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A STATISTICAL MODEL OF FRICTION IN ECONOMICS

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The maximum likelihood method for estimating relationships with limited dependent variables is generalized to include relationships in which the dependent variable, over some finite range, is not related to the independent variables.

THE STATISTICAL model presented in this paper is a generalization of the model of a limited dependent variable described by Tobin in *Econometrica*, January 1958.[1] The Tobin model deals with relationships in which it is known that the conditional cumulative distribution function of the dependent variable has a mass point at some lower or upper limiting value of the dependent variable. The more general model includes cases in which the mass point is anywhere in the conditional cumulative distribution function.

For example, if one were to examine the effects of changes in yield on changes in the holdings of a particular asset by a certain class of investors, it might be found that small changes in yield have no effect because of transaction cost. Figure 1 represents such a relationship. $\triangle A$ is change in asset holdings and $\triangle r$ is change in yield. For any change in yield $\triangle r_1 \leq \triangle r \leq \triangle r_2$, the change in asset holdings is zero.

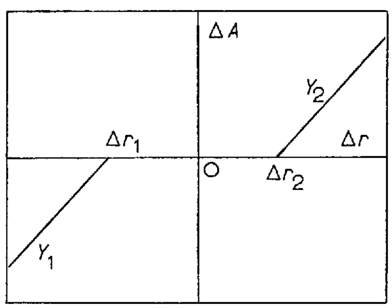

FIGURE 1

Since this is not an exact relationship, it is necessary to specify the distribution of errors. One would expect that at $\triangle r_1$, a negative error in $\triangle A$ would be more likely than a positive error and that at $\triangle r_2$ the reverse would be true. Further, if the relationship, for a given $\triangle r$, specifies a non-zero $\triangle A$ and an error $u = -\triangle A$ would give an observed value of $\triangle A = 0$, one would expect that if u were of slightly greater magnitude, the observation would still be $\triangle A = 0$.

[1] James Tobin, "Estimation of Relationships for Limited Dependent Variables," *Econometrica*, Vol. 26, No. 1, January 1958.

While the most obvious examples of this phenomenon are those involving transactions costs, there are many reasons for insensitivity to small changes in the state of the world. Insensitivity of this sort has traditionally been called friction. This, therefore, is a model of relationships in which the dependent variable is subject to friction.

The Model. Let W be a dependent variable subject to friction, and let $X_1, \ldots, X_m$ be a set of independent variables which are related to W. If the effects of the independent variables are assumed to be linear and additive, the relationship can be specified as follows:

$$\text{(1)} \quad \begin{aligned} &\text{Let} && Y_1 = \beta_0' + \sum_{i=1}^{m} \beta_i X_i \\ &\text{and} && Y_2 = \beta_0'' + \sum_{i=1}^{m} \beta_i X_i \qquad \text{where } \beta_0' > \beta_0''. \end{aligned}$$

If deviations (u) from the true relationship are taken to be random and normally distributed, then W is determined by

$$\text{(2)} \quad \begin{aligned} W &= Y_1 - u && (Y_1 - u < L), \\ W &= L && (Y_1 - u > L \text{ and } Y_2 - u < L), \\ W &= Y_2 - u && (Y_2 - u > L). \end{aligned}$$

$P(x)$ is the value of the unit-normal cumulative distribution function, $Q(x) = 1 - P(x)$, and $Z(x)$ is the value of the unit-normal density function.

$$\text{(3)} \quad \begin{aligned} &Pr\,(W > x, x > L \mid X_1, \ldots, X_m, L) = Pr(u < Y_2 - x) = P\left(\frac{Y_2 - x}{\sigma}\right), \\ &Pr(W = L \mid X_1, \ldots, X_m, L) = Pr(Y_1 - u > L \text{ and } Y_2 - u < L) \\ &\quad = Pr(Y_1 - L > u > Y_2 - L) = Q\left(\frac{Y_2 - L}{\sigma}\right) - Q\left(\frac{Y_1 - L}{\sigma}\right), \\ &Pr\,(x < L, x > W \mid X_1, \ldots, X_m, L) = Pr(u > Y_1 - x) = Q\left(\frac{Y_1 - x}{\sigma}\right). \end{aligned}$$

The distribution function of W is

$$\text{(4)} \quad \begin{aligned} F(x; X_1, \ldots, X_m, L) &= Q\left(\frac{Y_1 - x}{\sigma}\right) && (x < L), \\ F(L; X_1, \ldots, X_m, L) &= Q\left(\frac{Y_2 - L}{\sigma}\right) - Q\left(\frac{Y_1 - L}{\sigma}\right), \\ F(x; X_1, \ldots, X_m, L) &= Q\left(\frac{Y_2 - x}{\sigma}\right) && (x > L). \end{aligned}$$

