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ON THE STABILITY OF THE COMPETITIVE EQUILIBRIUM, II

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This paper is a sequel to "On the Stability of the Competitive Equilibrium, I," by K. J. Arrow and L. Hurwicz. It extends the results of "I" in several directions. In particular, it provides a proof of stability in the large (and not merely locally) when all goods are gross substitutes; this result is found to be valid for processes where the price adjustment rate is a continuous sign-preserving, but not necessarily proportionate, function of excess demand. The paper deals both with systems where one of the commodities plays the role of numéraire and with systems where all commodities are treated symmetrically.

INTRODUCTION

IN THIS paper we present several extensions of the results on the stability of the competitive equilibrium contained in [**2**].

One such extension establishes the stability in the large, when substitution prevails in the system, for commodity spaces of arbitrary finite dimensionality; in [**2**], the results obtained for the case of substitutability were local, or limited to spaces of low dimensionality, or valid for special initial positions only.

Horizons are also broadened with regard to the adjustment processes considered. In [**2**], only processes with a numéraire commodity singled out (its price fixed throughout the process) were treated (the "normalized process"). Here we obtain in virtually every case parallel results for the "non-normalized process" where all commodities are treated symmetrically and there is no numéraire.

Furthermore, in some cases (including that of substitutability) we find it possible to relax the assumptions concerning the dependence of the rate of change in prices on the excess demand for the relevant commodity: instead of postulating simple proportionality, we only require that the rate of price change be a single-valued sign-preserving function of the excess demand. This is an interesting example of a situation where only the "qualitative" features of the dynamic process are of importance.

Finally, we prove that competitive equilibrium is stable in a class of cases ("dominant diagonal") where the demand for each particular commodity is more sensitive (in a sense to be specified) to a change in the price of that commodity itself than it is to a price change in any other commodity.

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1. THE MODEL (STATICS)

We assume that there are $m+1$ commodities, labeled $0,1,\ldots,m$. The (non-normalized) price of the kth commodity is denoted by P_k and is always assumed to be nonnegative, i.e.,

$$P_k \geqq 0 \qquad (k = 0,1,\ldots,m). \tag{1}$$

The excess demand (= demand — supply) of the ith individual ($i = 1,2,\ldots,n$) for the kth commodity is denoted by x_k^i. The budget constraint for the ith individual is written in the form

$$\sum_{k=0}^{m} P_k x_k^i = 0, \tag{2'}$$

or, in the inner product notation,

$$P \cdot x^i = 0, \tag{2''}$$

where

$$P = (P_0,P_1,\ldots,P_m) \quad \text{and} \quad x^i = (x_0^i, x_1^i, \ldots, x_m^i). \tag{2'''}$$

The budget constraint (2″) does not exclude the possibility that the excess demands for some products with zero prices may be infinite.

The aggregate excess demand for the kth commodity is defined as

$$x_k = \sum_{i=1}^{n} x_k^i. \tag{3}$$

The summation of the budget constraints (2) for all individuals yields the so called *Walras Law* (W):

$$\sum_{k=0}^{m} P_k x_k \equiv P \cdot x = 0,$$

where

$$x = (x_0,x_1,\ldots,x_m). \tag{4''}$$

Each individual's excess demand is a function of the prices, written

$$x_k^i = F_k^i(P). \tag{5}$$

It is frequently assumed that the F_k^i are positively homogeneous of degree zero in P, so that

$$F_k^i(\lambda P) = F_k^i(P) \qquad \text{for any } \lambda > 0. \tag{6}$$

(The homogeneity property can be derived from the assumption that the individual is maximizing his satisfaction (utility) subject to the budget constraint (2).)

The aggregate excess demand function is defined by

$$F_k(P) = \sum_{i=1}^{n} F_k^i(P). \tag{7}$$

If the individual excess demand functions are positively homogeneous of degree zero, then so is the aggregate function, and we have

(8) $$F_k(\lambda P) = F_k(P) \qquad \text{for any} \quad \lambda > 0.$$

The aggregate excess demand function vector

(9) $$F = (F_0, F_1, \ldots, F_m)$$

is said to possess an equilibrium price (vector) $\bar{P}$ if the relations

(10) $$F(\bar{P}) \leqq 0, \bar{P} \geq 0, \bar{P}_k = 0 \quad \text{if} \quad F_k(\bar{P}) < 0,$$

are satisfied.[2]

It is known from recent contributions[3] that, with certain additional assumptions concerning the nature of the economy, F will possess an equilibrium price vector if every component of F is a continuous function single-valued except at the origin; in what follows we shall assume that F has the latter two properties (so that the differential equation systems being studied are meaningful and possess solutions) and that at least one equilibrium price vector exists. If positive homogeneity is assumed, any positive (scalar) multiple of an equilibrium price vector is also an equilibrium price vector.

In the present paper we shall confine attention to the case where the equilibrium is not characterized by any "corner" components. [k is a "corner" component if $F_k(\bar{P}) < 0, \bar{P}_k = 0$]. Then (10) is replaced by

(10′) $$F(\bar{P}) = 0, \ \bar{P} \geq 0.$$

It is sometimes convenient to single out one commodity, say that bearing the subscript 0, as the "numéraire" in terms of which other prices are expressed. Assuming $P_0 > 0$, we write

(11′) $$p_k = P_k/P_0 \qquad (k = 0,1, \ldots, m)$$

and

(11″) $$p = (p_1, p_2, \ldots, p_m).$$

p and its components are referred to as *normalized*, while P and its components are called *non-normalized*.

When the F_k are assumed positively homogeneous in P and $P_0 > 0$, we have

(12) $$F_k(P) = F_k(P_0, P_1, \ldots, P_m) = F_k(1, p_1, \ldots, p_m).$$

[2] We follow the usual conventions for vector inequalities. For a vector v, $v \geqq 0$ means that all components of v are nonnegative; $v \geq 0$ means $v \geqq 0$ and $v \neq 0$ (i.e., at least one component of v is positive and none is negative); $v > 0$ means that every component of v is positive.

[3] See [1], [4], [7], [8], [11], [13].

We shall write

(13′) $$f_k(p) = f_k(p_1, \ldots, p_m) = F_k(1, p_1, \ldots, p_m)$$

and

(13″) $$\bar{p}_k = \bar{P}_k / \bar{P}_0$$

for $k = 0, 1, \ldots, m$.

2. THE MODEL (DYNAMICS)

We shall consider two classes of price adjustment processes: (I) non-normalized and (II) normalized.

The *non-normalized* process (I) is governed by the differential equation system (with the symbols P, F as defined in the preceding section)

(1) $$dP_k/dt = H_k[F_k(P)] \qquad (k = 0, 1, \ldots, m)$$

where t denotes time. A *solution of* (1) *through* P^o is an $m + 1$ dimensional function $\Psi^I(t; P^o)$ of time such that

(2.1) $$\Psi^I(0; P^o) = P^o$$

and the kth component Ψ_k^I of Ψ^I satisfies the identity

(2.2) $$d\Psi_k^I/dt = H_k\{F_k[\Psi^I(t; P^o)]\} \qquad \text{for all } t \geqq 0.$$

The normalized process (II) is governed by the differential equation system (with the symbols p, f as defined in the preceding section)

(3) $$dp_j/dt = h_j[f_j(p)] \qquad (j = 1, 2, \ldots, m).$$

A *solution of* (3) *through* p^o is an m-dimensional function $\psi^{II}(t; p^o)$ of time such that

(4.1) $$\psi^{II}(0; p^o) = p^o$$

and the jth component ψ_j^{II} of ψ^{II} satisfies the identity

(4.2) $$d\psi_j^{II}/dt = h_j\{f_j[\psi^{II}(t; p^o)]\} \qquad \text{for all } t \geqq 0.$$

The functions H_k and h_j are always assumed to be continuous and *sign-preserving*. (A function g is said to be sign-preserving if $\text{sgn}(g(s)) = \text{sgn}\, s$.)

A differential *system* is said to be *stable* (*in the large*) if, given any initial value, every solution of the system through that value converges to some equilibrium point of the system.

It may be helpful to point out that one can study the behavior of the normalized prices under the non-normalized process and vice versa. Thus, under the non-normalized process (I), we have from the definition of the normalized price

$$p_j = \frac{\Psi_j^I(t; P^o)}{\Psi_0^I(t; P^o)} \equiv \psi_j^I(t; p^o)$$

where $$p^o = (P_1^o/P_0^o, \ldots, P_m^o/P_0^o).$$

On the other hand, we may "embed" (3) in an expanded differential system

$$\text{(3')} \qquad \begin{cases} d(P_j/P_0)/dt = h_j[f_j(P_1/P_0, \ldots, P_m/P_0)], \\ dP_0/dt = 0, \end{cases}$$

which yields the same behavior of $p(t)$ as (3), but also defines the behavior of $P(t)$, to be written, say, as $\Psi^{II}(t;P^o)$.

We note that, in general, Ψ^{II} is different from Ψ^{I}, even though they both describe the behavior of the non-normalized price vector P. Similarly, ψ^{II} is, in general, different from ψ^{I}. I.e., the behavior of the normalized price vector differs depending on whether process (I) or process (II) is assumed, and the same is true of the behavior of the non-normalized price vector.

In what follows, we shall confine ourselves to the study of the behavior of the non-normalized price vector P under the non-normalized process (Ψ^{I}) and of the normalized price vector p under the normalized process (ψ^{II}).

