

## AN INVENTORY POLICY FOR REPAIR PARTS\*

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### 1. INTRODUCTION

This paper considers an inventory problem for a manufacturer who maintains a stock of repair parts of his own products. The demand for any given part is generated by failures of that part in the existing stock of products. Such failure distributions are known for typical items. In the simplest case considered here, the conditional probability of failure is independent of the lifetime of the part. The number of parts demanded during a time interval of given length is then Poisson distributed. Another simple case is that of a high probability of failure during the first moments of use and a constant smaller conditional probability of failure thereafter. In this case, the number of times a demand arises in any fixed time interval is still approximately Poisson, while the number of units per demand is geometrically distributed. Suppose that each failure leads to a demand for a repair part from the manufacturer, except in a constant proportion of cases (say) in which the product is scrapped. We then have a simple description of the demand that is exercised on the manufacturer's stock of repair parts.

The stock of parts may be replenished by placing an order with the production department. Each batch of a repair part requires a certain setup cost in addition to the proportional cost of production. In general, processing of an order will be subject to various delays. We will assume that the time required for delivery is a random variable of known distribution. Because of this source of uncertainty, the stock department will wish to postpone specification of the size of the batch until the manufacturing has started, that is (roughly speaking) until the time of delivery. This implies that not more than one order will be outstanding at a time. -- Unfilled demand from customers will be backlogged at a penalty to the firm, reflecting the loss of good will.

The inventory problem for repair parts as outlined is of some importance, notably in connection with military logistics, but it also has some methodological interest. For it falls into a class which is most naturally analyzed in terms of continuous time. And the study of inventory problems of this nature is still in its beginning. The type of stochastic inventory model which has been perfected in a recent monograph by Arrow, Karlin, and Scarf [1] assumes a division of time into discrete periods at the beginnings of which all sections are taken. "The

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transition to continuous time in such models will require formulation of demands as a continuous-time stochastic process. We believe that such a reformulation will turn out to be not merely more realistic but also ultimately simpler to handle analytically, but it has not been accomplished" [1, p. 24]. The interested reader may refer to a recent paper on inventory policy for fixed delivery time by Beckmann and Muth [2] and to a monograph on queues by Morse [3].

Morse has considered the following cases on the assumption that unfilled demand is lost, demand is Poisson distributed, and delivery times are exponential:

1. An order for one unit is sent whenever an item is sold [p. 139].
2. An order for  $S$ , the maximum stock, is sent when stock is zero [p. 146].
3. An order for  $D$  units is sent when the stock level falls into a reorder point  $s$ . If thereafter stock falls to zero, another order is placed for  $s$  such that the two outstanding orders equal the maximum stock  $S$  [p. 148].
4. Order size  $D$  is technologically fixed, maximum inventory  $S$  is an unknown multiple (up to 4) of this size. An order for  $D$  is sent whenever stock falls to a multiple of  $D$  [p. 153].

These policies are interesting, but not always optimal. The present paper adds an important case, but there remains a great number of unanalyzed alternatives.

In our paper on inventory policy [2], R. Muth and I have considered a simple model in which demand is described by a Poisson process, delivery times are constant, and order sizes are specified in advance. The situation differs from the present problem, because it is then optimal to allow several orders to be outstanding.

To sum up our assumptions:

1. One unit is demanded at a time.
2. The probability distribution of demand is Poisson.
3. The probability density of the arrival of one shipment is a function of the time elapsed since the placing of the order.
4. At most, one order is outstanding.
5. The order size is specified at the time of the delivery.

For mathematical convenience we add the following assumptions:

6. An optimal starting stock exists.
7. The inventory policy is independent of the initial state of the system.

Our first object is to derive the cost of an inventory-control system from the "inventory equation" as formulated for continuous time. We shall then calculate the optimal  $s$  and  $D$  for the case of Poisson-distributed demand, exponentially distributed lag times, and proportional costs of storage and depletion.

