit-lows" and a concave segment for "deposit-lows" near unity. Since the assumption of zero covariance between

\[
(D-2) \quad T \leq \frac{1 - s_d}{(1 - s_d)(1 - s_t)}
\]

the frequency distribution of "deposit-lows" of total deposits, \( f(u) \), will be:

\[
f(u) = \frac{2}{3} \frac{\left[ u - Ts_t - (1-T)s_d \right]^3}{T^2(1-T)^2(1-s_d)(1-s_t)} \quad \text{where: } T(1-T)s_d < u < T^* + (1-T)s_d
\]

\[
= \frac{2}{3} \frac{3[u - Ts_t - (1-T)s_d] - 2T(1-s_t)}{(1-T)^2(1-s_d)^2} \quad \text{where: } T + (1-T)s_d < u < (1-T) + Ts_t
\]

\[
= 2\left[ u - Ts_t - (1-T)s_d \right] \left[ \frac{1}{(1-T)^2(1-s_d)^2} + \frac{1}{T^2(1-s_t)^2} \right] - \frac{4(1-T)(1-s_d)}{3T^*^2(1-s_t)^2}
\]

\[
\frac{4T(1-s_t)}{3(1-T)^2(1-s_d)^2} - \frac{2\left[ u - Ts_t - (1-T)s_d \right]^3}{3T^*(1-T)^2(1-s_d)^2(1-s_t)^2} \quad \text{where: } (1-T) + Ts_t < u < T^* + (1-T)s_d
\]

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'deposit-lowe's' of time and demand deposits leads to an understatement of frequencies in the low 'deposit-low' segment, and an overstatement in the high range, it will be convenient, and probably less unrealistic, to assume that the aggregate 'deposit-low' distribution is also 'triangular' with the point of zero probability, s, occurring at:

\[ s = T_s + (1-T)s_d \]  \hspace{1cm} (D-4)

The costs of time deposits, iT, are subtracted from equation (8) of the text to give the expected addition to net worth when the bank considers a time deposit policy:

\[ E[\triangle N] = gB + e(1+N-k-B) - \frac{3}{2} (x-B)^3 - iT \]  \hspace{1cm} (D-5)

where, for simplicity, bond prices are not fluctuate (i.e., a equal to zero) and, for brevity, x is written for (1-s)(1-k). This expression, (D-5), is maximized by the bank not only with respect to B, which yields equation (12) of the text (with a equal to zero), but also with respect to i, for the interest rate paid on time deposits is now a variable under the control of the bank. Differentiation of (D-5) with respect to i yields, after substitution for the profit-maximizing value of B by means of equation (12):

\[ \frac{\partial E[\triangle N]}{\partial i} = e(k_d - k_t) \frac{dT}{dt} - (e-g) \left[ 1 - \frac{2}{3} \sqrt{\frac{e-g}{q}} \right] \frac{dx}{dT} \cdot \frac{dT}{dt} - \frac{dT(iT)}{di} \]  \hspace{1cm} (D-6)

where

\[ \frac{dx}{dT} - \frac{d(1-s)(1-k)}{dT} = (1-s_d)(k_d - k_t) - (1-k_d)(s_t - s_d) - 2T(s_t - s_d)(k_d - k_t) \]  \hspace{1cm} (D-7)

Clearly, if i becomes sufficiently large that dT/di is zero (i.e., time deposits are at their maximum level, T0), equation (D-6) will be negative; for some high values of i, expected profits will be reduced if time deposit rates are raised. At the other extreme, when i, and hence T, is zero, equation (D-6) is positive; costs are not increased as much as expected profits when i is increased slightly from zero. Thus, expected profits at first increase and later decrease as i, and hence T, is raised. We can conclude, on the above assumptions, that the optimum proportion of total deposits in time deposit accounts will always be greater than zero. Moreover

The distribution, f(u), will be "triangular" only if s is one, i.e., time deposits never decline. If the direction of the inequality of (D-2) is reversed, the exact form of the distribution is altered, but the general shape is the same (and the distribution is "triangular" only if s is one).

1. This value of s is the lowest conceivable "deposit-low" of the distribution (D-3).

2. W. L. Smith uses a simplified version of equation (D-6), on the assumption that e equals g (i.e., neglecting the liquidity differences between assets), to show that it is profitable for the bank to induce switches from demand to time deposits, by raising time deposit rate, if:

\[ e > \frac{\frac{1}{k_d - k_t}}{[1 + \frac{1}{\eta}]} \]

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this conclusion still follows if time deposits are assumed to be reserve-saving though not deposit-stabilizing (i.e., \( k_d > k_t \), \( s_d = s_h \)), or vice versa (i.e., \( k_d = k_t \), \( s_d < s_h \)). Of course, there are many reasons beyond the scope of the model why the optimum amount of time deposits might be zero for a particular bank. Most obvious of these are that administrative considerations place a minimum on the amount of such deposits which the bank will wish to attract and/or that a time deposit rate markedly above zero is required to induce any time deposits at all. On the other hand, there are reasons other than expected profits (as defined here) for desiring such relatively stable deposits even if not profitable—for example, concern by the bank for the variability of its expected profits (see Appendix B).

