

Cardinal Utility for Even-chance Mixtures of Pairs of Sure Prospects^{*}

This short note had its origin in conversations that the author was privileged to have with Jacob Marschak. It is respectfully dedicated to him on the occasion of his sixtieth birthday.

In the utility theory of J. von Neumann and O. Morgenstern [8] a subject is faced with a set of prospects on which a mixture operation is defined, i.e., two prospects x, y and a probability p define a prospect denoted $(px, (1-p)y)$ and interpreted as the prospect of having either prospect x with probability p or prospect y with probability $1-p$. Given a complete preference preordering¹ on the set of prospects, axioms (on the mixture operation and on the preference preordering) which insure the existence on that set of a real-valued, order-preserving function v satisfying the relation :

$$v(px, (1-p)y) = pv(x) + (1-p)v(y)$$

have been offered by J. von Neumann and O. Morgenstern [8], J. Marschak [7], I. N. Herstein and J. Milnor [6] and others.

Some of the difficulties encountered in the testing of that theory originate in the inability of subjects to grasp the significance of complex prospects. This has led D. Davidson and J. Marschak [3] to study a situation where the subject is faced only with the simplest type of uncertain prospect. Let S be a set of sure prospects (e.g., commodity bundles) which will be assumed, for the sake of conceptual simplicity, to be a subset of R^n . Given two elements a and b of S , the symbol ab denotes the prospect of having either a with probability $\frac{1}{2}$ or b with probability $\frac{1}{2}$. The set $S \times S$ of prospects is completely preordered by the relation \preceq which is read "is not preferred to." As usual, \sim is read "is indifferent to," and $>$ is read "is preferred to." In this context,

A utility function is a real-valued, order-preserving function u on $S \times S$ such that

$$u(ab) = \frac{1}{2}[u(aa) + u(bb)] \text{ for every } a \text{ and } b \text{ in } S.$$

The object of this note is to give sufficient conditions for the existence of such a function.

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¹ A preordering is a reflexive and transitive binary relation.

The first axiom requires that the set S be of one piece :

- (1) S is a connected subset of R^n .

The second imposes continuity properties on the preference preordering :

- (2) \leq is a complete preordering of $S \times S$ such that the sets $\{ab \in S \times S \mid ab \succsim a'b'\}$ and $\{ab \in S \times S \mid ab \preceq a'b'\}$ are closed in $S \times S$ for every $a' b'$ in $S \times S$.

The last axiom is :

- (3) $[a_1 b_2 \preceq a_2 b_1 \text{ and } a_2 b_3 \preceq a_3 b_2] \longrightarrow [b_3 a_1 \preceq b_1 a_3]$.

It is clearly a necessary condition for the existence of a utility function.

THEOREM. Under assumptions (1), (2), (3), there is a continuous utility function determined up to an increasing linear transformation.

First a few consequences of (3) will be established. Making a_1, b_1 equal to a and a_2, b_2, a_3, b_3 equal to b in (3), one obtains $[ab \preceq ba \text{ and } aa \preceq aa] \longrightarrow [ba \preceq ab]$. Therefore $[ab \preceq ba] \longrightarrow [ba \preceq ab]$. Hence :

- (4) $ab \sim ba$ for every a and b in S .

From this and (3) one can easily show that

- (5) $[a_1 b_2 \sim a_2 b_1 \text{ and } a_2 b_3 \sim a_3 b_2] \longrightarrow [a_3 b_1 \sim a_1 b_3]$.

Making now a_1, b_2 equal to a ; a_2, b_1 equal to b ; a_3, b_3 equal to c in (3), one obtains $[aa \preceq bb \text{ and } bc \preceq ca] \longrightarrow [ca \preceq bc]$. Hence $[aa \preceq bb] \longrightarrow [ca \preceq bc]$. Making then a_1, b_3 equal to a ; a_3, b_1 equal to b ; a_2, b_2 equal to c in (3), one obtains $[ac \preceq cb \text{ and } ca \preceq bc] \longrightarrow [aa \preceq bb]$. Hence, by (4), $[ca \preceq bc] \longrightarrow [aa \preceq bb]$. Summing up, and using (4) again,

- (6) $[aa \preceq bb] \longleftrightarrow [ca \preceq cb]$.

Finally, from (6) and (4), $[aa \sim a' a'] \longrightarrow [ab \sim a' b]$ and $[bb \sim b' b'] \longrightarrow [a' b \sim a' b']$. Hence,

- (7) $[aa \sim a' a' \text{ and } bb \sim b' b'] \longrightarrow [ab \sim a' b']$.