## 2. THE INVENTORY EQUATION

Notations used herein are as follows:

- $x$  = stock
- $t$  = time since an outstanding order was placed
- $\lambda$  = probability density of a demand
- $\mu(t)$  = probability density of delivery
- $\alpha$  = (continuous) discount factor

$$f(x) = \begin{array}{ll} hx & \text{for } x > 0 \text{ storage cost} \\ -gx & \text{for } x < 0 \text{ shortage penalty} \end{array}$$

$k$  = fixed ordering cost

$S$  = optimal starting stock

$s$  = reordering point

$$D = S - s$$

$u(x)$  = loss function when no order is outstanding

$v(x, t)$  = loss function when an order placed  $t$  units previously is outstanding

$L(s, D)$  = expected value of discounted cost for a starting stock  $s+D$ , when the order size is  $D$  and the reordering point is  $s$

$$\beta = \alpha + \lambda$$

$$\gamma = \alpha + \lambda + \mu(t)$$

Suppose that stock is  $x$  and no order is outstanding. Compare the discounted expected cost of this state,  $u(x)$ , with what will prevail after a short time interval of length  $\Delta t$ . During  $\Delta t$  costs  $f(x) \cdot \Delta t$  will arise. With probability  $\lambda \Delta t$ , a sale will be made changing stock to  $x-1$ . Whether this will result in an order depends on whether  $u(x-1)$  or  $v(x-1, 0) + k$  is smaller. Applying appropriate discount factors, we now have the identity

$$(2.1) \quad u(x) = f(x) \Delta t + (1 - \lambda \Delta t) e^{-\alpha \Delta t} u(x) + \lambda \Delta t e^{-\alpha \Delta t} \text{Min} [u(x-1), v(x-1, 0) + k].$$

Consider secondly the case where stock is  $x$  and time  $t$  has elapsed since an undelivered order was first placed. Then, with probability  $\mu(t) \Delta t$ , the order will be delivered, changing the cost function to  $u(S)$ ; again, with probability  $\lambda \Delta t$ , a unit will be demanded, resulting in  $v(x-1, t + \Delta t)$ . Thus

$$(2.2) \quad \begin{aligned} v(x, t) = & f(x) \Delta t + (1 - \lambda \Delta t - \mu \Delta t) e^{-\alpha \Delta t} v(x, t) \\ & + \lambda \Delta t e^{-\alpha \Delta t} v(x-1, t + \Delta t) + \mu \Delta t e^{-\alpha \Delta t} u(S), \end{aligned}$$

In the limit, when  $\Delta t \rightarrow 0$

$$(2.3) \quad \beta u(x) = f(x) + \lambda \text{Min} [u(x-1), v(x-1, 0) + k],$$

$$(2.4) \quad \gamma v(x, t) - \frac{\partial v}{\partial t}(x, t) = f(x) + \lambda v(x-1, t) + \mu u(S).$$

Suppose that we start the system at a sufficiently high level of  $x$  and with no order outstanding. As  $x$  is decreased through sales, either it will never pay to order—a trivial case which we rule out—or a point  $s$  will be reached such that

$$\begin{aligned} & \text{Min } [u(s), v(s, 0) + k] = v(s, 0) + k \\ \text{while} & \\ & \text{Min } [u(x), v(x, 0) + k] = u(x) \qquad \text{for all } x > s.* \end{aligned}$$

Equation (2.3) may now be written more explicitly:

$$(2.5) \qquad \beta u(x) = f(x) + \lambda u(x-1), \qquad x > s+1;$$

$$(2.6) \qquad \beta u(s+1) = f(s+1) + \lambda v(s, 0) + \lambda k.$$

Through successive substitution in (2.4),  $u(s)$  may be expressed in terms of  $v(s, 0)$  as follows:

$$(2.7) \qquad u(s) = \frac{1}{\beta} \sum_{i=0}^{D-1} f(s-i) a^i + a^D [k + v(s, 0)], \qquad \text{where } a = \frac{\lambda}{\alpha + \lambda}.$$

To calculate  $v(s, 0)$ , we turn to the differential-difference equation (2.4). Multiply both sides by

$$\exp\left(-\int_0^t (\alpha + \lambda + \mu(t_1)) dt_1\right) = \exp\left(-(\alpha + \lambda)t - \int_0^t \mu dt_1\right) = \phi(t), \text{ say.}$$

Now,

$$\begin{aligned} \frac{-\partial v}{\partial t} \cdot \phi + (\alpha + \lambda + \mu) v \phi &= -\frac{\partial v}{\partial t} \cdot \phi - \frac{\partial \phi}{\partial t} \cdot v \\ &= \frac{-\partial}{\partial t} (v \phi). \end{aligned}$$

$$(2.8) \qquad \frac{-\partial}{\partial t} [v(x, t) \phi(t)] = f(x) \phi(t) + \mu(t) \phi(t) u(s) + \lambda \phi(t) v(x-1, t).$$