3. THE CONCEPT OF SUBSTITUTABILITY AND SOME OF ITS STATIC PROPERTIES

The concept of substitutability used here (as also in [**2**]) is "gross" (as in Metzler [**9**] and Mosak [**10**]) rather than "net" (as in Hicks [**5**]). The basic idea of the gross substitutability concept is this: if one commodity price goes up while all other prices remain unchanged, there will be an increase in excess demand for every commodity whose price has remained constant. In symbols, for two (non-normalized) price vectors

$$P' = (P'_0, \ldots, P'_m), \qquad P'' = (P''_0, \ldots, P''_m)$$

and any integer k_0 in $\{0,1,2,\ldots,m\}$, we have the following condition[4] (*gross substitutability, finite increment form*):

$$(\text{S}_\text{F}) \begin{cases} \text{the relations} \begin{bmatrix} P'_r = P''_r & \text{for all } r \neq k_0, \\ P'_{k_0} < P''_{k_0} & \end{bmatrix} \\ \text{imply} \quad F_r(P') < F_r(P'') \qquad \text{for all } r \neq k_0. \end{cases}$$

Now (S_F) as just defined is implied by, but not equivalent to, the condition (*gross substitutability, differential form*):[5]

(S_D) the functions F_k and all the partial derivatives $F_{rk} = \partial F_r/\partial P_k$ $(r,k = 0,1,\ldots,m)$ exist and are continuous (though possibly infinite-valued, and,
for all P, $F_{rs} = \partial F_r/\partial P_s > 0$, for all $r \neq s$ $(r,s = 0,1,\ldots,m)$.

[4] See Wald [**14**, pp. 385-7], where it is shown that, under (S_F) and homogeneity, $F(P') = F(P'') = 0$ implies $P' = \lambda P''$ for some $\lambda > 0$, which means that the normalized equilibrium price vector is unique. See also Gale [**4**, p. 163], and Lemma **4**, below.

[5] Labeled, in a slightly different form, "strong gross substitutability" in [**2**, p. 546, Theorem 10].

The advantage of (S_D) is that, in addition to implying (S_F), it also yields (with the help of the Walras Law) another needed property, viz. the positiveness of the equilibrium price vector. Because of its importance for subsequent developments, we state this as a corollary to

LEMMA 1. *If the excess demand functions are positive homogeneous* (H) *and satisfy the differential form* (S_D) *of gross substitutability, then the excess demand for any free good is positive infinite, provided not all goods are free; i.e.,*

$$\textit{if } P \geq 0 \textit{ and, for some } r \geqq 0,\ P_r = 0, \textit{ then } F_r(P) = +\infty.$$

PROOF: For any price vector P, whether or not it has zero components, and any component r, define

$$g(p_1,\ldots,p_m) = F_r(1, p_1,\ldots,p_m). \tag{1}$$

By homogeneity (H), for $P_0 > 0$,

$$F_r(P) = F_r(P_0,\ldots,P_m) = g(P_1/P_0),\ldots,P_m/P_0), \tag{2}$$

so that, for $r > 0$,

$$\begin{cases} F_{rj} = g_j/P_0 \quad \text{for} \quad j > 0, \\ F_{r0} = \sum_{s=1}^{m} (-g_s)\,(P_s/P_0^2), \end{cases} \tag{3}$$

and, therefore,

$$F_{r0} = -\left(\sum_{s=1}^{m} P_s F_{rs}\right)/P_0. \tag{4}$$

Let P^o be any point for which some but not all components are zero; without loss of generality we suppose $P_0^o > 0$. Let

$$A = \{s\colon P_s^o = 0\},\ B = \{s\colon P_s^o > 0,\ s > 0\}.$$

Define a variable point P^t, $t \geqq 0$, by the conditions

$$P_s^t = t \quad \text{for} \quad s\varepsilon A, \quad P_s^t = P_s^o \text{ otherwise}. \tag{5}$$

Choose any fixed r in A; then by the assumption (S_D), $F_{r0} > 0$, $F_{rs} > 0$ for $S\varepsilon B$. With the aid of (3) and (4), we have, for $P = P^t$, $t \geqq 0$,

$$\sum_{s\varepsilon A} P_s^t F_{rs} < -\sum_{s\varepsilon B} P_s^t F_{rs} < 0;$$

hence, because of (5),

$$t \sum_{s\varepsilon A} F_{rs} < 0. \tag{6}$$

The left-hand side is clearly $t[dF_r(P^t)/dt]$. Since the inequality holds for all $t \geqq 0$ and since the partial derivatives are all continuous, we can find c and t_0 such that,

$$t[dF_r(P^t)/dt] < c < 0 \quad \text{for} \quad 0 \leqq t \leqq t_0. \tag{7}$$

Divide through in (7) by t and integrate from ε to t_0. After rearrangement, we have

$$F_r(P^\varepsilon) > F_r(P^{t_0}) + c \log \varepsilon - c \log t_0. \tag{8}$$

If we now let ε approach zero and recall that $c < 0$, we see that $F_r(P^o) = +\infty$, Q.E.D.

COROLLARY TO LEMMA 1.[6] *Under the conditions of Lemma* 1, *if* $\bar{P}$ *is an equilibrium price vector, then* $\bar{P} > 0$, *and* $F(\bar{P}) = 0$.

PROOF: If $\bar{P}$ is an equilibrium price vector, then by definition (10), section 1, at least one component is positive. If some component $\bar{P}_r$ is zero, then $F_r(\bar{P}) = +\infty > 0$ by Lemma 1, contrary to (10), Section 2.

In deriving Theorem 2 below we find it convenient to use a condition of substitutability, to be denoted by (S_F') which is shown to be equivalent to (S_F) in

LEMMA 2. *The condition*

$$(S_F') \begin{cases} \text{the relations } P' \leq P'',\ P'_r = P''_r \text{ for all } r \,\varepsilon\, R \subset \{0,1,\ldots,m\} \\ \text{imply} \quad F_r(P') < F_r(P'') \quad \text{for all } r \,\varepsilon\, R, \end{cases}$$

is equivalent to the condition (S_F) *as defined above.*

PROOF. That (S_F') implies (S_F) is seen by taking $R = \{0,1,\ldots,m\} - k_o$. To see that ($S_F$) implies ($S_F'$), let

$$P''_k = P'_k + h_k$$

with

$$h_k > 0 \text{ for } k \,\varepsilon\, \{k_1,k_2,\ldots,k_v\} = \{0,1,\ldots,m\} - R.$$

Consider the sequence of price vectors defined by

$$\begin{cases} P^o = P', \\ \ldots \\ P^1 = P^o + (0,\ldots,0,h_{k_1},0,\ldots,0), \\ P^s = P^{s-1} + (0,\ldots,0,h_{k_s},0,\ldots,0), \qquad s = 1,2,\ldots,v, \\ \ldots \\ P^v = P''. \end{cases}$$

By (S_F), $\qquad F_w(P^s) > F_w(P^{s-1})$ for $w \neq k_s$,

and, since R and $\{k_1,\ldots,k_v\}$ are disjoint, we have, for any $r \,\varepsilon\, R$,

$$F_r(P'') = F_r(P^v) > F_r(P^{v-1}) > \ldots > F_r(P') > F_r(P^o) = F_r(P'),$$

and the conclusion of (S_F') follows.

[6] The authors were led to a reformulation of an earlier version of this Lemma as a result of correspondence with F. H. Hahn of the University of Birmingham.

In deriving Theorem 1 we use yet another condition related to substitutability (meaningful only when $\bar{P} > 0$):

$$(\mathrm{S}^*)\quad \begin{cases} \text{if } F(P) \neq 0,\ F(\bar{P}) = 0,\ P_{K'}/\bar{P}_{K'} = \max\limits_{k \varepsilon \{0,1,\ldots,m\}} (P_k/\bar{P}_k), \\ \text{and } P_{K''}/\bar{P}_{K''} = \min\limits_{k \varepsilon \{0,1,\ldots,m\}} (P_k/\bar{P}_k), \\ \text{then } F_{K'}(P) < 0 \text{ and } F_{K''}(P) > 0. \end{cases}$$

Since the proof of stability of equilibrium in Theorem 1 uses (S*) as its hypothesis, it becomes crucial to know under what conditions (S*) holds. This question is answered by

LEMMA 3. *Substitutability* ($\mathrm{S_F}'$), *together with positiveness of the equilibrium price vector* (E^+) *and positive homogeneity* (H) *of the excess demand functions, implies condition* (S*), *provided* $P > 0$.[7,8]

PROOF. Let $F(P) \neq 0$, $F(\bar{P}) = 0$, and $P_{K'}/\bar{P}_{K'} \geqq P_k/\bar{P}_k$ for all k in $\{0,1,\ldots,m\}$. We must have

$$P_{K'}/\bar{P}_{K'} > P_{k_o}/\bar{P}_{k_o} \quad \text{for some } k_o,$$

since otherwise

$$P = \lambda\bar{P} \quad \text{for some } \lambda > 0,$$

and hence (by (H)), $F(P) = 0$, which contradicts the hypothesis. Since $P_{K'} > 0$, we can define

$$P^* = (\bar{P}_{K'}/P_{K'})\, P.$$

Then, by hypothesis, $P^* \leq \bar{P}$ and $P^*_{K'} = \bar{P}_{K'}$. Hence, by (H) and ($\mathrm{S_F}'$).

$$F_{K'}(P) = F_{K'}(P^*) < F_{K'}(\bar{P}) = 0.$$

Similarly, with K'' as in (S*), we define

$$P^{**} = (\bar{P}_{K''}/P_{K''})\, P$$

and obtain $\bar{P} \leq P^{**}$, $\bar{P}_{K''} = P^{**}_{K''}$, so that, by (H) and ($\mathrm{S_F}'$),

$$0 = F_{K''}(\bar{P}) < F_{K''}(P^{**}) = F_{K''}(P).$$

Since the uniqueness of the "equilibrium price ray" is used in the proofs of Theorems 1 and 2, it is of interest to see that

LEMMA 4. (S*) *and* (H) *imply uniqueness of the equilibrium price ray, i.e.*,

(U) *if* $F(\bar{P}'') = F(\bar{P}') = 0$, *then* $\bar{P}'' = \lambda\bar{P}'$ *for some* $\lambda > 0$.

