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Rational Choice Functions and Orderings[1]

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1. INTRODUCTION

The language of the theory of consumers' demand is still somewhat confused despite the great progress that has been made in recent years.[2] The basic purpose of the theory is to explain the demand vector $d\,(p,\ M)$ chosen by an individual when faced with a price vector p and an income M. Cournot, who introduced the concept of the demand function, and others, simply postulated some properties such as monotonic decrease of demand for any commodity with respect to its own price. The development of utility theory in the second half of the 19th century by Gossen, Jevons, Menger, and Walras and its subsequent reinterpretation on an ordinal basis by Pareto led to an alternative formulation in terms of an ordering of all conceivable commodity bundles. The demand vector for a particular p and M is that vector among all those compatible with the budget limitation which is most preferred.

The derivation of demand functions from orderings (expressed as indifference maps or utility functions) became standard and its fruitfulness in yielding implications for demand functions was made evident by the work of Slutzky [14], Hicks and Allen [7], Hotelling [8], and Roy [12]. Apart from the problems raised by the integrability question,[3] the first distinctly novel approach was the revealed preference approach of Samuelson [13]. Here again an assumption is made on the demand function (see C5 below). From this assumption a number of properties of the demand function can be deduced, though so far not as many as from the assumption of an underlying ordering.

A good deal of effort has gone into finding assumptions on the demand function which would imply the existence of an ordering from which it could be derived. So far, Samuelson's original assumption (now generally known as the Weak Axiom of Revealed Preference) has not been found sufficiently strong.[4] Independently, Ville [16] and Houthakker [9] have shown that a modification of this axiom, referred

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[2] For recent restatements of the theory, see Wold [17], Part II, and Hicks [6].

[3] Georgescu-Roegen ([5], pp. 567–8) conclusively showed that the real issue behind the integrability problem was the question of transitivity. In terms of the present paper, the integrability problem is a somewhat misleading way of putting the problem of the relation between assumptions on the demand functions and the existence of an ordering which generates them.

[4] Despite a common opinion, it has not yet been shown that the Weak Axiom is not sufficient to ensure the desired result. The question is still open.

to as the Strong Axiom of Revealed Preference (see *C*1 below) is sufficient to ensure the desired result.

Both demand functions and orderings can be regarded as special cases of choice functions. For any set of alternatives X let $C(X)$ be the set of alternatives chosen (we admit the possibility that more than one alternative be chosen). The function $C(X)$ is of course not necessarily defined for all possible sets X. Let ß be the class of sets for which $C(X)$ is defined. The revealed-preference and other demand-function approaches essentially deal with the case where ß is the class of sets defined by budget constraints of the form,

(1) $$\Sigma p_i x_i \leqq M, \; x_i \geqq 0.$$ [1]

On the other hand, an ordering can be interpreted as a series of statements about choices from sets containing two elements. Choices from larger sets, including those defined by (1), are defined in terms of the binary choices. Thus the choice functions defined by an ordering are defined for a class ß which includes not only sets of form (1) but many others, including in particular two-element sets and in fact all finite sets.

It is the suggestion of this paper that the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets. Indeed, as Georgescu-Roegen has remarked, the intuitive justification of such assumptions as the Weak Axiom of Revealed Preference has no relation to the special form of the budget constraint sets but is based rather on implicit consideration of two-element sets (see [4], p. 125, fn. 29).

2. Definitions

The investigation suggested in the Introduction has already been begun in a significant paper by Uzawa [15] and his notation is followed here with minor modifications.

A binary relation R is said to be a *weak ordering* if

(R1) for all x and y, xRy or yRx,

(R2) for all x, y and z, xRy and yRz imply xRz.

A choice function $C(X)$ maps a non-null set X into a non-null subset. It is defined for all X in some class ß. We make the following

Assumption: The domain of definition ß of any choice function $C(X)$ contains all finite sets.

Definition 1. For any binary relation R, we define

$$C(X) = \{ x \mid x \in X, x R y \text{ for all } y \in X \},$$

to be the *choice function derived from R.*

Definition 2. For any choice function $C(X)$, we define

$$x R y = df.\; x \in C(\{x, y\}),$$

to be the *relation generated by* $C(X)$.

[1] The budget constraint is here written in the form used by current theory, which stresses inequalities and non-negativity constraints.

Here $\{x, y\}$ is the set consisting of the two elements x and y. The motivation of these definitions is obvious.

We now wish to make assumptions about choice functions. Since these assumptions are those of rational behaviour, we follow Uzawa in referring to choice functions satisfying them as *rational choice functions*. We give altogether five definitions of a rational choice function and study the relations of implication among them and to the existence of a weak ordering. First, we introduce two more relations defined by a given choice function.