Write  $v(x, t) \phi(t) = y(x, t)$  and consider this to be our new unknown. Integrating (2.8) from  $t$  to  $\infty$  we have

$$(2.9) \qquad y(x, t) - \lim_{t_1 \rightarrow \infty} y(x, t_1) = f(x) \int_t^\infty \phi dt_1 + u(s) \int_t^\infty \mu \phi dt_1 + \lambda \int_t^\infty y(x-1, t_1) dt_1.$$

To estimate the limit, observe that  $v(x, t)$  cannot exceed the expected discounted cost when no deliveries occur. The latter is bounded by

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\*It is a fine point of mathematics to show that this  $s$  is independent of whatever initial stocks  $x > s$  we started with; in other words, that there are no arbitrarily large  $s$ . For this to be true it is sufficient that disposal of stock is costless. For then  $u(x)$  must be a nonincreasing function. This implies that, from a certain stock level on, disposal actually takes place. But  $s$  can be no larger than this maximal stock level. Incidentally, it follows also that an optimal starting stock  $S$  must then exist.

$$(2.10) \quad g\left(\frac{|x|}{\alpha} + \int_0^\infty \lambda t e^{-\alpha t} dt\right) = \text{const.} + \frac{g}{\alpha} |x|.$$

Therefore,

$$|y(x, t)| \leq |(C_0 + C_1) |x| e^{-(\alpha + \lambda)t}$$

and

$$\lim_{t_1 \rightarrow \infty} y(x, t_1) = 0.$$

Substituting (2.9) with  $x - 1$  in the last integral yields

$$\begin{aligned} y(x, t) = & f(x) \int_t^\infty \phi dt_1 + f(x-1) \lambda \int_t^\infty \int_{t_1}^\infty \phi dt_2 dt_1 + u(S) \int_t^\infty \mu \phi dt_1 \\ & + u(S) \lambda \int_t^\infty \int_{t_1}^\infty \mu \phi dt_2 dt_1 + \lambda^2 \int_t^\infty \int_{t_1}^\infty y(x-2, t_2) dt_2. \end{aligned}$$

Successive substitution leads to

$$\begin{aligned} y(x, t) = & \sum_{i=0}^n f(x-i) \lambda^i \int_t^\infty \dots \int_{t_i}^\infty \phi dt_{i+1} \dots dt_1 + u(S) \sum_{i=0}^n \lambda^i \int_t^\infty \dots \int_{t_i}^\infty \phi \mu dt_{i+1} \dots dt_1 \\ & + \lambda^n \int_t^\infty \dots \int_{t_n}^\infty y(x-n, t_{n+1}) dt_{n+1} \dots dt_1. \end{aligned}$$

We now recall the formula for the iterated integral (assuming the integral to exist),

$$\int_t^\infty \dots \int_{t_i}^\infty g(t_{i+1}) dt_{i+1} \dots dt_1 = \int_t^\infty \frac{(t_1 - t)^i}{i!} g(t_1) dt_1.$$

For the last term in (2.9) we now obtain

$$(2.11) \quad \frac{\lambda^n}{n!} \int_t^\infty (t_1 - t)^n y(x-n, t_1) dt_1.$$

By applying the estimate (2.10), it is easy to show that (2.11) is bounded by

$$\left(\frac{\lambda}{\alpha + \lambda}\right)^n \cdot e^{-(\alpha + \lambda)t} |C_0 + C_1 |x| + C_1 n|,$$

and therefore converges to zero uniformly in  $t (\geq 0)$  as  $n \rightarrow \infty$ . We therefore have, putting  $t = 0$ ,

$$(2.12) \quad y(x, 0) = \sum_{i=0}^{\infty} f(x - i) \int_0^{\infty} \lambda^i \frac{t^i}{i!} \exp\left[-(\alpha + \lambda)t - \int_0^t \mu dt_1\right] dt \\ + u(S) \sum_{i=0}^{\infty} \int_0^{\infty} \frac{\lambda^i}{i!} t^i \mu(t) \exp\left[-(\alpha + \lambda)t - \int_0^t \mu dt_1\right] dt.$$

Now  $\mu(t) e^{-\int_0^t \mu dt_1}$  is the probability density that delivery time equals  $t$ , which will be written  $q(t)$ . The last term assumes, then, the form

$$u(S) \cdot \sum_{i=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-(\alpha + \lambda)t} q(t) dt = u(S) \cdot \int_0^{\infty} e^{-\alpha t} q(t) dt,$$

after an interchange of summation and integration—which is legitimate because the sum converges uniformly in  $T$  if  $T$  denotes the upper limit of integration.

