

A NOTE ON EXPECTATIONS AND STABILITY<sup>1</sup>

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Demand and supply depend in general not only on current prices but also on expectations. If it is assumed that expectations are formed from past prices in the adaptive way suggested by Cagan and others, it is shown that under certain circumstances the resulting system of multiple markets has dynamic stability.

IN A RECENT ARTICLE, Enthoven and Arrow [5] examine the relationship between extrapolative expectations and the dynamic stability of a multiple market system. They show, for a simple expectations function, that a stable dynamic system can absorb the effects of some extrapolation of price movements and remain stable. This note discusses the relationship between dynamic stability and a somewhat more complicated expectations function suggested by Hicks' definition of the elasticity of expectations.

## 1. HICKSIAN EXPECTATIONS

In the notation of Enthoven and Arrow,  $P_i$  represents the current price of the  $i$ th commodity and  $P_i'$ , the expected future price of the  $i$ th commodity. It is assumed that changes in the expected future price of the  $i$ th good, induced by changes in actual prices, are governed by the relationship

$$(1) \quad P_i' = P_i + \eta_i \dot{P}_i,$$

where  $\dot{P}_i$  is the derivative of  $P_i$  with respect to time, and  $\eta_i$  is a constant.<sup>2</sup> When  $\eta_i = 0$ , current prices are expected to persist, and we say that expectations are static. When  $\eta_i > 0$ , some multiple of the change in prices is added to current price in arriving at expected price; hence, expectations may be described as extrapolative. When  $\eta_i < 0$ , expected prices do not change as much as actual prices.<sup>3</sup>

In their discussion Enthoven and Arrow assume that expectations "... for all future time periods can be represented by one 'expected' price" ([5], p. 289). As they indicate, this frequently made assumption is difficult to justify. Justification becomes easier, however, if we consider, not expectations of

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<sup>2</sup> Equation (1) is related to the naive models used to test the forecasts of econometric models. (See Christ [3], pp. 55-59.) It is also related to an expectations model presented by Metzler in his analysis of inventory cycles [8].

<sup>3</sup> In a model formulated in terms of periods, the case  $\eta_i < 0$  corresponds to taking expected price as a weighted average of actual prices lagged one and two periods.

particular future prices, but expectations of the average level about which future prices are expected to fluctuate. It then becomes quite natural to think of changes in current price as affecting people's expectation of this level of future prices.<sup>4</sup> In what follows we call the average level about which future prices are expected to fluctuate "expected normal price."

If more specific information is not available, it seems reasonable to suppose that expected normal price depends in some way on what actual prices have been in the past. Such an assumption corresponds to the distinction made by Enthoven and Arrow between the "induced" and "autonomous" components of a change in expected price. Induced changes in expected price, or the dependence of expected normal price on past prices, are the only things we can study outside the context of a specific situation. Factors, other than past prices, which influence expected prices cannot readily be incorporated into a discussion of dynamic stability.

Each past price represents only a short-run market phenomenon, an equilibrium of those forces present in the market at the time. It is for precisely this reason that the assumption of static expectations (i.e., the assumption that people expect current prices to persist) is not plausible. This does not mean, however, that the past or the present has no relevance for the future. Past and present prices reflect forces which determine the level about which future prices may be expected to fluctuate: the more recent the past price the more it expresses the operation of those forces relevant to expectations.<sup>5</sup> Hence, we assume that the influence of more recent prices on expectations should be greater than the influence of less recent prices. A more specific assumption would be that expected normal price is a weighted average of past prices, where the weights decline as one goes back in time.<sup>6</sup>

Hicks may have had this notion in mind when he defined "... the elasticity of a particular person's expectations of the price of a commodity  $x$  as the ratio of the proportional rise in [all] expected future prices of  $x$  to the proportional rise in its current price" ([7], p. 205). Hicks, it will be remembered, distinguished two pivotal cases: (1) an elasticity of zero, implying no effect of a change in current price upon expected future prices; and (2) an

<sup>4</sup> See Nerlove [11], pp. 41-54.