[7] Surprisingly enough, the Walras Law is not used. However, one may regard (W) as the rationale underlying the formulation of ($\mathrm{S_F}$).

[8] Lemma 3 shows that (given (E^+) and (H)) (S*) is no stronger than ($\mathrm{S_F}'$). In fact, (S*) is weaker. This can be seen when the functions $F_K(P)$ are replaced by $H_k[F_k(P)]$ where the H_k are sign-preserving but not monotone; in this case (S*) still holds, but ($\mathrm{S_F}'$) can easily be violated.

PROOF. We follow Wald [**14**, pp. 385-7]. Suppose $F(\bar{P}') = F(\bar{P}'') = 0$, and let $\bar{P}_K'/\bar{P}_K'' = \min_{k\varepsilon\{0,1,\ldots,m\}} (\bar{P}_k'/\bar{P}_k'')$. By (H), we may replace $\bar{P}''$ by $\bar{P}'''$ such that $\bar{P}''' = \mu\bar{P}''$, $\mu > 0$, and $\bar{P}_K' = \bar{P}_K'''$. If $\bar{P} \neq \lambda\,\bar{P}''$ for all $\lambda > 0$, $\bar{P}''' \leq \bar{P}'$, and hence by (S*), $F_K(\bar{P}''') > 0$, which contradicts the assumption that $\bar{P}''$ (and, therefore, $\bar{P}'''$) is an equilibrium price vector.

3.1. In this section we shall establish another (static) result, to be used in the proof of Theorem 2 in section 4.2. It is included here because of its static nature, but it is not used in the proof of Theorem 1 of section 4.1 and its reading may be postponed until after section 4.1 is completed.

LEMMA 5. *If the equilibrium vector* $\bar{P} > 0$ *and gross substitutability* (S_F') *prevails and the Walras Law* (W) *together with positive homogeneity* (H) *hold, then, for any non-equilibrium vector* $P > 0$, *we have*

(1) $$\bar{P}\cdot F(P) \equiv \sum_{k=0}^{m} \bar{P}_k F_k(P) > 0.$$ [9]

3.1.1. *Proof of Lemma* 5.

3.1.1.0. Suppose the Lemma has been shown to hold for the special case where,[10]

$$\bar{P} = \lambda\mathbf{1} \equiv (\lambda,\lambda,\ldots,\lambda), \qquad \lambda > 0,$$

so that

(2) $$F(P) \neq 0 \quad \text{implies} \quad \sum_{k=0}^{m} F_k(P) > 0.$$

We shall show that this implies the validity of the Lemma in the general case. This is accomplished by showing that the general case can be reduced to the special. Let asterisks denote the entities in the general case and transform the variables by changing to new units of measurement in such a way that in the new units of measurement (entities without asterisks) $P_k = P_k^*/\bar{P}_k^*$, so that $\bar{P} = \lambda\mathbf{1}$, $\bar{P}_k F_k(P) = \bar{P}_k^* F_k^*(P^*)$ and $\bar{P}\cdot F(P) = \bar{P}^*\cdot F^*(P^*)$. But then $\bar{P}^*\cdot F^*(P^*) = \lambda \sum_{k=0}^{m} F_k(P)$ and the latter expression is positive by (2) if $F^*(P^*) \neq 0$. Hence it will suffice to prove (2) and the Lemma will follow.

3.1.1.1. We now assume that the equilibrium vector is of the form $(\lambda,\lambda,\ldots,\lambda)$ and it is our purpose to show that, for a positive non-equilibrium P,

(3) $$\mathbf{1}\cdot F(P) \neq \sum_k F_k(P) > 0$$

[9] The relation (1) may be interpreted as stating that, under gross substitutability, the weak revealed preference axiom holds for any pair of price vectors, one of which is an equilibrium price vector.

[10] $\mathbf{1} = (1,1,\ldots,1)$.

Without loss of generality we may so number the commodities that

(4) $$P_0 \leqq P_1 \leqq \ldots \leqq P_m.$$

Furthermore, since P is non-equilibrium, it cannot have all components equal (by (H)), and hence one of the inequalities in (4) must be strict, say

(5) $$P_v < P_{v+1} \quad \text{for some } 0 \leqq v < m.$$

We now define a sequence of $m + 1$ dimensional vectors $P^o, P^1, \ldots, P^m$ by the relations

(6) $$\begin{cases} P^o = (P_0,P_0,P_0, \ldots,P_0), \\ P^1 = (P_0,P_1,P_1, \ldots,P_1), \\ P^2 = (P_0,P_1,P_2, \ldots,P_2), \\ \ldots \\ P^s = (P_0,P_1, \ldots,P_{s-1},P_s,P_s, \ldots,P_s), \\ \ldots \\ P^m = (P_0,P_1, \ldots,P_m), \end{cases}$$

where the P_j are the components of P, so that

(7) $$P^m = P.$$

The excess demand vector corresponding to P^s will be denoted by x^s, i.e.,

(8) $$x^s = F(P^s).$$

In particular, since P^o has equal components,

(9) $$x^o = F(P^o) = 0.$$

The inequality (3) we are about to prove may be rewritten as

(10) $$\mathbf{1} \cdot x^m > 0.$$

Now

(11) $$x^m = (x^m - x^{m-1}) + (x^{m-1} - x^{m-2}) + \ldots + (x^2 - x^1) + (x^1 - x^o) + x^o,$$

and, because of (9), the last term on the right drops out. Hence (10) will have been proved if we show that the sum of components of every difference in parentheses on the right of (11) is nonnegative and at least one is positive. Specifically, we shall show that

(12) $$\mathbf{1} \cdot (x^{s+1} - x^s) \geqq 0 \quad \text{for } s = 0,1, \ldots, m-1$$

and $$\mathbf{1} \cdot (x^{v+1} - x^v) > 0.$$

Before proceeding with the proof, we introduce some additional notation. We define

(13) $$\begin{cases} b_o = 1, \\ b_j = (1/P_0)\,(P_j - P_{j-1}), \end{cases}$$

so that

(14) $$\begin{cases} b_k \geqq 0 \ \text{for}\ k = 0,1, \ldots,m, \ \text{and} \\ b_{v+1} > 0. \end{cases}$$

For $s<m$, let k' range over $\{0,1,\ldots,s\}$ and k'' over $\{s+1,s+2,\ldots,m\}$.[11] We then have

$$P_{k'}^{s+1} = P_{k'}^{s}, \tag{15.1}$$

$$P_{k''}^{s+1} = P_{k''}^{s} + P_0 b_{s+1} = P_m^{s+1}, \tag{15.2}$$

$$P_{k'}^{s} \leqq P_{k''}^{s} = P_m^{s}. \tag{15.3}$$

Now because of zero degree homogeneity (H) of the excess demand functions,

$$x^{s+1} = F(P^{s+1}) = F(Q^{s+1}) \tag{16}$$

where

$$Q^{s+1} = [(1 + b_1 + \ldots + b_s)/(1 + b_1 + \ldots + b_s + b_{s+1})]P^{s+1}. \tag{17}$$

We then have

$$Q_{k''}^{s+1} = P_{k''}^{s} \tag{18.1}$$

while

$$Q_{k'}^{s+1} \leqq P_{k'}^{s}; \tag{18.2}$$

moreover,

$$Q_{w'}^{v+1} < P_{w'}^{v} \quad \text{for } w' \,\varepsilon\, \{0,1,\ldots,v)\}. \tag{18.3}$$

It then follows from the assumption of substitutability (S_F') that

$$x_{k''}^{s+1} \leqq x_{k''}^{s} \quad (\text{so that} \quad x_{k''}^{s+1} \leqq x_{k''}^{s} \leqq \ldots \leqq x_{k''}^{0} = 0) \tag{19.1}$$

and

$$x_{w''}^{v+1} < x_{w''}^{v} \quad (\text{so that} \quad x_{w''}^{v+1} < x_{w''}^{v} \leqq \ldots \leqq x_{w''}^{0} = 0 \quad \text{for } w'' \,\varepsilon\, \{v+1, v+2, \ldots, m\}). \tag{19.2}$$

Also, since $P_{k''}^{s+1} \geqq P_{k''}^{s}$, while $P_{k'}^{s+1} = P_{k'}^{s}$ substitutability implies

$$x_{k'}^{s+1} \geqq x_{k'}^{s}. \tag{19.3}$$

Write now, for $s < m$,

$$P_* = (P_0, \ldots, P_s), P_{**} = (P_{s+1}, \ldots, P_m) \tag{20}$$

with the corresponding symbols for the x's. (This partitioning depends on s, even though the dependence is not shown by the symbols.) From (6), (13) and (15) it then follows that

[11] The ranges of k' and k'' depend on s, but this dependence is not shown in the symbols.