Consider now the first integral on the right-hand side of (2.12),

$$\int_0^{\infty} \frac{(\lambda t)^i}{i!} \left( \exp -\lambda t - \alpha t - \int_0^t \mu dt_1 \right) dt.$$

Now  $[(\lambda t)^i / i!] e^{-\lambda t} = p(i, t)$  is the (Poisson) probability that  $i$  units are demanded during an interval  $t$ . The probability that no delivery is made within a time  $t$  after placing an order is

$$\text{given by } e^{-\int_0^t \mu dt} = 1 - Q(t).$$

Equation (2.12) therefore takes the form

$$(2.13) \quad y(s, 0) = \sum_{i=0}^{\infty} f(x - i) \int_0^{\infty} p(i, t) [1 - Q(t)] e^{-\alpha t} dt \\ + u(S) \int_0^{\infty} q(t) e^{-\alpha t} dt = v(s, 0).$$

If the probability density for a demand depends on the time elapsed since the last demand, but this time is not remembered, a calculation along the same lines, but more complex in detail, leads to the same formula. However,  $p(i, t)$  is then no longer Poisson.

Substituting (2.13) into (2.7) and ordering terms, we have

$$(2.14) \quad L(s, D) = \frac{k + \frac{1}{\beta} \cdot \sum_{i=1}^D f(s+i) a^i + \sum_{i=0}^{\infty} f(s-i) \int_0^{\infty} p(1-Q) e^{-\alpha t} dt}{a^{-D} + \int_0^{\infty} q e^{-\alpha t} dt}$$

### 3. MINIMIZING COST PER TIME

Since  $\alpha$  is small relative to  $\lambda$  and provided that  $q(t)$  is small for large  $t$ , the denominator may be approximated as follows:

$$\begin{aligned} 1 + \frac{\alpha}{\lambda} D - \int_0^{\infty} q(t) [1 - \alpha t] dt \\ = \frac{\alpha}{\lambda} D + \alpha \int_0^{\infty} t q(t) dt \\ = \alpha \bar{t} + \frac{D}{\lambda}, \end{aligned}$$

where  $\bar{t} + D/\lambda$  is the average length of a stock cycle from S to S.

On the other hand, in the numerator we may suppress the discount factors, provided we absorb the interest cost in the charges for storage and depletion,  $f(x)$ . The numerator represents the total inventory cost per cycle. The formula,

$$(3.1) \quad \frac{k + \frac{1}{\lambda} \sum_{i=1}^D f(s+i) + \sum_{i=0}^{\infty} f(s-i) \int_0^{\infty} p(i, t) [1 - Q(t)] dt}{\frac{D}{\lambda} + \int_0^{\infty} t q(t) dt} = \bar{L}(s, D),$$

now gives the average cost of the inventory system per unit time. Up to a factor of proportionality—which is irrelevant for purposes of minimization—the formula (2.14) for the discounted cost of the system may, therefore, be approximated by the simpler expression (3.1) for the average cost per unit time.

We now specify

$$(3.2) \quad \begin{aligned} f(x) &= \begin{cases} hx & x > 0 \\ -gx & x < 0 \end{cases} \\ p(i, t) &\text{ Poisson} \end{aligned}$$

Writing

$$\bar{t} = \int_0^{\infty} t q(t) dt \quad \text{expected delivery time}$$

$$\sigma^2 = \int_0^{\infty} (t - \bar{t})^2 q(t) dt \quad \text{variance of delivery time}$$

$$a_i = \int_0^{\infty} p(i, t) [1 - Q(t)] dt,$$

we obtain, after some calculation,

$$(3.3) \quad \begin{aligned} \bar{L} &= \frac{\lambda}{D + \lambda \bar{t}} \left[ k + \frac{h}{\lambda} sD + \frac{hD(D+1)}{2\lambda} + (h+g) \sum_{i=0}^{s-1} (s-i) a_i \right. \\ &\quad \left. + g \frac{\lambda}{2} (\sigma^2 + \bar{t}^2) - g s \bar{t} \right]. \end{aligned}$$