<sup>5</sup> It should be noted that this statement may not hold in specific situations, although we might expect it to hold "on the average". For example, the sharp increase in the prices of certain commodities after the outbreak of the Korean conflict probably did not have much to do with shaping people's long-run price expectations. If we examine expectations over long periods, however, and abstract from changes in price expectations due to factors which are exogenous, the statement in the text is plausible.

<sup>6</sup> As indicated in footnote 3, the model expressed by equation (1) allows a representation of expected price in terms of only two past prices, when time is treated as a discontinuous variable. If expectations for all future time periods are to be represented by one "expected price," the formulation in (1) does not, therefore, seem plausible.

elasticity of one, implying that if prices were previously expected to remain constant, i.e., were at their long-run equilibrium level, they will now be expected to remain constant at the level of current price. By allowing for a range of elasticities between the two, Hicks implicitly recognized that a particular past price may have something, but not everything, to do with people's notion of what the normal price will be.

Hicks' definition of the elasticity of expectations implies that prices have actually been normal (i.e., that the system has been in equilibrium) up to the time when some change occurred. But, of course, we know that conditions are seldom normal in the real world; furthermore, "normality" itself is a subjective matter. It is useful, therefore, to express current price, not as a deviation from what prices have been in the past, but from what people had previously thought of as the normal (i.e., people's previous expected normal price). If we think of time as a discontinuous variable, and let  $P'_i(t)$  be the normal price of the  $i$ th commodity during period  $t$  expected at the start of the period and  $P_i(t)$  be the actual price, we may express the expectations model suggested by Hicks' definition of the elasticity of expectations as

$$(2) \quad \frac{P'_i(t) - P'_i(t-1)}{P_i(t-1) - P'_i(t-1)} = \beta_i,$$

where the prices are expressed in logarithms and  $\beta_i$  is the elasticity of expectations of the price of the  $i$ th commodity, assumed to be constant. When  $\beta_i = 0$ , changes in actual price have no effect on expected normal price; when  $\beta_i = 1$ , current price is projected forward as people's expectations of the level of future prices.

If we drop the assumption that prices are expressed in logarithms, the differential equation analogue of (2) is

$$(3) \quad \dot{P}'_i = \beta_i (P_i - P'_i), \beta_i \geq 0.^7$$

When  $\beta_i = 0$ , changes in current price have no effect on expected normal price. Note, however, that the case of static expectations is now  $\beta_i = +\infty$ , as can be seen more rigorously by letting  $\beta_i$  approach  $+\infty$  in (4) below. For a given time path of prices  $P_i(t)$ , equation (3) has the solution

$$(4) \quad P'_i(t) = P'_i(0)e^{-\beta_i t} + e^{-\beta_i t} \int_0^t \beta_i P_i(u) e^{\beta_i u} du,$$

where  $P'_i(0)$  is the initial value of expected normal price. If the origin is in the sufficiently distant past, the term in (4) involving  $P'_i(0)$  is negligible;

<sup>7</sup> The model in this form has been used by Phillip Cagan in his study of hyperinflation [2], and by Milton Friedman in his study of the consumption function [6]. The model has also been applied in the form suggested by equation (2) to the study of supply functions for agricultural commodities by Nerlove [10] and [11].

hence,  $P_i'(t)$  may be taken as an exponentially weighted average of past prices.<sup>8</sup> Consequently, the model of expectation formation suggested by Hicks' definition of the elasticity of expectations leads to a reasonable representation of expected normal price in terms of past prices. In what follows we call expectations generated by equations such as (3) *adaptive* expectations.

## 2. STABILITY UNDER ADAPTIVE EXPECTATIONS

Enthoven and Arrow [5] assume that there are no "cross-effects" of price expectations on excess demand, i.e., that only the expected price of the  $i$ th good significantly affects the excess demand for that good. Under adaptive expectations, however, we may allow for possible cross-effects; consequently we may write the excess demand functions for the  $n$  goods in our system as

$$(5) \quad x_i = x_i(P_1, \dots, P_n, P_1', \dots, P_n') \quad (i = 1, \dots, n),$$

where  $x_i$  represents the excess demand for the  $i$ th commodity. In equilibrium demand equals supply in each of the  $n$  markets and all expectations are fulfilled, i.e.,

$$(6) \quad x_i = 0 \quad (i = 1, \dots, n)$$

and

$$(7) \quad P_i' = P_i \quad (i = 1, \dots, n).$$

As in [5], we approximate the dynamic behavior of prices by equations of the form

$$(8) \quad \dot{P}_i = K_i x_i \quad (i = 1, \dots, n),$$

where the  $K_i$  are positive constants. We assume that expectations are generated by equations such as (3) above.