(21.1) $$P_{*}^{s+1} = P_{*}^{s},$$

(21.2) $$P_{**}^{s+1} = P_{**}^{s} + P_0\, b_{s+1}\, \mathbf{1} = P_{m}^{s+1}\, \mathbf{1}, \qquad \mathbf{1} = (1,1,\ldots,1).$$

Now consider the expression

(22) $$D = P^{s+1}\cdot x^{s+1} - P^{s}\cdot x^{s}.$$

By the Walras Law both inner products in (22) vanish, hence

(23) $$D = 0.$$

Therefore, partitioning according to (20) and using (22) and (23), we have

(24) $$\begin{aligned} 0 = D &= P_{*}^{s+1}\cdot x_{*}^{s+1} + P_{**}^{s+1}\cdot x_{**}^{s+1} - P_{*}^{s}\cdot x_{*}^{s} - P_{**}^{s}\cdot x_{**}^{s} \\ &= P_{*}^{s}\cdot(x_{*}^{s+1} - x_{*}^{s}) + P_{**}^{s+1}\cdot x_{**}^{s+1} - P_{**}^{s}\cdot x_{**}^{s} \ [\text{(by 21.1)}] \\ &= P_{*}^{s}\cdot(x_{*}^{s+1} - x_{*}^{s}) + P_{**}^{s+1}\cdot x_{**}^{s+1} - (P_{**}^{s+1} - P_0 b_{s+1}\, \mathbf{1})\cdot x_{**}^{s} \\ &\qquad [\text{by } 21.2)] \\ &= P_{*}^{s}\cdot(x_{*}^{s+1} - x_{*}^{s}) + P_{**}^{s+1}\cdot(x_{**}^{s+1} - x_{**}^{s}) + P_0 b_{s+1}\, (\mathbf{1}\cdot x_{**}^{s}) \\ &= P_{*}^{s}\cdot(x_{*}^{s+1} - x_{*}^{s}) + P_{m}^{s+1}[\mathbf{1}\cdot(x_{**}^{s+1} - x_{**}^{s})] + P_0 b_{s+1}\, (\mathbf{1}\cdot x_{**}^{s}) \\ &\qquad [\text{by } (21.2)] \\ &\leqq P_{m}^{s+1}[\mathbf{1}\cdot(x_{*}^{s+1} - x_{*}^{s})] + P_{m}^{s+1}[\mathbf{1}\cdot(x_{**}^{s+1} - x_{**}^{s})] + P_0 b_{s+1}(\mathbf{1}\cdot x_{**}^{s}) \\ &\qquad [\text{by } (15.1)\ (15.3)\ (19.3)] \\ &= P_{m}^{s+1}[\mathbf{1}\cdot(x^{s+1} - x^{s})] + P_0 b_{s+1}(\mathbf{1}\cdot x_{**}^{s}). \end{aligned}$$

This may be written as

(25) $$\mathbf{1}\cdot(x^{s+1} - x^{s}) \geqq -(P_0/P_{m}^{s+1})\, b_{s+1}(\mathbf{1}\cdot x_{**}^{s}).$$

It follows that the left member of (25) is nonnegative, since the prices are positive, $b_{s+1} \geqq 0$ by (14) and $(\mathbf{1}\cdot x_{**}^{s}) \leqq 0$ by (19.1). Furthermore, $b_{v+1} > 0$, and $(\mathbf{1}\cdot x_{**}^{v}) < 0$ by (19.2). Hence (12) follows and the proof is complete.

4. STABILITY OF EQUILIBRIUM UNDER SUBSTITUTABILITY

4.0. As seen from Lemma 4, if a positive equilibrium price vector $\bar{P}$ exists and gross substitutability (even in its weakest form (S*)) is assumed, it follows that all other equilibrium price vectors are positive scalar multiples of $\bar{P}$. We express this geometrically by saying that there is a unique "equilibrium ray" $E = \{\lambda \bar{P} \colon \lambda > 0\}$. It follows that the normalized equilibrium price vector $\bar{p} = (\bar{P}_1/\bar{P}_0, \ldots, \bar{P}_m/\bar{P}_0)$ is also unique.

To establish the stability of the normalized process (II),[12] we shall show

[12] Because it has a unique equilibrium point, its stability properties are somewhat simpler to discuss. Therefore we start with this case.

that the distance $D(t)$ from the variable point $p(t) = \psi^{II}(t;p^o)$ to the (unique) normalized equilibrium price vector $\bar{p}$ tends to zero as time tends to infinity. Where this distance has a continuous time derivative $\dot{D}(t)$, the convergence $p(t) \rightarrow \bar{p}$ is established by showing that $\dot{D}(t) < 0$ unless $f[p(t)] = 0$, i.e., when $p(t)$ is the equilibrium vector.

In the non-normalized process (I) we proceed in a similar fashion, except that here, because of the non-uniqueness of the equilibrium vector, we consider the distance from the variable point $P(t) = \Psi^I(t;P^o)$ to *any* (arbitrarily selected) equilibrium price vector $\bar{P}$. Where this distance has a continuous time derivative $\dot{D}(t)$, we show that $\dot{D}(t) < 0$ unless $F[P(t)] = 0$, i.e., unless $P(t)$ is an equilibrium price vector (possibly different from $\bar{P}$), and it follows that $P(t) \rightarrow \bar{P}'$ where $\bar{P}'$ is some equilibrium price vector.

Interestingly enough, there is some latitude in the selection of the norm (metric) in terms of which distance is measured. If we use the Euclidean[13] norm $\| \ \|_2$ and postulate (H), (W), (E^+) and (S_F'), we find that the (Euclidean) distance function $D_2(t)$ does have a continuous derivative for all $t \geqq 0$ in the class of non-normalized differential processes given by

$$\text{(I}'\text{)} \qquad \dot{P}_k = A_k F_k(P), \qquad A_k > 0,$$

as well as in the class of normalized differential processes given by

$$\text{(II}'\text{)} \qquad \dot{p}_j = a_j f_j(p), \qquad a_j > 0.$$

Because of the continuity of $\dot{D}_2$ it is then relatively simple to prove convergence (hence stability of equilibrium) from the fact that $\dot{D}_2 = -\bar{P}\cdot F(P) < 0$ (by Lemma 5) always. Furthermore $\dot{D}_2 < 0$ means that the convergence is *monotone*, i.e., the Euclidean distance from $p(t)$ to $\bar{p}$ decreases throughout the process.

On the other hand, we may use the "maximum norm" $\| \ \|_M$, where the length of a vector equals the largest of the absolute values of its components.[14] The corresponding distance is denoted by D_M.

We assume[15] (H), (E^+), and (S^*) and consider the processes[16] given by

(I) $\dot{P}_k = H_k[F_k(P)]$, where H_k is continuous and sign-preserving,

(II) $\dot{p}_j = h_j[f_j(p)]$, where h_j is continuous and sign-preserving.

[13] Actually a "weighted" Euclidean distance is used:
$\| P \|_2 = [\sum_k P_k^2/A_k]^{1/2}$, where the A_k's are the constants in (I′) below.

[14] $\| P \|_M = \max_k | P_k |$.

[15] These assumptions are weaker than those for $\| \ \|_2$, since (W) is not used at all, and (S^*) in conjunction with (H) and(E^+) is weaker than (S_F'). See footnote 8 above.

[16] These are, of course, respectively more general than (I′) and (II′).

Here it is not true that $\dot{D}_M$ always exists, but when it does we have, for the non-normalized process $\dot{D}_M = -|H_K[F_K(P)]/\bar{P}_K|$ where K is such that

$$\left|\frac{P_K}{\bar{P}_K} - 1\right| = \max_k \left|\frac{P_k}{\bar{P}_k} - 1\right|,$$

with an analogous formula for the normalized process. By Lemma 3, $\dot{D}_M < 0$. With some complications due to the fact that $\dot{D}_M$ does not exist everywhere, we again are able to conclude convergence and also find this convergence to be monotone in the $\| \|_M$ norm, i.e., the distance to equilibrium decreases throughout the process.

Since the $\| \|_M$-approach uses weaker assumptions and applies to a broader class of processes than the $\| \|_2$-approach, it is important to realize that this does not make the Euclidean approach superfluous, since monotonicity of convergence in the $\| \|_M$ norm does not imply monotonicity of convergence in $\| \|$.[17]

4.1. THEOREM 1. (Stability of equilibrium under substitutability; monotone convergence in the maximum norm). *Assume that the excess demand functions* $F_k(P)$ *are single-valued, continuous, and positively homogeneous* (H) *of degree zero; assume further that there exist positive equilibrium price vectors* ($\bar{P} > 0$) *and that substitutability* (S*) *prevails.*[18]

Then, for $P^o > 0$,

(a_1) *every*[19] *solution through* P^o *of the non-normalized process* (I) *with the functions* H_k *continuous and sign-preserving, i.e., every* $\Psi^I(t;P^o)$, *converges to some equilibrium price vector* $\bar{P}$, *and*

(a_2) *the convergence of* $\Psi^I(t;P^o)$ *is strictly monotone in the norm* $\| \|_M$ *defined below;*

and for $p_r^o > 0$,

(b_1) *every*[19] *solution through* p^o *of the normalized process* (II) *with the functions* h_j *continuous and sign-preserving, i.e., every* $\psi^{II}(t;p^o)$, *converges to the* (*unique*) *normalized equilibrium price vector* $\bar{p}$, *and*

(b_2) *the convergence of* $\psi^{II}(t;p^o)$ *is strictly monotone in the norm* $\| \|_M$ *defined below.*

($\Psi^I(t;P^o)$ *and* $\psi^{II}(t;p^o)$ *are respectively solutions of* (1) *and* (3) *in section* 2).