#### 4. ORDER SIZE D AND REORDERING POINT s

It remains to determine the best values for  $s$  and  $D$ . These are given by the conditions that the total discounted cost of the inventory system be minimal. To a first degree of approximation the average inventory cost per unit time,  $\bar{L}$ , may be minimized instead. To this end we set the first differences of  $\bar{L}$  with respect to  $s$  and  $D$  equal to zero:

$$(4.1) \quad \begin{aligned} \Delta_s \bar{L} &\sim \frac{hD}{\lambda} + (h+g) \sum_{i=0}^{s-1} a_i - g\bar{t} = 0. \\ \sum_{i=0}^{s-1} a_i &= \frac{g\bar{t} - \frac{hD}{\lambda}}{g+h}. \end{aligned}$$

To take the first difference with respect to  $D$ , write  $\bar{L}$  in the form

$$\bar{L} = \frac{1}{D + \lambda t} \left[ \hat{k} \lambda + h s D + \frac{h}{2} D(D+1) \right],$$

where  $\hat{k}$  is independent of  $D$ .

Disappearance of  $\Delta_D \bar{L}$  means that

$$h(s + D)(D + \lambda \bar{t}) = \hat{k}\lambda + hsD + \frac{h}{2}D(D + 1)$$

or

$$D + (\lambda \bar{t} - 1/2) = \sqrt{\frac{2\hat{k}\lambda}{h} - 2s\lambda\bar{t} + (\lambda \bar{t} - 1/2)^2}.$$

Since  $D$  must be integral, terms of the order of  $1/2$  may be neglected.

$$(4.2) \quad D = -\lambda \bar{t} + \sqrt{\frac{2K\lambda}{h} + (\lambda \bar{t})^2},$$

where

$$(4.3) \quad \begin{aligned} K &= \hat{k} - hst = k + h \sum_{i=0}^{s-1} (s-i) a_i + g \sum_{i=s}^{\infty} (i-s) a_i - hs\bar{t} \\ &= k + (g+h) \sum_{i=s}^{\infty} (i-s) a_i - \frac{h\lambda}{2} (\sigma^2 + \bar{t}^2), \end{aligned}$$

where  $\sum_{i=s}^{\infty} (i-s) a_i$  represents the expected size of the shortage per cycle.

##### 5. POISSON DEMAND—EXPONENTIAL DELIVERY TIME

The simplest case is, of course, that in which both  $\lambda$  and  $\mu$  in Eq. (2.1) are independent of time. This means that, during any fixed-time interval, demand is Poisson distributed and that the distribution of lag times is exponential with  $\bar{t} = 1/\mu$ . Now

$$\begin{aligned} a_i &= \int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t - \mu t} dt = \frac{1}{\lambda + \mu} \left( \frac{\lambda}{\lambda + \mu} \right)^i \\ &= \frac{\hat{a}^i}{\lambda + \mu} \text{ (say)}. \end{aligned}$$

Formula (4.1) for  $s$  becomes

$$\frac{1}{\lambda + \mu} \frac{1 - \hat{a}^s}{1 - \hat{a}} = \frac{\frac{g}{\mu} - \frac{hD}{\lambda}}{g + h}.$$

$$(5.1) \quad \hat{a}^s = \frac{h}{g+h} \left( 1 + \frac{\mu}{\lambda} D \right)$$

$$(5.2) \quad s = \frac{\log \frac{h}{g+h} + \log \left( 1 + \frac{\mu}{\lambda} D \right)}{\log \frac{\lambda}{\lambda + \mu}}$$

To calculate  $D$  a direct approach is more convenient than the use of formula (4.2). First,

$$(5.3) \quad \bar{L} = \frac{\lambda}{D + \frac{\lambda}{\mu}} \left[ k + \frac{h}{\lambda} \left( sD + \frac{D(D+1)}{2} \right) + \frac{h+g}{\lambda + \mu} \sum_{i=0}^{s-1} (s-i) \hat{a}^i + q \frac{\lambda}{\mu^2} - \frac{gs}{\mu} \right]$$

The difference with respect to  $D$  is proportional to

$$k - \frac{h}{2\lambda} D^2 + \frac{h}{2\lambda} D + \frac{(h+g)}{\lambda + \mu} \frac{s \cdot \frac{\mu}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} (1 - a^s)}{\frac{\mu^2}{(\lambda + \mu)^2}} - (h+g) \frac{s}{\mu} - \frac{hD}{\mu} + \frac{g\lambda}{\mu^2} = 0.$$

By the substitution of (5.1) and clearing terms,

$$k - \frac{h}{2\lambda} D^2 + \left( \frac{h}{2\lambda} - \frac{h}{\mu} \right) D - \frac{\lambda}{\mu^2} \left( g - hD \frac{\mu}{\lambda} \right) + \frac{g\lambda}{\mu^2} = 0,$$

$$(D - 1/2)^2 = \sqrt{\frac{2\lambda k}{h} + \frac{1}{4}}.$$

Since  $D$  must be an integer, terms of the order of  $1/2$  may be neglected,

$$(5.4) \quad D = \sqrt{\frac{2\lambda k}{h}},$$

and this is the well-known Wilson formula for optimal lot size [4, p. 33]. Notice that the order size turns out to be independent of the reordering point  $s$  and, in particular, of the size of the shortage penalty  $g$ .