Definition 3. We say that x is *revealed preferred* to y (symbolized by $x \tilde{P} y$) if, and only if, for some $X \in \mathfrak{B}$, $x \in C(X)$ and $y \in X - C(X)$.

Definition 4. The element x is *indirectly revealed preferred* to y (xP^*y) if, and only if, there exist x^i $(i=0, \ldots, n)$ such that $x^0=x$, $x^n=y$, and $x^{i-1}\tilde{P}x^i$ $(i=1, \ldots, n)$.

Uzawa has suggested the following two definitions for rational choice functions:

($C1$) For all x and y, if there exists an X for which $x \in C(X)$, $y \in C(X)$, then $\overline{xP^*y}$. (The bar denotes negation.)

($C2$) If $X \subset Y$, then $X - C(X) \subset Y - C(Y)$.

(By $X \subset Y$ is meant that every alternative in X also belongs to Y. By $A - B$ is meant the set of alternatives in A but not in B). ($C1$) is a form of the Strong Axiom of Revealed Preference due to Ville and Houthakker. ($C2$), as will be seen below, is a weaker assumption. It asserts that any element not chosen from a set of alternatives X will not be chosen if the range of alternatives is widened.

It will be shown below that the following condition is equivalent to ($C2$):

($C3$) *If* $X \subset Y$, *then* $C(Y) \cap X \subset C(X)$.[1]

(By $A \cap B$ is meant the set of alternatives in both A and B.) Because ($C2$) and its equivalent ($C3$) are weak assumptions, a slight strengthening is called for.

($C4$) If $X \subset Y$ and $C(Y) \cap X$ is non-null, then $C(X) = C(Y) \cap X$.

This can be given the following intuitive interpretation: if some elements are chosen out of a set Y and then the range of alternatives is narrowed to X but still contains some previously chosen elements, no previously unchosen element becomes chosen and no previously chosen element becomes unchosen.[2]

Finally, we introduce the definition of rationality by the Weak Axiom of Revealed Preference.

($C5$) If $x\tilde{P}y$, then there exists no Y such that $x \in Y$, $y \in C(Y)$.

[1] This is the same as postulated in Chernoff [2], pp. 429–30.

[2] Definition ($C4$) was originally introduced in [1], pp. 4–8, and the properties presented in this paper were first proved there.

3. Interrelations Among Definitions of Rational Choice Functions

We first establish relations of logical implication among the definitions of rational choice functions.

Theorem 1. Definition ($C1$) implies definitions ($C2$–5); ($C4$) and ($C5$) are equivalent and imply ($C2$) and ($C3$); ($C2$) and ($C3$) are equivalent.

It suffices to prove (a) ($C1$) implies ($C5$); (b) ($C4$) and ($C5$) are equivalent; (c) ($C4$) implies ($C3$), and (d) ($C2$) and ($C3$) are equivalent.

(a) Suppose ($C1$) holds and ($C5$) does not. Then we can choose x, y, Y so that $x\tilde{P}y$, $x \in Y$, $y \in C(Y)$. If $x \in Y - C(Y)$, then $y\tilde{P}x$; by Definition 4, yP^*y, which contradicts ($C1$). If $x \in C(Y)$, then again ($C1$) is contradicted, since $x\tilde{P}y$ implies xP^*y, while both x and y belong to $C(Y)$.

(b) First we prove that ($C4$) implies ($C5$). Suppose ($C4$) true and ($C5$) false. Then there exist x, y, X, Y, such that

(2) $x \in C(X)$,

(3) $y \in X - C(X)$,

(4) $x \in Y$,

(5) $y \in C(Y)$.

From (2–3), $\{x, y\} \subset X$, and $\{x, y\} \cap C(X)$ contains the single element x. From ($C4$),

(6) $C(\{x, y\}) = \{x\}$.

But from (4–5), $\{x, y\} \subset Y$, and from (5), $\{x, y\} \cap C(Y)$ is non-null and contains y. By ($C4$), $y \in C(\{x, y\})$, which contradicts (6), unless $y=x$; but this last is impossible from (2) and (3).

Now suppose that ($C5$) and the hypothesis of ($C4$) hold but that $C(X) \neq C(Y) \cap X$. Then either (i) $C(X) - [C(Y) \cap X]$ is non-null, or (ii) $[C(Y) \cap X] - C(X)$ is non-null.

(i) Let $y \in C(X) - [C(Y) \cap X]$, $x \in C(Y) \cap X$. Then $x \in C(Y)$, $y \in Y$ (since $y \in C(X) \subset X \subset Y$), and $y \notin C(Y)$, so that $y \in Y - C(Y)$, and therefore $x\tilde{P}y$. But $y \in C(X)$, $x \in X$, a contradiction to ($C5$).

