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STOCHASTIC CHOICE AND CARDINAL UTILITY [1]

BY GERARD DEBREU

D. DAVIDSON AND J. MARSCHAK [**5**] consider the following problem. A set S of actions is given; an agent is presented with a pair (a, b) of actions in S and asked to choose one; he chooses a with probability $p(a, b)$ and b with probability $p(b, a) = 1 - p(a, b)$. The relation $p(a, b) > 1/2$ is read "a is preferred to b." One may further suggest that the relation $p(a, b) > p(c, d)$ be read "a is preferred to b more than c is preferred to d." One is thus led to seek a real-valued (cardinal utility) function u on S such that $p(a, b) > p(c, d)$ be equivalent to $u(a) - u(b) > u(c) - u(d)$. Formally:

(1) *S is a set, p is a function from $S \times S$ to* $[0, 1]$ *such that $p(a, b) + p(b, a) = 1$ for every (a, b) in $S \times S$.*

DEFINITION: *A utility function for (S, p) is a real-valued function u on S such that*

$$[p(a, b) \leqq p(c, d)] \rightleftarrows [u(a) - u(b) \leqq u(c) - u(d)].$$

The problem is to find additional assumptions on (S, p) which will insure the existence of a utility function.

Since $u(a) - u(b) \leqq u(c) - u(d)$ is equivalent to $u(a) - u(c) \leqq u(b) - u(d)$, if there is a utility function for (S, p), then ([**5**], section II)

$$(2) \qquad [p(a, b) \leqq p(c, d)] \rightleftarrows [p(a, c) \leqq p(b, d)].$$

This necessary condition for the existence of a utility function will be taken as an axiom. It has the immediate consequence

$$(2') \qquad [p(a, b) = p(a', b') \text{ and } p(b, c) = p(b', c')] \rightarrow [p(a, c) = p(a', c')].$$

PROOF: By (2), $p(a, a') = p(b, b') = p(c, c')$. Applying (2) again to the equality of the first term and the third term, one obtains the conclusion.

The dependence of $p(a,c)$ on $p(a,b)$ and $p(b,c)$ can be described by a function P. Consider a point (x, y) of the square $[0, 1] \times [0, 1]$. That point belongs to the domain of P if and only if there are three elements a, b, c of S such that $p(a, b) = x$ and $p(b, c) = y$. Then the value of P at (x, y) is $P(x, y) = p(a, c)$. This definition is legitimate since, by (2'), a different triple a', b', c' of points of S such that $p(a', b') = x$ and $p(b', c') = y$ would give the same value for P. It is easy to check, using (2), that P is increasing in each one of its two variables.

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The last, and least satisfactory, axiom is

(3) *If* $p(b, a) \leqq q \leqq p(c, a)$, *then there is* d *in* S *such that* $p(d, a) = q$.

THEOREM. *Under assumptions* (1), (2), (3), *there is for* (S, p) *a utility function determined up to an increasing linear transformation.*

The trivial case where p is constant on $S \times S$ is solved by taking a constant utility function on S; it will now be excluded.

The theorem will be proved by means of a representation of S in [0, 1]. Let k be an arbitrary element of S which will be kept fixed. The generic element a of S is represented by the number $\alpha = p(a, k)$. According to (3), the image of the set S is an *interval* Σ, contained in [0, 1]. Given two elements a, b of S, one has $p(a, b) = P[p(a, k), p(k, b)] = P(\alpha, 1-\beta)$. The last term will be denoted by $\pi(\alpha, \beta)$. The function π from $\Sigma \times \Sigma$ to [0, 1] defined in this way satisfies

$$p(a, b) = \pi(\alpha, \beta) \tag{4}$$

and thus corresponds to p in the representation.

It is clear that finding a utility function u for (S, p) is *equivalent* to finding a utility function v for (Σ, π), the two utility functions being related by

$$u(a) = v(\alpha).$$

The second problem, however, is notably easier than the first.

Summing up the data of the new problem: Σ is a non-degenerate interval in [0, 1]; π is a function from $\Sigma \times \Sigma$ to [0, 1] such that $\pi(\alpha, \beta) + \pi(\beta, \alpha) = 1$; moreover π is increasing in α, decreasing in β, and it satisfies (2) and (3). It follows, without difficulty, that π is continuous.