Incidentally, this formula remains valid for the more general delivery-time distribution

$$q(t) = \sum_i c_i \mu_i e^{-\mu_i t},$$

where

$$\sum_i c_i = 1 \quad c_i > 0,$$

known sometimes as the hyper-exponential distribution. Returning to formula (4.1), we obtain the reordering point  $s$  in the final form

$$\left(\frac{\lambda}{\lambda + \mu}\right)^s = \frac{h}{g + h} \left(1 + \mu \sqrt{\frac{2k}{\lambda h}}\right),$$

or, explicitly,

$$(5.5) \quad s = \frac{\log \frac{g+h}{h}}{\log \left(1 + \frac{\mu}{\lambda}\right)} - \frac{\log \left(1 + \mu \sqrt{\frac{2k}{\lambda h}}\right)}{\log \left(1 + \frac{\mu}{\lambda}\right)}.$$

When lags are large relative to the spacing of demand, then  $\mu$  is small compared with  $\lambda$ , and we have approximately

$$(5.6) \quad s = \frac{g}{h\mu} \cdot \lambda - \left(1 + \frac{g}{h}\right) \sqrt{\frac{2k}{h}} \cdot \sqrt{\lambda}.$$

A straightforward calculation shows that the value of the minimum i.e., the average inventory cost per unit time, is equal to

$$\bar{L} = h(s + D) + \frac{h}{2} \cdot \frac{1}{1 + \frac{\lambda}{\mu D}},$$

or, from (5.4) and (5.6),

$$\bar{L} = \frac{g}{\mu} \cdot \lambda - g \cdot \sqrt{\frac{2k}{h}} \cdot \lambda + \frac{\mu \sqrt{\frac{kh}{2}}}{\mu \sqrt{\frac{k}{2h}} + \sqrt{\lambda}}.$$

In the case that the failure probability of a part is high during a small initial period of its use and constant thereafter, we may approximate by assuming that the probability of failure has an exceptional value,  $r$ , in the first instant and is exponential after that. To replace one part will thus require  $n$  items with probability  $(1 - r) r^{n-1}$ . The distribution of demand during a random interval of length  $t$  is now obtained by compounding a Poisson-distributed number of trials with the geometric distribution of numbers per trial.

When both  $r$  and  $\mu$  are small the solution is approximately as in the case of Poisson demand, with a demand density  $\lambda/(1 - r)$ . That is to say, for small probabilities of initial failure and long delivery times, the Poisson model applies with the appropriate demand density. The following case, based on data supplied by a manufacturing firm in Chicago, may serve as an example:

$\lambda$	1
$\mu$	0.01
$k$	1.80 \$
$h$	0.002 \$
$g$	2.00 \$

$$(5.4) \quad D = \sqrt{\frac{2 \cdot 1.8}{0.002}} = \sqrt{3600} = \underline{60}$$

$$(5.2) \quad s = \frac{\log \frac{0.002}{2.002} + \log (1+0.01 \cdot 60)}{\log \frac{1}{1.01}}$$

$$\approx \frac{\log 0.001 + \log 1.6}{-\log 1.01}$$

$$= \frac{-3 + .204}{-0.04} = \underline{70}.$$

#### SUMMARY

Assume that the demand for a repair part is Poisson, that the costs of storage and shortage per part and of reordering are constant, that the probability distribution of delivery times for stock replacement is arbitrary but known, and that the order size is specified at the time of delivery. The inventory equation for this problem is formulated and solved; the optimal policy is found to be of the  $s, S$  type (Section 2). Since the loss function is closely proportional to the average cost per unit time (Section 3), it is found convenient to minimize average cost in determining order size and reordering point rather than discounted future cost (Section 4). The solution turns out to be simple in the case of exponential delivery time; in fact, the minimal order size is that given by the Wilson square-root formula (Section 5). These formulae are approximately valid also in the case where the initial probability of failure is somewhat higher than the conditional-failure probability at later times.

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