(ii) Let $x \in C(X)$, $y \in [C(Y) \cap X] - C(X)$. Then $x \in C(X)$, $y \in X - C(X)$, so that $x\tilde{P}y$. But $x \in C(X) \subset X \subset Y$, $y \in C(Y)$, which again contradicts ($C4$).

(iii) If $C(Y) \cap X$ is non-null, then clearly ($C4$) implies ($C3$). If $C(Y) \cap X$ is null, then ($C3$) holds in any case.

(iv) Suppose $X \subset Y$. If ($C2$) holds, then $X - C(X)$ is disjoint from $C(Y)$, that is, $C(Y) \cap [X - C(X)]$ is null. This is equivalent to the statement that $[C(Y) \cap X] - C(X)$ is null, which in turn is equivalent to the conclusion of ($C3$). Conversely, if ($C3$) holds, then $[C(Y) \cap X] - C(X)$ is null, which is equivalent to saying $C(Y) \cap [X - C(X)]$ is null, and therefore $X - C(X)$ is disjoint from $C(Y)$. But $X-C(X) \subset X \subset Y$, so that the conclusion of ($C2$) follows.

We will see (Corollary to Theorem 3) that a slightly stronger statement can be made than that contained in Theorem 1, namely that ($C1$), ($C4$), and ($C5$) are all equivalent.

4. Weak Orderings and Derived Choice Functions

Theorem 2. If R is a weak ordering, let $C(X)$ be the choice function derived from it and R' the relation generated by $C(X)$. Then $C(X)$ satisfies ($C1$–5) and R' is identical with R.

Proof: That $C(X)$ satisfies ($C1$) is shown by Uzawa ([15], Theorem 1). By Theorem 1, then, $C(X)$ satisfies ($C2$–5). By Definition 2, $xR'y$ if, and only if, $x \in C(\{x, y\})$ and, by Definition 1, if, and only if, both xRx and xRy. But xRx holds for all x, from ($R1$).

Theorem 2 is in the usual direction of implications from assumptions about orderings or indifference maps to properties of the demand functions, or the choice functions in a more generalized setting.

5. Rational Choice Functions and Relations Generated by Them

Theorem 3. If $C(X)$ satisfies ($C1$), ($C4$), or ($C5$), let R be the relation generated by $C(X)$ and $C'(X)$ the choice function derived from R. Then R is a weak ordering, and $C'(X) = C(X)$.

Proof: It suffices to assume that $C(X)$ satisfies ($C4$) in view of Theorem 1.

For all x and y, $C(\{x, y\})$ is defined, since $\{x, y\}$ is a finite set; either $x \in C(\{x, y\})$, or $y \in C(\{x, y\})$, that is, either xRy or yRx, so that ($R1$) holds.

Now suppose xRy and yRz. If $y=z$ or if $x=y$, the conclusion xRz follows immediately. If $x=z$, then xRz is the same as xRx, which follows from ($R1$), already established. Now suppose x, y, z distinct. Let X consist of the elements x, y, z and suppose $x \notin C(X)$. If $y \in C(X)$, $\{x, y\} \cap C(X)$ is non-null so that $C(\{x, y\}) = \{x, y\} \cap C(X)$. Since $x \in C(\{x, y\})$ by assumption, $x \in C(X)$, a contradiction.

(7) $$x \notin C(X) \text{ implies } y \notin C(X).$$

By exactly the same argument, $y \notin C(X)$ implies that $z \notin C(X)$. But with the aid of (7), $x \notin C(X)$ implies that $C(X)$ is null, which is contrary to the basic assumption. Hence $x \in C(X)$. Since $x \in \{x, z\} \cap C(X)$, $\{x, z\} \cap C(X)$ is non-null; by (C4), $x \in C(\{x, z\})$, or xRz. Therefore ($R2$) holds, and R is a weak ordering.

Let $x \in C(X)$, y be any element of X. Then $x \in \{x, y\} \cap C(X)$ and therefore $x \in C(\{x, y\})$, or xRy. From Definition 1, $x \in C'(X)$, therefore

(8) $$C(X) \subset C'(X).$$

Let $x \in C'(X)$, $y \in C(X)$. Then $\{x, y\} \cap C(X)$ is non-null. Since xRy, by Definition 1, $x \in C(\{x, y\}) = \{x, y\} \cap C(X) \subset C(X)$, so that $C'(X) \subset C(X)$. In conjunction with (8), the theorem is proved.

Corollary. Definition ($C1$) is equivalent to Definitions ($C4$) and ($C5$).

Proof: That ($C1$) implies ($C4$) and ($C5$) is already stated in Theorem 1. Suppose $C(X)$ satisfies ($C4$) or ($C5$). Then, by Theorem 3, $C(X) = C'(X)$, where $C'(X)$ is derived from a weak ordering R and therefore satisfies ($C1$), by Theorem 2.