The corresponding density function is

$$\text{(5)} \quad \begin{aligned} f(x; X_1, \ldots, X_m, L) &= \frac{1}{\sigma} Z\left(\frac{Y_1 - x}{\sigma}\right) && (x < L), \\ f(x; X_1, \ldots, X_m, L) &= \frac{1}{\sigma} Z\left(\frac{Y_2 - x}{\sigma}\right) && (x > L). \end{aligned}$$

For ease of exposition and computation, it is convenient to normalize on σ. Let $(a_0', a_0'', a_1, \ldots, a_m, a)$ be estimates of $(\beta_0'/\sigma, \beta_0''/\sigma, \beta_1/\sigma, \ldots, \beta_m/\sigma, 1/\sigma)$.

$$I_1 = a\,Y_1 = a_0' + \sum_{i=1}^{m} a_i X_i, \tag{6}$$
$$I_2 = a\,Y_2 = a_0'' + \sum_{i=1}^{m} a_i X_i.$$

Each observation in a sample consists of a set of values for the independent variables and the dependent variable. The value of L may be different from observation to observation, but it must be known. Since the distribution function has a mass point, the likelihood function $\varphi(a_0', a_0'', a_1, \ldots, a_m, a)$ will be a mixture of densities and probabilities. Assume that a sample contains n observations, and that for p of these observations $W < L$, for q observations $W = L$, and for r observations $W >$ L. Then

$$\varphi(a_0', a_0'', a_1, \ldots, a_m, a) = \prod_{j=1}^{p} aZ(I_{1j} - aW_j) \cdot \tag{7}$$
$$\prod_{k=1}^{q} [Q(I_{2k} - W_k) - Q(I_{1k} - W_k)] \cdot \prod_{l=1}^{r} aZ(I_{2l} - W_l).$$

The natural logarithm of φ is

$$\varphi^*(a_0', a_0'', a_1, \ldots, a_m, a) = -\frac{1}{2} \sum_{j=1}^{p} (I_{1j} - aW_j)^2 +$$
$$\sum_{k=1}^{q} \ln [Q(I_{2k} - W_k) - Q(I_{1k} - W_k)]$$
$$-\frac{1}{2} \sum_{l=1}^{r} (I_{2l} - a\,W_l)^2 + (p + r) \ln a - \frac{p+r}{2} \ln 2\pi.$$

Let $X_0 = 1$ for all observations. Taking the derivatives of φ^* with respect to all parameter estimates gives the following system of $m + 3$ equations:

$$\varphi^*_{a_0'} = -\sum_{j=1}^{p} (I_{1j} - aW_j)\, X_{0j} + \sum_{k=1}^{q} \frac{Z(I_{1k} - aW_k)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)} X_{0k} = 0,$$

$$\varphi^*_{a_0''} = -\sum_{k=1}^{q} \frac{Z(I_{2k} - W_k)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)} X_{0k} - \sum_{l=1}^{r} (I_{2l} - a\,W_l)\, X_{0l} = 0,$$

$$\varphi^*_{a_i} = -\sum_{j=1}^{p} (I_{ij} - aW_j)\, X_{ij} + \sum_{k=1}^{q} \frac{Z(I_{1k} - W_k) - Z(I_{2k} - W_k)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)} X_{ik} \tag{8}$$
$$- \sum_{l=1}^{r} (I_{1l} - aW_l)\, X_{il} = 0 \qquad (i = 1, 2, \ldots, m),$$

$$\varphi^*_{a} = \sum_{j=1}^{p} (I_{1j} - aW_j)\, W_j - \sum_{k=1}^{q} \frac{Z(I_{1k} - aW_k) - Z(I_{2k} - aW_k)}{Q(I_{2k} - aW_k) - Q(I_{1k} - W_k)} W_k$$
$$+ \sum_{l=1}^{r} (I_{2l} - a\,W_l)\, W_l + \frac{p+r}{a} = 0.$$

As is the case of the Tobin model, these equations are nonlinear, and the suggested method of solution is exactly the same as that described by Tobin.