By Taylor's theorem, we can approximate (5) in the neighborhood of equilibrium by the linear expressions

$$(9) \quad x_i = \sum_j a_{ij} (P_j - P_j^o) + \sum_j b_{ij} (P_j' - P_j^o) \quad (i = 1, \dots, n),$$

where the  $P_j^o$  are prices which simultaneously satisfy equations (5)-(7), and where  $a_{ij} = \partial x_i / \partial P_j$  and  $b_{ij} = \partial x_i / \partial P_j'$ , both evaluated in the neighborhood of equilibrium. Substituting (9) into (8), we have

$$(10) \quad \dot{P}_i = \sum_j K_i a_{ij} (P_j - P_j^o) + \sum_j K_i b_{ij} (P_j' - P_j^o) \quad (i = 1, \dots, n).$$

Adding and subtracting  $\beta_i P_i^o$  on the right-hand side of equation (3) we obtain

$$(11) \quad \dot{P}_i' = \beta_i (P_i - P_i^o) - \beta_i (P_i' - P_i^o) \quad (i = 1, \dots, n).$$

<sup>8</sup> Since  $e^{-\beta_i t} \int_0^t \beta_i e^{\beta_i u} du \approx 1$ , if  $t$  is sufficiently large.

Equations (10) and (11) constitute the dynamic system, the properties of which we wish to investigate.

In order to simplify matters, we assume that all present commodities are gross substitutes for one another and also that each future commodity is a gross substitute for each present commodity (see Metzler [9]). That is, we assume that

$$(12) \quad \begin{array}{ll} a_{ij} \geq 0 & \text{for all } i \neq j \\ \text{and } b_{ij} \geq 0 & \text{for all } i \text{ and } j. \end{array}$$

As indicated above  $K_i$  and  $\beta_i$  are assumed to be positive. The dynamic system (10)-(11) has a matrix of the form

$$(13) \quad C = \begin{pmatrix} A & B \\ \beta & -\beta \end{pmatrix},$$

where  $A$  is a matrix  $(K_i a_{ij})$ ,  $B$  is a matrix  $(K_i b_{ij})$ , and  $\beta$  is a diagonal matrix with the elements  $\beta_i$  along the diagonal and 0 elsewhere. Under the assumptions all the off-diagonal elements of  $C$  are nonnegative.

When the  $\beta_i$  are infinitely large equations (3) reduce to

$$P_i' = P_i^s \quad (i = 1, \dots, n),$$

which, as we have indicated, defines static expectations. Subtracting  $P_i$  from both sides of these equations, we have

$$(14) \quad P_i' - P_i^s - P_i = P_i^s - P_i \quad (i = 1, \dots, n)$$

Substituting (14) into (10) we have

$$(15) \quad \dot{P}_i = \sum_j K_j (a_{ij} + b_{ij}) (P_j - P_j^s) \quad (i = 1, \dots, n).$$

Equations (15) define a dynamic system under static expectations; the matrix of this system is  $A + B$ .

Consider the case in which no expected prices enter the excess demand functions, i.e., in which the excess demand function may be written

$$(16) \quad x_i = x_i(P_1, \dots, P_n) \quad (i = 1, \dots, n).$$

The matrix of the dynamic system defined by (16) and (8) is the matrix  $A$  defined above. Metzler [9] has shown that if all present goods are gross substitutes, i.e., if all off-diagonal elements of the matrix  $A$  are positive, the system with matrix  $A$  is stable if and only if the principal minors of  $A$  alternate in sign, with sign  $(-1)^n$  where  $n$  is the order of the minor. This, of course, implies that all the elements of  $A$  along the diagonal are negative, since any commodity can be taken as the first. It is suggested below that  $A + B$  is stable when derived from competitive supply and demand functions.