Remark 1. Let P be the relation of preference derived from a weak ordering R, that is, xPy if, and only if, xRy and not yRx. Then Uzawa has shown ([15], Theorem 3) that under the assumption $P = P^*$, so that under any of the definitions of a rational choice function ($C1$, 4, or 5) the relation of preference coincides with that of indirect revealed preference.

Remark 2. Conditions ($C2$) or ($C3$) are not sufficient for Theorem 3, as can easily be seen from the following example: Let R be an ordering, $C(X)$ the choice function derived from R, and $C'(X) = C(X)$ for all sets X other than those containing two elements, for which $C'(X) = X$. By Theorem 2, $C(X)$ satisfies ($C2$). On the other hand, if X has one or two elements, X-$C'(X)$ is the null set so that

$$\text{(9)} \qquad \text{if } X \subset Y, \text{ then } X - C'(X) \subset Y - C'(Y),$$

whenever X has not more than two elements. If X has more than two elements, then Y must also, and (9) holds since then $C'(X) = C(X)$ and $C'(Y) = C(Y)$. As can be seen from ($C3$), the problem is that when passing from Y to a subset X, the choice set $C(X)$ may be relatively too big.

Uzawa ([15], Theorem 4) has shown that if $C(X)$ contains only a single element for all X, then ($C2$) is a sufficient condition for the results of Theorem 3; however, the assumption that $C(X)$ contains only a single element rules out the possibility of indifference.

6. General Remark

The most interesting conclusion is the complete equivalence of the Weak Axiom of Revealed Preference with the existence of an ordering from which the choice function can be derived. This equivalence is demonstrable by very elementary means provided we concede that choices should be definable from finite sets as well as budget constraint sets. It is true that very interesting mathematical problems are bypassed by this point of view, but this should not perhaps be a compelling consideration.

It has already been argued that requiring the choice functions to be defined for finite sets is thoroughly consistent with the intuitive arguments underlying revealed preference. It should also be observed that any hope of using experimental methods for studying preference will require inferring from choices on finite sets to choices on infinite ones (see May [10], Papandreou [11], and for a somewhat different but parallel situation, Davidson and Suppes [3]).

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REFERENCES

[1] Arrow, Kenneth J., " The Possibility of a Universal Social Welfare Function," Project RAND, RAD(L)-289, 26 October, 1948, Santa Monica, California (hectographed).

[2] Chernoff, Herman, " Rational Selection of Decision Functions," *Econometrica*, 22 (1954): 422–443.

[3] Davidson, Donald, and Patrick Suppes, " A Finitistic Axiomatization of Subjective Probability and Utility," *Econometrica*, 24 (1956): 264–275.

[4] Georgescu-Roegen, Nicholas, " Choice and Revealed Preference," *Southern Economic Journal*, 21 (1954): 119–130.

[5] Georgescu-Roegen, Nicholas, " The Pure Theory of Consumer's Behaviour," *Quarterly Journal of Economics*, 50 (1936): 545–593.

[6] Hicks, John R., *A Revision of Demand Theory*. Oxford: Clarendon Press, 1956.

[7] Hicks, John R., and R. G. D. Allen, " A Reconsideration of the Theory of Value," *Economica*, N.S. 1 (1934): 52–76, 196–219.

[8] Hotelling, Harold, " Demand Functions with Limited Budgets," *Econometrica*, 3 (1935): 66–78.

[9] Houthakker, Hendrik S., " Revealed Preference and the Utility Function," *Economica*, N.S.17 (1950): 159–174.

[10] May, Kenneth O., " Transitivity, Utility, and Aggregation in Preference Patterns," *Econometrica*, 22 (1954): 1–13.

[11] Papandreou, Andreas G., " A Test of a Stochastic Theory of Choice," *University of California Publications in Economics*, 16 (1957): 1–18.

[12] Roy, René, *De l'utilité*. Paris: Herman et Cie., *Actualités Scientifiques et Industrielles*, no. 930, 1942.

[13] Samuelson, Paul A., " A Note on the Pure Theory of Consumer's Behaviour," *Economica*, N.S. 5 (1938): 61–71, 353-4.

[14] Slutzky, Eugene, " Sulla teoria del bilancio del consumatore," *Giornale degli Economisti*, 51 (1951): 1–26.

[15] Uzawa, Hirofumi, " Note on Preference and Axioms of Choice," *Annals of the Institute of Statistical Mathematics*, 8 (1956): 35–40.

[16] Ville, Jean, " Sur les conditions d'existence d'une ophélimité totale et d'un indice du niveau des prix," *Annales de l'Université de Lyon*, 9 (1946): Section A(3), 32–39.

[17] Wold, Herman, and Lars Jureen, *Demand Analysis*. New York: John Wiley & Sons, 1953.