OUTLINE OF PROOF. Assertions (a_1), (a_2) are proved in 4.1.2; their proof is based on Lemma 6 in 4.1.1. Assertions (b_1) and (b_2) are proved in 4.1.4 on the basis of Lemma 7 given in 4.1.3.

17 Most stability proofs in [2] were carried out in terms of $\| \|_2$. On the other hand, the method used in [2] for proving stability in the large for $m = 2$ under substitutability has a close relationship to the approach using $\| \|_M$.

18 We recall that, for instance, under homogeneity, the differential form of substitutability (S_D) implies both (S*) and $\bar{P} > 0$.

19 We do not assume or assert the uniqueness of these solutions, thus relaxing the conditions under which some of the results in [2] had been obtained.

4.1.1. LEMMA 6.[20,21] *Let* $Q = (Q_0, Q_1, \ldots, Q_m)$ *denote an* $m+1$*-tuple of real numbers. For* $k = 0, 1, \ldots, m$, *let* G_k *be a continuous real-valued function defined on the space of positive Q's (i.e., Q's whose every component is positive). It is assumed that the functions* G_k *satisfy Condition* (S):

$$\text{Condition (S):} \quad \begin{cases} \text{if } Q \text{ is such that } G(Q) \neq 0 \text{ and } Q \neq \lambda \mathbf{1} \text{ for all } \lambda > 0 \\ \text{(where } \mathbf{1} \text{ denotes a vector all of whose components are 1's),} \\ \text{then}^{22}\ Q_{K'} = \max_k Q_k \text{ implies } G_{K'}(Q) < 0 \text{ and} \\ Q_{K''} = \min_k Q_k \text{ implies } G_{K''}(Q) > 0. \end{cases}$$

Then the differential system

(1) $$dQ/dt = G(Q) \quad \text{for } t \geqq 0, \text{ with } Q(0) = Q^o > 0$$

(*A*) *has a solution* $Q(t)$ *for all* $t \geqq 0$ *which*

(*B*) *is contained in the "cube"* $B = \{Q\colon 0 < \alpha \leqq Q_k \leqq \beta \text{ for all } k\}$, *where* $\alpha = \min_k Q_k^o$ *and* $\beta = \max_k Q_k^o$. *Furthermore,*

(*C*) *let* $Q(t)$ *be a solution of* (1) *satisfying the condition* $Q(t) \notin E$ *for all t in the interval* $0 \leqq t' \leqq t \leqq t'' \leqq \infty$, *with E denoting the "equilibrium ray,"* $E = \{\lambda \mathbf{1}\,;\ \lambda > 0\}$, *and, for some fixed positive* $\bar{\lambda}$, *write* $D_M(t) = \max_k |Q_k(t) - \bar{\lambda}|$; *then* $D_M(t)$ *is a continuous strictly decreasing function of t on the closed interval* $[t', t'']$, *and*

(*D*) *as* $t \to \infty$, $Q(t) \to \lambda_o \mathbf{1}$ *for some* $\lambda_o > 0$.

PROOF. In the "cube" $B' = \{Q\colon 0 < \alpha/2 \leqq Q_k \leqq \beta + \alpha/2 \text{ for all } k\} \supset B$ the functions G_k are bounded, say

$$\max_k \max_{Q \varepsilon B'} |G_k(Q)| \leqq M.$$

By the Cauchy-Peano existence Theorem[23] the differential system has, therefore, a solution remaining within B' for $0 \leqq t \leqq \beta/M \equiv \nu$. However, in virtue of condition (S), the solution will actually stay within the smaller cube B. For, by condition (S), $\dot{Q}_{K'} < 0$, so that $Q_{K'}(t) \leqq \beta$ implies $Q_{K'}(t+h) \leqq \beta$ for $h > 0$. The reasoning for $Q_{K''}$ is analogous.

Because the solution stays within B, the initial conditions for the time-interval $(\nu, 2\nu)$ are the same as those for the interval $(0, \nu)$ and the solution can be continued, still within B. In this manner a solution staying in B can be found for all $t \geqq 0$. This establishes assertions (A) and (B) of the Lemma.

[20] In developing the proof of Lemmas 6 and 7 we benefited from a suggestion due to Professor J. Blackman of Syracuse University.

[21] While Lemma 6 is stated abstractly, the reader may find it helpful to think of Q as a non-normalized price vector, $G_k(Q)$ as the excess demand function for the kth commodity, with the units of measurement selected in such a manner that every equilibrium price vector has equal components, i.e., is of the form $(a, a, \ldots, a)$.

[22] It is understood throughout 4.1.1 that k ranges over $\{0, 1, \ldots, m\}$.

[23] See, e.g., Coddington and Levinson [3, Theorem 1.2, p. 6 and remarks on p. 19].

Now let $\bar{\lambda}$ be a fixed positive number and define $D_M(t)$ as in assertion (C) of the Lemma. That $D_M(t)$ is a *continuous* function of t, follows from its definition. To show that it is *strictly decreasing* when $Q(t) \notin E$, we proceed as follows. First, we consider a special relatively simple case where $\max_k |Q_k(t) - \bar{\lambda}|$ is achieved for a unique value K of k. Then we revert to the general case where K need not be unique. (The consideration of the special case may be omitted without invalidating the proof; it is included for expository purposes only.)

When K is unique, then in a sufficiently small neighborhood of t, say $(t - \Delta t, t + \Delta t)$ it remains, by continuity, the unique maximizer, so that $D_M(t)$ has a derivative

$$dD_M/dt = [\operatorname{sgn}(Q_K(t) - \bar{\lambda})]\, dQ_K/dt = [\operatorname{sgn}(Q_K(t) - \bar{\lambda})]G_K. \tag{2}$$

If $Q_K(t) - \bar{\lambda} > 0$, it follows that $Q_K(t)$ is maximal among the $Q_k(t)$ and, by condition (S), $G_K < 0$; therefore, the rightmost member of (2) is negative. When $Q_K(t) - \bar{\lambda} < 0$, we see that $Q_K(t)$ is minimal among the $Q_k(t)$ and, by condition (S), $G_K > 0$; here again the rightmost member of (2) is negative. Since $Q_K(t) - \bar{\lambda} = 0$ is prohibited by the requirement that $Q \notin E$, we have established that when K is unique,

$$dD_M/dt = -|G_K| < 0 \qquad \text{for } Q \text{ not in } E. \tag{3}$$

Now we drop the assumption of a unique K and thus return to the general case, where the existence of a derivative for D_M cannot be assured. Consider the value $Q(t)$ of a solution of (1) and let $\mathscr{K}$ denote the non-empty subset of $\{0,1,\ldots,m\}$ such that the maximum over k in $\{0,1,\ldots,m\}$ of $|Q_k(t) - \bar{\lambda}|$ is achieved for all the members of $\mathscr{K}$ and no others. Then, for $h > 0$,

$$\limsup_{h\to 0} \frac{D_M(t+h) - D_M(t)}{h} \leqq \max_{k\in\mathscr{K}} (-|G_k[Q(t)]|) < 0, \tag{4}$$

since

$$\limsup_{h\to 0} \left\{\max_{k\in\mathscr{K}} \left|\frac{Q_k(t+h) - \bar{\lambda}}{h}\right| - \max_{k\in\mathscr{K}} \left|\frac{Q_k(t) - \bar{\lambda}}{h}\right|\right\}$$

$$\leqq \limsup_{h\to 0} \left\{\max_{k\in\mathscr{K}} \left[\frac{|Q_k(t+h) - \bar{\lambda}| - |Q_k(t) - \bar{\lambda}|}{h}\right]\right\}$$

$$= \max_{k\in\mathscr{K}} \left\{(\operatorname{sgn}[Q_k(t) - \bar{\lambda}])\frac{dQ_k}{dt}\right\} = \max_{k\in\mathscr{K}} (-|G_k|).$$

By (4), for sufficiently small $h > 0$, $D_M(t+h) < D_M(t)$, and thus assertion (C) of the Lemma is established.

To prove convergence to some point on the "equilibrium ray" E, we consider the path $Q(t)$ as $t\to\infty$. There are two possibilities: either this path stays a finite distance away from E, or it comes arbitrarily close to E. We shall see below that the former supposition leads to a contradiction; hence we must examine the case where $Q(t)$ comes arbitrarily close to E as $t \to \infty$. Here for any positive ε_ν we can find a time-point t_ν such that $\max_k |Q_k(t_\nu) - \overline{\lambda}^\nu| < \varepsilon_\nu$ for some $\overline{\lambda}^\nu$. We note that, by the already established assertion (C) of the Lemma, $\max_k |Q_k(t) - \overline{\lambda}^\nu| < \varepsilon_\nu$ for all $t > t_\nu$. Therefore, by selecting a sequence of numbers $\varepsilon_\nu > 0$, we obtain a corresponding sequence of nested "parallelepipeds" with centers on the segment of $E = \lambda \mathbf{1}$ given by $0 < \alpha \leqq \lambda \leqq \beta$ and diameters tending to zero such that $Q(t)$ must lie in each for all t sufficiently large. The unique common point must obviously lie on E and may be written as, say, $\lambda_o \mathbf{1}$. It is the limit of $Q(t)$ as $t \to \infty$ and thus assertion (D) of the Lemma follows.