We shall prove the following theorem:

**THEOREM:** *The system (10)-(11), with matrix  $C$ , is stable if and only if the system (15) with matrix  $A + B$  is stable.*

That is, we shall prove that a system of multiple markets is stable under Hicksian expectations if and only if it is stable under static expectations. We argue below that stability under static expectations is plausible.

In order to prove the theorem, we must first prove two lemmas:

**LEMMA 1:** *If  $M$  is a matrix with nonnegative off-diagonal elements, then there exists a vector  $x \geq 0$  such that  $Mx = \varphi(M)x$ , where  $\varphi(M)$  is the largest of the real parts of the characteristic roots of  $M$ .*

**PROOF:** Choose a real number  $s$  so that  $s + m_{ii} \geq 0$  for all  $i$ . Then  $sI + M$  is a nonnegative matrix whose characteristic roots are  $s$  greater than those of  $M$ . If  $\lambda$  is a characteristic root of  $M$ ,  $\lambda + s$  is a characteristic root of  $sI + M$ .  $sI + M$  has a real characteristic root,  $\lambda_0$ , such that  $|\lambda + s| \leq \lambda_0$ , and with  $\lambda_0$  there can be associated an eigen-vector  $x \geq 0$ .<sup>9</sup> The real part of  $\lambda + s$  does not exceed  $|\lambda + s|$  and hence does not exceed  $\lambda_0$ . Thus the real part of  $\lambda$  does not exceed  $\lambda_0 - s$ ; since  $\lambda$  was any characteristic root of  $M$ , it follows by definition that  $\varphi(M) \leq \lambda_0 - s$ . But  $\lambda_0 - s$  is a real characteristic root of  $M$ , so that  $\varphi(M) = \lambda_0 - s$ ; since  $(sI + M)x = \lambda_0 x$ , the vector  $x$  fulfills the conditions of the lemma.

**LEMMA 2:** *If  $M$  is a matrix with nonnegative off-diagonal elements, then  $\varphi(M) \geq 0$  if and only if there exists a vector  $x \geq 0$  such that  $Mx \geq 0$ .*

**PROOF:** Consider the matrix  $sI + M$ . Suppose there exists an  $x$  as given by the hypothesis. Since  $sI + M$  is nonnegative by definition,

$$(17) \quad sI + M \geq 0.$$

Since  $Mx \geq 0$ , it follows by adding  $sx$  to both sides that

$$(18) \quad (sI + M)x \geq sx,$$

where  $x \geq 0$ . Lemma 2 in Enthoven and Arrow [5] states that for any non-negative matrix  $A$ , if  $Ax \geq \lambda x$  for some real  $\lambda$  and some  $x \geq 0$ , then  $\lambda \leq \varphi(A)$ . It follows from (17) and (18) that

$$(19) \quad s \leq \varphi(sI + M),$$

so that  $\varphi(M) \geq 0$ .

Conversely suppose that  $\varphi(M) \geq 0$ . By Lemma 1, there exists an  $x \geq 0$  such that  $Mx = \varphi(M)x$ , so that  $Mx \geq 0$ .

<sup>9</sup> See Debreu and Herstein [4], p. 600, Theorem I\*. For a vector  $x$ ,  $x \geq 0$  means  $x_i \geq 0$  for all  $i$ ,  $x \geq 0$  means  $x \geq 0$ ,  $x \neq 0$ .

We are now in a position to prove the theorem stated above.

Recall that a system with matrix  $M$  is stable if and only if all characteristic roots have negative real parts, i.e., if and only if  $\varphi(M) < 0$ .

To prove necessity, we shall show that the instability of  $A + B$  implies the instability of  $C$ . If  $A + B$  is unstable, then  $\varphi(A + B) \geq 0$ . The matrix  $A + B$  clearly has nonnegative off-diagonal elements. Hence, by Lemma 2, there exists a vector  $p \geq 0$  such that

$$(20) \quad (A + B) p \geq 0,$$

or,

$$(21) \quad Ap + Bp \geq 0.$$

Also,  $\beta p - \beta p = 0$ . Comparison with (13) shows that  $Cq \geq 0$ , where  $q$  is the vector  $\begin{pmatrix} p \\ p \end{pmatrix}$ , so that  $q \geq 0$ . It follows from Lemma 2 that  $\varphi(C) \geq 0$ , so that the system (10)-(11), with matrix  $C$ , is unstable.