The theorem will be proved by means of a representation of $S \times S$ in R^2 . According to [4], there is, by (2), a continuous real-valued, order-preserving function f on $S \times S$. Let ab be a generic element of $S \times S$. Using the notation $\alpha = f(aa)$ and $\beta = f(bb)$, one defines the representation by $ab \rightarrow (\alpha, \beta)$. Since S is connected, the range of α is a real interval Σ . The real number $f(ab)$ depends only on α and β on account of (7). Let φ be the function defined on $\Sigma \times \Sigma$ in that way :

$$f(ab) = \varphi(\alpha, \beta).$$

(4) implies that $\varphi(\alpha, \beta) = \varphi(\beta, \alpha)$. And (6) implies that φ is increasing in each one of its two variables. It follows, without great difficulty, that φ is continuous. One may also notice that $\varphi(\alpha, \alpha) = \alpha$. Two curves corresponding to two different values of φ have been drawn in fig. 1.

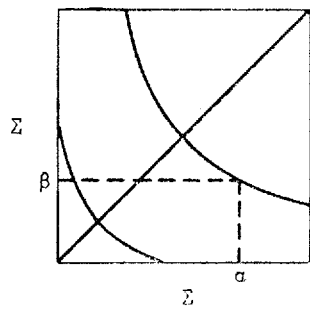


Figure 1

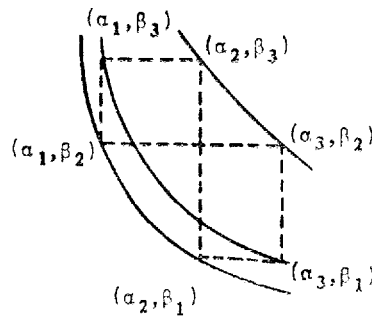


Figure 2

Finally (5) implies that :

$$[\varphi(\alpha_1, \beta_2) = \varphi(\alpha_2, \beta_1) \text{ and } \varphi(\alpha_2, \beta_3) = \varphi(\alpha_3, \beta_2)] \longrightarrow [\varphi(\alpha_3, \beta_1) = \varphi(\alpha_1, \beta_3)].$$

Fig. 2 illustrates this relation.

The existence of a utility function u on $S \times S$ is clearly equivalent to the existence of a real-valued function v on $\Sigma \times \Sigma$, derived from φ by an increasing transformation θ , and such that

$$v(\alpha, \beta) = \frac{1}{2}[v(\alpha, \alpha) + v(\beta, \beta)].$$

This, in turn, is equivalent, because of the relation $\varphi(\alpha, \alpha) = \alpha$, to the existence of an increasing transformation θ on both co-ordinates in fig. 1 which carries the level curves of φ into straight line segments perpendicular to the diagonal as in fig. 3. The close analogy

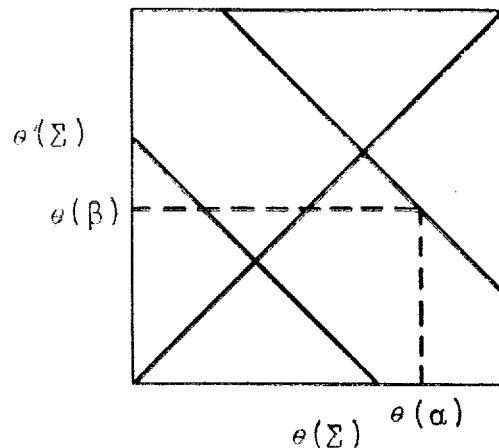


Figure 3

of the latter problem with the one treated in [5], both being particular cases of a problem of plane topology¹ studied by G. Thomsen [9] and W. Blaschke [1] (See also W. Blaschke and G. Bol [2]), makes it unnecessary to go into the details of the proof. The hexagonal

¹ The problem can be briefly described as follows. Given three families of curves in a plane, when does there exist a topological transformation carrying them into three families of parallel straight lines? In Fig. 1 the three families are the level curves of φ , the verticals, the horizontal. After the transformation, in fig. 3, they are the perpendiculars to the diagonal, the verticals, the horizontal.

configuration of fig. 2 allows one to show that there exists a *continuous* transformation θ of the wanted type. The continuous utility function u is defined by

$$u(ab) = \theta[\varphi(\alpha, \beta)].$$

It is obviously determined up to an increasing linear transformation.

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