It remains to be shown that the path cannot stay a finite distance away from E as $t \to \infty$. If it did, it would remain within a set

$$\tilde{S}_\varepsilon = \{Q: \inf_{\overline{Q} \varepsilon E} \max_k |Q_k - \overline{Q}_k| \geqq \varepsilon\}.$$

Now let us select (arbitrarily) a fixed positive number λ_1, and, for each k in $\{0,1,\ldots,m\}$ define the sets

$$T_k = \{Q: |Q_k - \lambda_1| = \max_{k'} |Q_{k'} - \lambda_1|\},$$
$$H_k = B \cap \tilde{S}_\varepsilon \cap T_k.$$

Note that, for Q not in E,

$$|Q_k - \lambda_1| = \max_{k'} |Q_{k'} - \lambda_1| \quad \text{implies} \quad G_k \neq 0,$$

as shown earlier (following equation (2)[24] with the help of condition (S). It follows that $|G_k| > 0$ on points of T_k that are not in E, and hence $|G_k| > 0$ on H_k. Since $|G_k|$ is continuous on H_k, we may conclude (because of the compactness of H_k) that

$$|G_k| \geqq \delta_k > 0 \quad \text{on} \quad H_k.$$

Now consider the distance from $Q(t)$ to $\lambda_1 \mathbf{1}$ on E. This distance is given by $D_M(t) = \max_{k'} |Q_{k'}(t) - \lambda_1|$. Because it is monotone and continuous (by assertion (C)) it has a time derivative dD_M/dt almost everywhere[25] and, as seen above, for a point of H_k, it is given by $-|G_k|$. Since every point in the intersection $B \cap \tilde{S}_\varepsilon$ belongs to H_k for some k in $\{0,1,\ldots,m\}$, it follows that, where the derivative exists,

$$|dD_M(t)/dt| \geqq \min_k \delta_k \equiv \delta > 0.$$

Using again the fact that D_M is a monotone decreasing continuous function of time we have[25]

[24] The argument given there does not depend on the uniqueness of K.

[25] See, for instance, Kestelman [**6**, Theorem 258, p. 178].

$$D_M(0) \geqq D_M(0) - D_M(t) \geqq \int_0^t [-\frac{d}{dt} D_M(t)]dt = \int_0^t |\frac{d}{dt} D_M(t)|dt \geqq \delta \cdot t$$

which yields a contradiction for t sufficiently large.

4.1.2. *Proof of assertions* (a_1) *and* (a_2) *of Theorem* 1. (Stability and monotone convergence of the non-normalized process under substitutability; maximum norm.)

Consider the non-normalized differential equation system

$$dP_k/dt = H_k[F_k(P)] \qquad (k = 0,1, \ldots,m),$$

with $P^o > 0$, $F(P^o) \neq 0$.

Let $\bar{P}$ be some fixed equilibrium price vector. Since $\bar{P}$ is positive by the hypothesis of the theorem, we may perform the transformation of variables

$$Q_k = P_k/\bar{P}_k \qquad (k = 0,1, \ldots,m).$$

We then obtain a differential equation system

$$\bar{P}_k\dot{Q}_k = H_k[F_k(Q_0\bar{P}_0, Q_1\bar{P}_1, \ldots, Q_m\bar{P}_m)]$$

or

$$(1) \qquad \dot{Q}_k = G_k(Q), \qquad Q_k^0 = P_k^0/\bar{P}_k,$$

where we define

$$G_k(Q) = (1/\bar{P}_k)\, H_k[F_k(Q_0\bar{P}_0, \ldots, Q_m\bar{P}_m)].$$

We shall now verify that the system (1) satisfies the hypotheses of Lemma 6. Clearly $Q^o > 0$ since $P^o > 0$ and $\bar{P} > 0$; also, G_k is continuous since both F_k and H_k are continuous. It remains for us to verify condition (S) of Lemma 6. Now if $Q_{K'} = \max_k Q_k$, we have (by definition of the Q_k)

$$P_{K'}/\bar{P}_{K'} \geqq P_k/\bar{P}_k \qquad \text{for all } k.$$

Now let $G(Q) \neq 0$. Since the H_k are sign-preserving, it follows that $F(P) \neq 0$ also. The substitutability condition (S*) then implies $F_{K'}(P) < 0$, and hence $G_{K'}(Q) = (1/\bar{P}_{K'})H_{K'}[F_{K'}(P)] < 0$ in view of the sign-preserving property of $H_{K'}$. The proof that $Q_{K''} = \min_k Q_k$ implies $G_{K''}(Q) > 0$ is analogous. Hence Lemma 6 applies to the system (1).

Therefore, there exists a positive number λ_o such that $Q(t)$ converges to $\lambda_0\mathbf{1}$ as $t \to \infty$; i.e., for each k,

$$P_k(t)/\bar{P}_k \to \lambda_o \qquad \text{as } t \to \infty,$$

or, equivalently,

$$P(t) \to \lambda_o\bar{P} \qquad \text{as } t \to \infty.$$

By homogeneity (H), $\lambda_o\bar{P}$ is an equilibrium price vector, since $\bar{P}$ is one.

Furthermore, this convergence is, by assertion (C) of Lemma 6, strictly monotone in the norm $||P||_M = \max_k (|P_k|/\bar{P}_k)$ where $\bar{P}$ is any equilibrium price vector. I.e., the distance $D_M(t)$ from $P(t)$ to any equilibrium price vector $\bar{P}$, given by $D_M(t) = \max_k (|[P_k(t) - \bar{P}_k]/\bar{P}_k|)$, decreases, since this decrease is equivalent to a decrease in $\max_k |Q_k(t) - 1|$ implied by Lemma 6.

4.1.3. Lemma 7.

This Lemma is used in the proof of convergence of the normalized process in the same way as Lemma 6 is used in the proof of convergence of the non-normalized process. As in the context of Lemma 6, the reader may find it helpful to have an economic interpretation: q is the normalized (m-dimensional price vector), $g(q)$ the corresponding excess demand function, and the equilibrium price vector is $(1,1,\ldots,1)$.

Lemma 7. *Let $q = (q_1,q_2,\ldots,q_m)$ denote an m-tuple of real numbers. For $j = 1,2,\ldots,m$, let g_j be a continuous real-valued function defined on the space of positive q's (i.e., q's whose every component is positive). It is assumed that the functions g_j satisfy Condition* (s):

$$\begin{cases} \text{If } q \neq \mathbf{1} \ (\textit{where } \mathbf{1} = (1,1,\ldots,1)) & \\ \textit{then}^{26}\ q_{J'} = \max\limits_j q_j \geqq 1 & \textit{implies } g_{J'}(q) < 0; \\ \textit{and} \quad q_{J''} = \min\limits_j q_j \leqq 1 & \textit{implies } g_{J''}(q) > 0. \end{cases}$$

Then the differential system

$$dq/dt = g(q) \quad \textit{for } t \geqq 0,\ q(0) = q^o > 0, \tag{1}$$

(*A*) *has a solution $q(t)$ for all $t \geqq 0$ which*

(*B*) *is contained in the "cube" $A = \{q: 0 < \alpha \leqq q_j \leqq \beta \text{ for all } j\}$ where $\alpha =$* min (min$_j$ q_j^o, 1) *and $\beta =$* max (max$_j$ q_j^o, 1). *Furthermore,*

(*C*) $D_M(t) = \max_j |q_j(t) - 1|$ *is a continuous strictly decreasing function in the interval $[t', t'']$, $0 \leqq t' \leqq t \leqq t'' \leqq \infty$, in which $q(t) \neq \mathbf{1}$, and*

(*D*) *as $t \to \infty$, $q(t) \to \mathbf{1}$.*

Proof. The proof is very similar to that of Lemma 6. By constructing a cube $A' = \{q: 0 < \alpha/2 \leqq q_j \leqq \beta + \alpha/2 \text{ for all } j\} \supset A$ and proceeding with A' and A, respectively, as we did with B' and B, respectively, in the proof of Lemma 6, we establish assertions (A) and (B) of the present Lemma. The continuity of D_M is again established by appeal to its definition. Its decreasing nature is again easy to show when the value J of j maximizing the expression $|q_j(t) - 1|$ over all j is unique. For in this case

$$dD_M/dt = \text{sgn}\,(q_{J'} - 1)\, g_{J'}$$

[26] In 4.1.3, it is understood that j ranges over $\{1,2,\ldots,m\}$.

and the right member of this derivative is negative by condition (s) (substitutability) when $q \neq \mathbf{1}$. In the general case (when J need not be unique) the argument is parallel to that in Lemma 6 and involves showing that

$$\limsup_{h \to 0} (1/h) [D_M(t+h) - D_M(t)] \leqq \max_{j \varepsilon \mathscr{J}} (-|g_j(q(t))| < 0$$

where $\mathscr{J}$ is the set of maximizing subscripts for $|q_j(t) - 1|$.

