

ON "AN IDENTITY IN ARITHMETIC"<sup>1</sup>

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H. D. Block and J. Marschak have presented in the Bulletin of the American Mathematical Society vol. 65 (1959) pp. 123-124, an identity which arose in a probability context. This note proves it by a probability theoretical argument.

Consider an experiment having the set of possible outcomes  $N = \{1, \dots, n\}$  with positive probabilities  $(u_1, \dots, u_n)$  and an infinite sequence of independent repetitions of that experiment. Let  $M$  be a subset of  $N$  and  $i$  be an element of  $M$ , and denote by  $A(i, M)$  the event that  $i$  is the first element of  $M$  which occurs in an infinite sequence of outcomes. If  $B_j(i, M)$  denotes the event that the first  $(j-1)$  outcomes are not in  $M$  and the  $j$ th outcome is  $i$ , one has

$$A(i, M) = \bigcup_{j=1}^{\infty} B_j(i, M).$$

Hence

$$\Pr[A(i, M)] = \sum_{j=1}^{\infty} \Pr[B_j(i, M)] = \sum_{j=1}^{\infty} (1 - u_M)^{j-1} u_i = u_i / u_M,$$

writing  $u_M$  for  $\sum_{j \in M} u_j$ , and putting  $0^0 = 1$  for the degenerate case  $M = N$ .

Let now  $r$  be a permutation of  $N$  and  $k_r$  be the element of  $N$  ranked  $k$ th by  $r$ . Let also  $C(r)$  be the event that the first occurrences of the  $n$  elements of  $N$  in an infinite sequence of outcomes appear in the order  $r = (1_r, \dots, n_r)$ . If  $D_{j_1, \dots, j_n}(r)$  denotes the event that  $k_r$  occurs for the first time at the  $j_k$ th performance of the experiment for every  $k = 1, \dots, n$  (therefore  $j_1 = 1$  in what follows), one has

$$C(r) = \bigcup_{j_1 < \dots < j_n} D_{j_1, \dots, j_n}(r).$$

Hence

$$\begin{aligned} \Pr[C(r)] &= \sum_{j_1 < \dots < j_n} \Pr[D_{j_1, \dots, j_n}(r)] \\ &= \sum_{j_1 < \dots < j_n} u_{1_r}^{j_2-j_1} u_{2_r} (u_{1_r} + u_{2_r})^{j_3-j_2-1} u_{3_r} (u_{1_r} + u_{2_r} + u_{3_r})^{j_4-j_3-1} \dots u_{n_r}. \end{aligned}$$

Received by the editors June 12, 1959.

<sup>1</sup> A technical report of research undertaken by the Cowles Foundation for Research in Economics under contract with the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government. I wish to thank S. Kakutani and C. B. McGuire for the conversations I had with them about this note.

Putting  $j_{k+1} - j_k - 1 = h_k$  and  $u_1 + \cdots + u_{k_r} = v_{k,r}$ , one obtains

$$\begin{aligned} \Pr[C(r)] &= \left( \prod_{j=1}^n u_j \right) \sum_{h_1, \dots, h_{n-1} \geq 0} v_{1,r}^{h_1} v_{2,r}^{h_2} \cdots v_{n-1,r}^{h_{n-1}} \\ &= \left( \prod_{j=1}^n u_j \right) \prod_{k=1}^{n-1} (1 - v_{k,r})^{-1}. \end{aligned}$$

Let finally  $R(i, M)$  be the set of permutations of  $N$  for which  $i$  is ranked first among the elements of  $M$ . Since the event that some element of  $N$  never occurs has probability zero, one has

$$\Pr[A(i, M)] = \sum_{r \in R(i, M)} \Pr[C(r)].$$

Hence by a simple transformation we obtain the desired identity:

$$u_i / \sum_{j \in M} u_j = \left( \prod_{j=1}^n u_j \right) \sum_{r \in R(i, M)} \prod_{k=1}^n \left( \sum_{j=k}^n u_{j_r} \right)^{-1}.$$

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