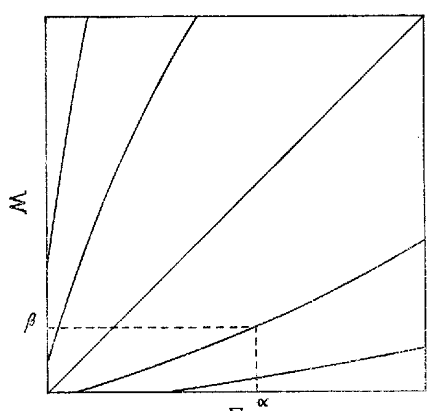

FIGURE 1

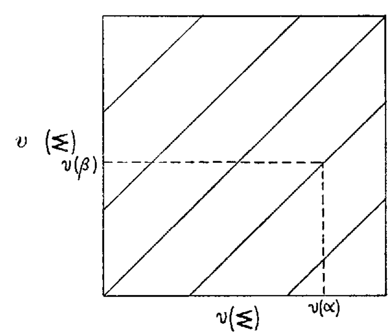

FIGURE 2

On Figure 1, the square $\Sigma \times \Sigma$ has been drawn and, in it, isoprobability lines corresponding to a few values of π. (The diagonal corresponds to the value $^1/_2$). The problem is to find an increasing transformation v of Σ into the reals such that these isoprobability lines become straight lines parallel to the diagonal of $v(\Sigma) \times v(\Sigma)$ (see Figure 2). This, however, is but a particular case of a problem of plane topology[2] solved in 1927 by G. Thomsen [**19**] and W. Blaschke [**3**] (See also W. Blaschke and G. Bol [**4**] pp. 1–42). Instead of proceeding to a painstaking identification of assumptions and conclusions it seems preferable to sketch a proof adapted to the present situation.

Select two arbitrary points α_0 and α_1 of Σ such that $\alpha_0 < \alpha_1$, and take $v(\alpha_0) = 0$ and $v(\alpha_1) = 1$. Consider the function f defined by $f(\alpha) = \pi(\alpha, \alpha_0) - \pi(\alpha_1, \alpha)$; it is continuous and increasing, $f(\alpha_0)$ is negative and $f(\alpha_1)$ is positive; hence there is a unique $\alpha_{1/2}$ where $f(\alpha_{1/2}) = 0$, i.e.,

$$\pi(\alpha_1, \alpha_{1/2}) = \pi(\alpha_{1/2}, \alpha_0).$$

The value of v at $\alpha_{1/2}$ is necessarily $1/2$. That dichotomy of the interval $[\alpha_0, \alpha_1]$ will be repeated *ad infinitum.* At the nth stage, one has points of the form $\alpha_{i/2^n}$ and the value of v at that point is necessarily $i/2^n$. In this way the function v is defined on the set $\triangle$ of points obtained by the preceding process. One must now check that v satisfies the definition of utility on $\triangle$. Notice first that $\pi(\alpha_{1/4}, \alpha_0) = \pi(\alpha_{3/4}, \alpha_{1/2})$, for, by (2′) and (2), one can have neither inequality sign. Therefore, by induction,

(5) $\pi(\alpha_{(i+1)/2^n}, \alpha_{i/2^n}) = \pi(\alpha_{(j+1)/2^n}, \alpha_{j/2^n})$ for every i, j from 0 *to* 2^n-1.

And, by repeated application of (2′),

$$\pi(\alpha_{(i+h)/2^n}, \alpha_{i/2^n}) = \pi(\alpha_{(j+h)/2^n}, \alpha_{j/2^n})$$

if the four subscripts are in $[0, 1]$.

The common value of the probabilities in (5) depends only on n; it will be denoted by $\theta(n)$. It is not difficult to show, using the continuity of π, that $\lim \theta(n)_{n\to+\infty} = 1/2$. It is then easy to derive from this fact that the set $\triangle$ is *dense in* $[\alpha_0, \alpha_1]$. The utility function v constructed on $\triangle$ is a one – to – one correspondence between $\triangle$ and the set of dyadic numbers of $[0, 1]$, which is *dense in* $[0, 1]$. The extension of v from $\triangle$ to $[\alpha_0, \alpha_1]$ is therefore immediate and the resulting function is clearly continuous. It remains

[2] The problem can be roughly described as follows. Given three families of curves in a plane, when does there exist a topological transformation carrying them into three families of parallel straight lines? On Figure 1 the three families are the isoprobability lines, the verticals, the horizontals. After the transformation, on Figure 2, they are the parallels to the diagonal, the verticals, the horizontals.

only to extend v from $[\alpha_0, \alpha_1]$ to Σ. For this, a procedure similar to that of Herstein-Milnor ([**8**], pp. 296–297) can be used.

The only arbitrariness in the construction comes from the choice of the values 0 and 1 for α_0 and α_1 respectively. Given two utility functions, one is derived from the other by an increasing linear transformation.

The discussion on cardinal utility in the thirties is well known (O. Lange [**9**], E. H. Phelps Brown, H. Bernardelli, O. Lange [**15**], R. G. D. Allen [**1**], F. Zeuthen [**20**], P. A. Samuelson [**17**]). However, the important paper by F. Alt [**2**] has generally been overlooked. Noteworthy in the recent revival of interest in this topic is the article by P. Suppes and M. Winet [**18**]. In connection with the problem of stochastic choice in economics the works of N. Georgescu-Roegen ([**6**] section VI, [**7**]), F. Mosteller and P. Nogee [**13**], K. O. May [**12**], A. G. Papandreou with the collaboration of O. H. Sauerlender, O. H. Brownlee, L. Hurwicz, and W. Franklin [**14**], J. Marschak [**11**], R. E. Quandt [**16**], and R. D. Luce and H. Raiffa [**10**, Appendix 1], must be mentioned.

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