It remains to establish (D). Here again we distinguish two cases, depending on whether $q(t)$ has $\mathbf{1}$ as a limiting point for $t \to \infty$. If $\mathbf{1}$ is a limiting point for $q(t)$, then it must be that $q(t)$ converges to $\mathbf{1}$ since D_M has just been shown to decrease in a monotone fashion. Hence the proof will be complete if we can show that $\mathbf{1}$ must be a limiting point of $q(t)$.

Suppose not. Then $q(t)$ will stay away a distance of at least ε, i.e., it will stay within the set $\tilde{S}_\varepsilon = \{q : \max_j |q_j - 1| \geqq \varepsilon\}$. Now define, for each j, the set $T_j = \{q : |q_j - 1| = \max_{j'} |q_{j'} - 1|\}$. Also, write $H_j = A \cap \tilde{S}_\varepsilon \cap T_j$.

Using condition (s) we show that $|g_j| > 0$ on H_j. By compactness of H_j and continuity of $|g_j|$ we conclude that $|g_j| \geqq \delta_j > 0$ on H_j. By steps identical with those for Lemma 7, it follows that $D_M(t)$ as defined in assertion (C) of the present Lemma has a time-derivative almost everywhere and that

$$|dD_M/dt| \geqq \min_j \delta_j > \delta > 0,$$

and, following the last phases of the proof of Lemma 7, we again have a contradiction.

4.1.4. *Proof of assertions* (b_1) *and* (b_2) *of Theorem* 1. (Stability and monotone convergence of the normalized process under substitutability; maximum norm.)

This proof is parallel to that of section 4.1.2; it involves going over from the differential system

$$\dot{p}_j = h_j[f_j(p)] \qquad (j = 1,2, \ldots, m) \tag{1}$$

to the system

$$\dot{q}_j = g_j(q) \qquad (j = 1,2, \ldots, m) \tag{2}$$

where $q_j = p_j/\bar{p}_j$. The assertions of the theorem are easily established for the p's if the hypotheses of Lemma 7 can be verified for the system (2). This verification raises no new problems except for justifying condition (s) of Lemma 7. This is accomplished by taking condition (S*), involving non-normalized prices P_k as the starting point and observing that, for instance,

$$P_{K'}/\bar{P}_{K'} \geqq P_k/\bar{P}_k \quad \text{for all } k \text{ in } \{0,1, \ldots, m\}$$

is equivalent to

$$p_{K'}/\bar{p}_{K'} \geqq p_k/\bar{p}_k \quad \text{for all } k \text{ in } \{0,1, \ldots, m\},$$

where $p_0 = 1$, and this in turn is implied by

$$q_{J'} = \max_j q_j \geqq 1 \qquad \text{(with } j \text{ ranging over } \{1,2,\ldots,m\})$$

if K' is put equal to J'.

The minimal subscript J'' is treated in an analogous manner.

4.2. THEOREM 2. (Stability of equilibrium under substitutability; and monotone convergence in the Euclidean norm.)

In this section we obtain a theorem similar to Theorem 1, but there are two important differences. The class of differential processes is narrower, in that the right members of the differential equations must be proportional to the excess demands instead of merely sign-preserving functions; also, the Walras Law is here included among the assumptions, while it was not used in the proof of Theorem 1. The proof of convergence uses one of the results of Lemma 6, hence is not completely independent of the results of section 4.1.

4.2.1. *The non-normalized process* (I′).

In this section we consider processes defined by the differential equation system

$$\text{(I}'\text{)} \qquad dP_k/dt = A_k F_k(P), \quad P^o > 0 \qquad (k = 0,1,\ldots,m),$$

where the A_k are positive constants. (This is a special case of $\dot{P}_k = H_k[F_k(P)]$ with sign-preserving H_k, i.e., of process (I).)

The (weighted) Euclidean norm of a vector P will be defined as

$$\text{(1)} \qquad ||P||_2 = (\sum_k P_k^2/A_k)^{1/2}.$$

Hence the square of the distance from the moving point $P(t) = P$ to an equilibrium point $\bar{P}$ is given by

$$\text{(2)} \qquad D_2 = \sum_k V_k$$

where

$$\text{(3)} \qquad V_k = (1/A_k)(P_k - \bar{P}_k)^2.$$

Differentiating the latter expression we obtain

$$\text{(4)} \quad \tfrac{1}{2}\, dV_k/dt = (1/A_k)(P_k - \bar{P}_k)(dP_k/dt) = (1/A_k)(P_k - \bar{P}_k)\, A_k F_k(P) = (P_k - \bar{P}_k)\, F_k(P).$$

Hence

$$\text{(5)} \qquad \tfrac{1}{2}\, dD_2/dt = \sum_k (P_k - \bar{P}_k)\, F_k(P) = -\sum_k \bar{P}_k F_k(P)$$

where the second equality follows from the Walras Law.

Now, by assertions (A) and (B) of Lemma 6, (I′) has a solution and $P^o > 0$ implies $P(t) > 0$ for all $t \geqq 0$. Therefore, the derivative dD_2/dt exists for all $t \geqq 0$, is continuous (by the continuity of $F_k(P)$ and $P(t)$), and, furthermore, by Lemma 5 applied to equation (5),

$$(6) \qquad dD_2/dt < 0 \quad \text{if} \quad \bar{P} > 0 \text{ and } F(P) \neq 0.$$

Hence the convergence of $P(t)$ to some equilibrium point (already guaranteed by Theorem 1) is monotone in the Euclidean norm. (One could also establish this convergence directly from the continuity of dD_2/dt as a function of time.) Thus we have

THEOREM 2.1. *Assume the excess demand functions to be continuous, single-valued, and positively homogeneous of degree zero* (H). *Suppose furthermore that the Walras Law* (W), *gross substitutability* (S_F) *prevails and* $\bar{P} > 0$ *exists. Then, for* $P^o > 0$, *the system* (I′) *is stable* (*i.e.*, *there is convergence to some equilibrium point*) *and the convergence to an equilibrium price vector is monotone in the weighted Euclidean norm given by equation* (1).

4.2.2. The normalized process (II′).

In this section we consider processes defined by the differential equation system

$$(\text{II}') \qquad dp_j/dt = a_j f_j(p), \quad p^o > 0 \qquad (j = 1,2,\ldots m),$$

where the a_j are positive constants. (This is a special case of process (II).) We proceed in a manner parallel to that of section 4.2.1, using, respectively, p, f, a for P, F, A and with summations on j ranging over $\{1,2,\ldots,m\}$ instead of those on k ranging over $\{0,1,\ldots,m\}$. For the squared distance here given by

$$(1) \qquad D_2 = \tfrac{1}{2} \sum_j (1/a_j)(p_j - \bar{p}_j)^2$$

we find

$$(2) \qquad dD_2/dt = \sum_{j=1}^{m} (p_j - \bar{p}_j) f_j(p).$$

Now, as indicated in section 2, we may embed (II′) in a process such as that given by equation (3′) of section 2. This amounts to replacing p_j by P_j/P_o and specifying a (constant) value for P_o. Because $P_o(t)$ is constant, we may, if we put $P_o = \bar{P}_o = 1$, rewrite equation (2) as

$$(3) \qquad dD_2/dt = \sum_{k=0}^{m} (P_k - \bar{P}_k) F_k(P).$$

From here on, we again proceed again as in 4.2.2, thus obtaining the counterpart of Theorem 2.1 above for the normalized process. This may be labeled

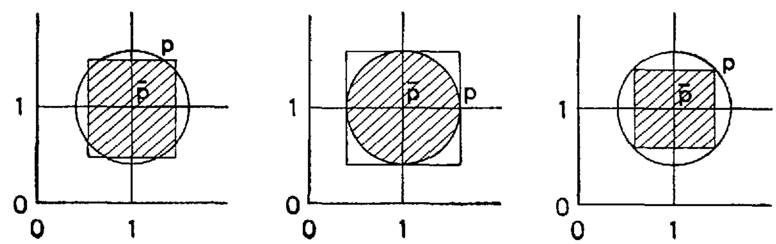
THEOREM 2.2. (The wording is exactly as that of Theorem 2.1, except that P is replaced by p and I′ by II′.) The two theorems 2.1 and 2.2 together are referred to as *Theorem* 2.

4.3. *Combining information provided by the two norms.*

It was mentioned in 4.0 that information concerning the path of convergence can be obtained from the use of the two norms in the proof of convergence. To see this, consider the special case where, for the normalized process (19), all the $a_j = 1$ and all the $\bar{p}_j = 1$. Suppose we are at the point p. Where shall we be a short time interval later?

To answer the question consider the following two sets: $S_2(P)$, the m-dimensional sphere with center at the equilibrium point and going through the point p; $S_M(P)$ the m-dimensional 'cube' with center at the equilibrium point and walls parallel to the coordinate planes. The proof using $\| \ \|_2$ tells us that p will travel into the interior of $S_2(p)$; the proof using $\| \ \|_M$ tells us that p will travel into the interior of $S_M(p)$. Hence p will have to travel into the interior of the intersection of the two sets. Since, in general, neither set contains the other, there is a clear gain of information in using both norms rather than either one alone. At special points, however, the cube is a subset of the sphere or vice versa and there is no gain of information.

The diagrams below show a few situations for the case $m = 2$, where the "sphere" is a circle and the "cube" a square. The shaded area is the intersection of the two sets.



5. DOMINANT NEGATIVE DIAGONAL

5.1. In this section we establish stability in the large for a class of cases closely related to that of substitutability.[27] We shall deal only with the normalized process. The method of proof used is in some respects analogous to that of 4.1 above.