Appendix E: Uniform Distribution of "Deposit-Lows"

In order to indicate that the results of the model of the text do not depend too critically upon the type of distribution assumed for the bank's "deposit-lows," the same model is here worked through briefly on the alternative assumption that the "deposit-lows" are uniformly distributed over the range from \( s \) to unity; that is,

\[
(E-1) \quad f(u) = 1/(1-s)
\]

The expected addition to net worth is then exactly as given by equation (7) of the text with the appropriate alteration in the distribution of \( u \). The analog to equation (8), expected profit after the integrations have been performed, is:

\[
(E-2) \quad E[\Delta N] = gB + eL - \frac{g}{2(1-s)(1-k)} \left[ (1-s)(1-k) - B \right]^2 - \frac{\frac{e-g}{q}}{6(1-s)(1-k)}
\]

And the optimum proportion of securities (see equation (12)) is:

\[
(E-3) \quad B = \frac{(1-s)(1-k)}{(1 + \frac{e-g}{q})} [1 - \frac{e-g}{q}]
\]

It should be noted that the parameters enter in a less complex, but similar, manner (see pages 22-23).

The first and second partial derivatives of \( B \) have the following signs:

<table>
<thead>
<tr>
<th>( \frac{\partial B}{\partial x} )</th>
<th>( \frac{\partial^2 B}{\partial x \partial y} )</th>
<th>where</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a )</td>
<td>( y = q )</td>
<td>( y = g )</td>
</tr>
<tr>
<td>( x = s )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( x = k )</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( x = e )</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( x = g )</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( x = q )</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( x = a )</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

where \( \eta \) is the time deposit interest elasticity of time deposits. Of course, the above conclusions follow in this special case as well. Cf. footnote 7, p. 544 of W. L. Smith, "Financial Intermediaries and Monetary Controls," Quarterly Journal of Economics, Vol. 73, No. 4 (November 1959), pp. 533-553.

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Appendix F: End-of-Period Deposits

In the text it is assumed that the "most radical" adjustment of the bank's portfolio is required at the moment when its deposits reach their lowest point of the period. If the bank is able to postpone this adjustment until the end of the period by means of short-term borrowing, then the relevant distribution for its profit calculations is that of end-of-period deposits. It is interesting to compare the implications for the bank's portfolio of this alternative assumption about the relevant deposit distribution.

In Appendix A it is shown, under certain assumptions, that the "deposit-low" distribution is simply a truncated (at the level of initial deposits) duplication (of twice the density) of the end-of-period deposit distribution. Since this "deposit-low" distribution is approximated in the text by a "triangular" distribution,

\[ f(u) = \frac{2(u-s)}{(1-s)^2} \quad \text{for} \quad 0 < s < u < 1, \]

comparison between the two assumptions is facilitated if the distribution of end-of-period deposits, \( f(v) \), is approximated by an "isosceles" distribution:

\[ f(v) = \frac{(v-s)}{(1-s)^2} \quad \text{for} \quad 0 < s < v < 1 \]

\[ f(v) = \frac{(2-s-v)}{(1-s)^2} \quad \text{for} \quad 1 < v < (2-s) \]

The distribution of \( v \) for values greater than unity need not detain us, for the possible earnings obtained from net increases in deposits are quite independent of the start-of-period portfolio decision.

Therefore the expected addition to net worth may be written, analogously to equation (7) of the text, as:

\[
(F-1) \quad E[\Delta N] = \int \int_{w=a}^{(1+w)B} \left[ gB + wB + eL \right] \left[ \frac{v-s}{2a(1-s)^2} \right] \, dv \, dw
\]

\[ + \int \int_{w=a}^{v=s} \left[ q(1-k)+(q(1+w)B+q(1-k)v) \right] \left[ \frac{v-s}{2a(1-s)^2} \right] \, dv \, dw \]

\[ + \frac{1}{2} \int_{w=a}^{gB + wB + eL} \left[ \frac{1}{2a} \right] \, dw \]

Maximization yields a value for \( B \), the optimum securities holding, analogously to equation (12) of the text:

\[
(F-2) \quad B = \frac{(1-s)(1-k)(1+a^2)}{1 + a^2} \left\{ 1 - \sqrt{1 - \frac{(1+a^2)(1-2e^{-\frac{2}{3}})}{(1 + a^2/3)^2}} \right\} 
\]

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The only difference between equation (F-2) and equation (12) is that the term, \((e-g)/q\), is multiplied by a factor of two in equation (F-2). For equal values of all parameters, the present formulation of the deposit distribution therefore implies a lower amount of securities and a larger amount of loans.¹ The directions of change in the optimum portfolio induced by variations in the parameters are, however, not affected by this concern for end-of-period deposits instead of the "deposit-low." Of course, the results would be different for other specific forms of the distributions, but this exercise indicates that the conclusions of the model are not critically dependent upon the deposit distribution assumption.

¹. A corner maximum with no securities held in the optimum portfolio, now occurs when \((e-g)/q\) is greater than .5, whereas, on the assumption of the text, the corner maximum occurred when \((e-g)/q\) was greater than 1.0.