To prove *sufficiency* we shall show that the instability of  $C$  implies the instability of  $A + B$ . Suppose that system (10)-(11) is unstable, then, by Lemma 2, there exists a vector  $q$  of length  $2n$  such that

$$(22) \quad Cq \geq 0,$$

and

$$(23) \quad q \geq 0.$$

Let  $p$  be the first  $n$  elements of  $q$ , and  $\bar{p}$  be the remaining  $n$  elements. (22) may then be expanded into

$$(24) \quad Ap + B\bar{p} \geq 0,$$

and

$$(25) \quad \beta p - \beta \bar{p} \geq 0.$$

(23) implies

$$(26) \quad \text{either } p \geq 0 \text{ or } \bar{p} \geq 0.$$

(25) can be rewritten

$$\beta_i (p_i - \bar{p}_i) \geq 0 \quad (i = 1, \dots, n)$$

which is equivalent to

$$(27) \quad p \geq \bar{p},$$

since  $\beta_i > 0$  for all  $i$ . By (27) we see that if  $\bar{p} \geq 0$ , then  $p \geq 0$ ; hence (26) and (27) together imply that

$$(28) \quad p \geq 0.$$

Since, under the assumptions, each future commodity is a gross substitute

for each present commodity, the matrix  $B$  is nonnegative. Hence, (27) implies that  $B\hat{p} \geq B\bar{p}$ . Consequently, (24) implies that

$$(29) \quad (A + B)\hat{p} \geq 0.$$

Since all present commodities are gross substitutes for each other, the matrix  $A$  has nonnegative off-diagonal elements and, hence, so has  $A + B$ . Hence, (28) and (29) together imply that  $\varphi(A + B) \geq 0$ , by Lemma 2. Consequently, the system under static expectations, with matrix  $A + B$ , is unstable.

We have shown that the instability of  $A + B$  is a necessary and sufficient condition for the instability of  $C$ ; or, what is the same, we have shown that  $C$  is stable if and only if  $A + B$  is stable.

### 3. CONCLUSIONS

Enthoven and Arrow [5] show that a stable dynamic system can absorb the effects of some extrapolation of price movements and yet remain stable. The critical amount of extrapolation is related to the inertia of the system, i.e., the size of the parameters  $K_i$  which enter equations (8). We show that under adaptive expectations, a dynamic system, stable under static expectations, remains stable no matter what the inertia of the system or the elasticities of expectations. Under the assumption that expectations for all future time periods can be represented by one "expected" price, adaptive expectations appear to be a more reasonable formulation than extrapolative expectations.

The relevance of the theorem proved above can be further extended: It can be argued that the stability of the dynamic system under static expectations, (15), is plausible;<sup>10</sup> hence, by the theorem proved, stability, under the more general assumption of adaptive expectations is plausible. Consider any possible way of dividing the  $n$  commodities into two groups. Suppose that the relative prices within each group remain constant, so that each group can be regarded as a composite commodity in the sense of Hicks.<sup>11</sup> Let the price ratio between the two composite commodities vary. If individuals maximize profit or utility under competitive conditions, and if the market demand and supply functions are the sums of the individual demand and supply functions, it can be shown that the excess demand for the first composite commodity will be negative for a sufficiently high relative price and positive for a sufficiently low relative price. If all present commodities are gross substitutes, it can then be shown that a system such as (16)-(8), which does not involve expectations of the future, is necessarily stable. Suppose now that expectations of future prices are introduced. In equilibrium

<sup>10</sup> See K. Arrow and L. Hurwicz [1].

<sup>11</sup> See Hicks [7], pp. 312-313.

the planned excess demands for any future time will be the same as the current excess demands if expectations are static. Under the same assumption expectations may differ from current prices, but expected future prices for the same commodity relative to one another are constant and indeed identical. Hence, the same commodity at different points in future time is a composite commodity in the sense of Hicks, and the same reasoning used to show that a system such as (16)-(8) is stable may be used to show that a system such as (15) is stable.

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