Let A be the vector of parameter estimates, M the matrix of second derivatives of φ^*, and V the vector of first derivatives. An initial trial value of A is obtained, $A^{(0)}$. An approximation to A is obtained by iteration. Each iteration consists of solving the equation

$$M(A^{(i)} - A^{(i-1)}) = -V, \tag{9}$$

where M and V are evaluated at $A^{(i-1)}$. Letting $u_1 = I_{1k} - aW_k$ and $u_2 = I_{2k} - aW_k$, the second derivatives of φ^* are

$$\varphi^*_{a'_0 a''_0} = \sum_{k=1}^{q} \frac{Z(u_1)\,Z(u_2)}{[Q(u_2) - Q(u_1)]^2} X^2_{0k},$$

$$\varphi^*_{a'_0 a_i} = -\sum_{j=1}^{p} X_{ij}X_{0j} - \sum_{k=1}^{q} \frac{u_1 Z(u_1)[Q(u_2) - Q(u_1)] + Z(u_1)[Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{0k}X_{ik},$$

$$\varphi^*_{a''_0 a_i} = \sum_{k=1}^{q} \frac{u_2 Z(u_2)[Q(u_2) - Q(u_1)] + Z(u_2)[Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{0k}X_{ij} - \sum_{l=1}^{r} X_{0l}X_{il},$$

$$\varphi^*_{a_i a_t} = -\sum_{j=1}^{p} X_{ij}X_{tj} - \sum_{k=1}^{q} \frac{[u_1 Z(u_1) - u_2 Z(u_2)][Q(u_2) - Q(u_1)] + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} X_{ik}.$$

$$\varphi^*_{a'_0 a} = \sum_{j=1}^{p} X_{0j}W_j + \sum_{k=1}^{q} \frac{u_1 Z(u_1)\,[Q(u_2) - Q(u_1) + Z(u_1)]\,[Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{0k}W_k,$$

$$\varphi^*_{a''_0 a} = -\sum_{k=1}^{q} \frac{u_2 Z(u_2) + Z(u_2)\,[Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{0k}W_k + \sum_{l=1}^{r} X_{0l}W_l,$$

$$\varphi^*_{a_i a} = -\sum_{j=1}^{p} X_{ij}W_j + \sum_{k=1}^{q} \frac{[u_1 Z(u_1) - u_2 Z(u_2)][Q(u_2) - Q(u_1)] + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} X_{ij}I$$
$$- \sum_{l=1}^{r} X_{il}W_l,$$

$$\varphi^*_{aa} = \sum_{j=1}^{q} W_j^2 - \sum_{k=1}^{r} \frac{[u_1 Z(u_1) - u_2 Z(u_2]\,[Q(u_2) - Q(u_1)] + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} W_k^2$$
$$+ \sum_{l=1}^{r} W_l^2 - \frac{p + r}{a^2}.$$

The computation of the first and second derivatives can be accomplished through a trivial modification of a program which has been written for obtaining estimates of the probit-regression model. A linear regression could be used as an initial trial value, A^0. Doubtless, better methods for obtaining initial trial values are available, but this problem has not yet been examined. The significance tests are exactly those described by Tobin. The negative inverse of the matrix of second derivatives evaluated at the maximum of φ^* is the matrix of large sample estimates of the variances and covariances of the parameter estimates, and likelihood ratio tests can be applied to the parameter estimates.

The locus of estimated expected values of W is not the relationship which is estimated, but can be computed as follows:

$$E(W; X_1, \ldots, X_m, L) = \int_{-\infty}^{L} \frac{x}{\sigma} Z\left(\frac{Y_1 - x}{\sigma}\right) dx + L\left[Q\left(\frac{Y_2 - L}{\sigma}\right) - Q\left(\frac{Y_1 - L}{\sigma}\right)\right]$$
$$+ \int_{L}^{\infty} \frac{x}{\sigma} Z\left(\frac{Y_1 - x}{\sigma}\right) dx = Y_1 Q\left(\frac{Y_1 - L}{\sigma}\right) - \sigma Z\left(\frac{Y_1 - L}{\sigma}\right) + L\left[Q\left(\frac{Y_2 - L}{\sigma}\right)\right.$$
$$\left. - Q\left(\frac{Y_1 - L}{\sigma}\right)\right] + Y_2 P\left(\frac{Y_2 - L}{\sigma}\right) + \sigma Z\left(\frac{Y_2 - L}{\sigma}\right)$$

The locus of expected values of W is represented in Figure 2.

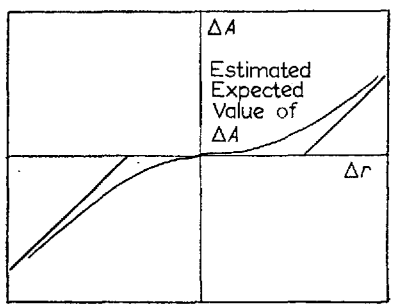

FIGURE 2

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