[27] This line of inquiry was suggested by the corresponding local results (unpublished), due to F. H. Hahn, and R. M. Solow, Massachusetts Institute of Technology.

THEOREM 3. *The normalized process* (II′) *is stable in the large if there exists a set of positive constants* $(c_1, c_2, \ldots, c_m)$ *such that, for each* $j = 1,2,\ldots,m$ *the inequalities*

$$(1) \qquad f_{jj} < 0,$$

$$(2) \qquad c_j|f_{jj}| > \sum_{s \neq j} c_s|f_{js}|$$

(*where* $f_{js} = \partial f_j/\partial p_s$) *are satisfied.*

PROOF. We show that, for

$$(3) \qquad V = \max_j |a_j f_j|/c_j$$

we have, wherever dV/dt exists,

$$(4) \qquad dV/dt < 0 \quad \text{except at equilibrium.}$$

Let

$$(5) \qquad |a_J f_J|/c_J \geqq |a_j f_j|/c_j \quad \text{for all } j,$$

so that

$$(6) \qquad V = |a_J f_J|/c_J.$$

Therefore

$$(7) \qquad \frac{dV}{dt} = \left(\frac{a_J}{c_J}\right) (\text{sgn } f_J) \sum_j f_{Jj} \left(\frac{dp_j}{dt}\right) = \left(\frac{a_J}{c_J}\right) (\text{sgn } f_J) \sum_j f_{Jj} f_j a_j.$$

Clearly, $dV/dt = 0$ at equilibrium where all the f_j vanish. But suppose we are not at equilibrium. Then at least one of the f_j must be different from zero. Hence by (5), $|f_J| > 0$. In this case, using (2) for $j = J$, and multiplying by $|f_J|$ on both sides of the inequality, we obtain

$$(8) \qquad c_J|f_{JJ}|\cdot|f_J| > \sum_{s \neq J} c_s |f_{Js}|\cdot|f_J| \geqq \sum_{s \neq J} c_s |f_{Js}| (c_J a_s/a_J c_s)\cdot|f_s| = (c_J/a_J) \sum_{s \neq J} |f_{Js}|\cdot|f_s| a_s \qquad \text{(by (5))}.$$

Hence

$$(9) \qquad |f_{JJ}|\cdot|f_J|a_J > \sum_{s \neq J} |f_{Js}|\cdot|f_s|a_s,$$

i.e., because of (1)

$$(10) \qquad -f_{JJ} (\text{sgn } f_J) f_J a_J > \sum_{s \neq J} |f_{Js}|\cdot|f_s|a_s \geqq (\text{sgn } f_J) \sum_{s \neq J} f_{Js} f_s a_s.$$

Writing this as

$$(11) \qquad (\operatorname{sgn} f_J) \sum_j f_{Jj} f_j a_j < 0$$

we see that the right member of (7) is negative. The case in which dV/dt does not exist may be dealt with as in the proofs of Lemmas 6 and 7.

5.2. The domain of applicability of the preceding result remains to be explored. In particular, the results in section 4 concerning the stability *in the large* under gross substitutability do not seem to follow from Theorem 3. Locally, on the other hand, the stability under substitution can be derived from Theorem 3 through the utilization of the Euler equation $\sum_{k=0}^{m} P_k F_{kr} = 0$ (implied by the homogeneity of the F's): the $\bar{P}_j$ can be used as the c_j of the theorem. (In the large, one is tempted to try the use of the P_j, even though variable, in the same way as the c_j's. This introduces additional terms in dV/dt which make trouble when $f_J < 0$.)

One could obtain the analogue of Theorem 3 for the non-normalized process (II′) by a procedure similar to that of 4.1, but here even the local version of the substitution case does not seem to follow.

5.3. For purposes of economic interpretation, it is easiest to deal with the special case of (2) where all the $c_j = 1$, so that

$$|f_{jj}| > \sum_{s \neq j} |f_{js}|$$

and, also, $f_{jj} < 0$. (The c_j's can be made $= 1$ by a suitable change of measurement of the commodities.) Suppose now that the price of the jth commodity (non-numéraire) is changed, either up or down, by an amount K, while the other $m - 1$ (non-numéraire) commodities undergo changes (either up or down) whose magnitude does not exceed K. (They need not all move in the same direction, and some may remain unchanged.) Then the excess demand for the jth commodity will have gone up or down according to whether the change in p_j had been downward or upward.

Walras' argument [**15**, lesson 12, 127-130, pp. 170-172] is perhaps based on such an assumption.

6. OTHER RESULTS ON STABILITY FOR THE NON-NORMALIZED PROCESS (I′)

In this section it is shown that three of the results obtained in [**2**] for the normalized process (II′) can also be obtained for the non-normalized process (I′). Without loss of generality, we put, in (I′), $A_k = 1$ for $k = 0,1,\ldots,m$, so that the process may now be written as

$$(1) \qquad dP_k/dt = F_k(P) \qquad (k = 0,1,\ldots,m).$$

Using inner product notation, the Walras Law is written as

$$P \cdot F(P) = 0. \tag{2}$$

6.1. *The aggregate excess demand functions satisfy the Weak Axiom of Revealed Preference.* (See [**2**, Theorem 2].)

We say that the aggregate excess demand functions satisfy the Weak Revealed Preference Axiom if

$$P' \cdot [F(P'') - F(P')] \leqq 0,\ F(P'') \neq F(P') \text{ imply } P'' \cdot [F(P'') - F(P')] < 0. \tag{3}$$

Now let $P'' = \bar{P}$ and $P' = P$. Then the first inequality of (3) is fulfilled because of the Walras Law (2) and by definition of equilibrium $F(\bar{P}) = 0$. Also, if P is not an equilibrium price vector, we have $F(P) \neq 0$, hence the second inequality of (3) is also satisfied. It follows that

$$-\bar{P} \cdot F(P) < 0 \quad \text{if } P \text{ is not an equilibrium price vector.} \tag{4}$$

Now define, as before,

$$D_2 = (1/2)\,(P - \bar{P}) \cdot (P - \bar{P}). \tag{5}$$

Then

$$\begin{aligned} dD_2/dt &= (P - \bar{P}) \cdot F(P) = P \cdot F(P) - \bar{P} \cdot F(P) \\ &= -\bar{P} \cdot F(P) \quad [\text{by } (2)] \\ &< 0 \quad [\text{by } (4)] \end{aligned} \tag{6}$$

which implies stability.

When there is only one individual ($n = 1$), or when all individuals are identical, the preceding results also imply stability.

6.2. *The case of "no trade" at equilibrium.*

This case (see [**2**, Theorem 1]) arises when

$$F^i(\bar{P}) = 0 \qquad (i = 1,2, \ldots n) \tag{7}$$

where the superscript refers to the individual. By methods very close to those of the preceding section, the proof of Theorem 1 in [**2**] can be adapted to non-normalized process (I'). For every individual i, we find that (7) implies the fulfillment of the antecedent of the (individual) weak revealed preference axiom, viz.

$$P \cdot [F^i(\bar{P}) - F^i(P)] \leqq 0, \tag{8}$$

and $F^i(P) \neq F^i(\bar{P})$ since we assume that P is non-equilibrium. If follows that

$$\bar{P} \cdot [F^i(\bar{P}) - F^i(P)] < 0, \tag{9}$$

i.e., because of (7),

$$-\bar{P}\cdot F^i(P) < 0 \tag{10}$$

and, by aggregation,

$$-\bar{P}\cdot F(P) < 0. \tag{11}$$

Using D_2 as defined in (5), we see that the left member of (11) equals dD_2/dt which completes the proof.

6.3. *The case of two commodities* ($m = 1$). (See [**2**, Theorem 6].)

First, let us make the following observation, valid for *any m*:

$$\sum_{k=0}^{m} P_k^2 = \text{constant} \quad \text{along a solution of (1) subject to (2).} \tag{12}$$

(This is because $d(\sum_k P_k^2)/dt = 2\sum_k P_k(dP_k/dt) = 2\sum_k P_k F_k(P)$ and the last expression vanishes by (2).)

Secondly, also for any m, by zero degree homogeneity,

$$F_k(P) = f_k(p) \qquad (k = 0,1,\ldots,m) \tag{13}$$

where $p = (P_1/P_0, \ldots, P_m/P_0)$.

Now put $m = 1$, so that $p = P_1/P_0$. Then, by the Walras Law, the two functions $f_0(p)$ and $f_1(p)$ have the same zeros and opposite signs when they are not zero.

Suppose that, at the initial point $(P_0^o, P_1^o) = P^o$, $f_1(p^o) > 0$ where $p^o = P_1^o/P_0^o$. Then $F_1(P^o) > 0$ and $F_0(P^o) < 0$ by (13) and the Walras Law; hence P_1 will be increasing and P_0 decreasing by (1), and thus p will increase towards the zero of $f_1(p)$ next above p^o, say $\bar{p}$. Hence the ratio P_1/P_0 converges to $\bar{p}$, while the sum $P_0^2 + P_1^2$ remains constant by (12). It follows that $P(t)$ converges to some $\bar{P}$ as t tends to infinity. The rest of the proof is carried out in terms of the argument used in establishing Theorem 6 in [**2**].

Remark: The results for $m = 1$ can easily be extended to the generalized adjustment processes of section **4** above, since the signs of $f_k(p)$ are again decisive.

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