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A TARGET-ASSIGNMENT PROBLEM*

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This paper is concerned with a target assignment model of a probabilistic and nonlinear nature, but nevertheless one which is closely related to the 'personnel-assignment' problem. It is shown here that, despite the apparent nonlinearities, it is possible to devise a linear programming formulation that will ordinarily provide a close approximation to the original problem.

AT A RECENT CONFERENCE,† MERRILL FLOOD informally described a target-assignment model that he considered to be of some military relevance. He pointed out that although the mathematical form involved optimization subject to constraints similar to those of the 'personnel assignment' problem,[5] the minimand involved probabilities, and was of a distinctly nonlinear form. From these facts he felt tempted to conclude that linear programming would be of no avail to him.

This note is to suggest that by a fairly minor modification of the original problem, and then a transformation of variables, it is possible to recast the model into linear programming form—and indeed, into a special case of linear programming under uncertainty. In this form, specific numerical solutions should require little more than clerical talent.

THE ASSIGNMENT PROBLEM

IT IS ASSUMED that there are m guns, labelled $i=1, 2, \cdots, m$, and that these are to be assigned against n targets, indicated by the subscript $j=1, 2, \cdots, n$. The objective is stated as one of minimizing the expected value of the surviving targets. More formally, if one lets x_{ij} denote the probability with which we assign the ith gun to the jth target, a_j the unit worth of the jth target, and p_{ij} the conditional probability that the ith gun will destroy the jth target—given that the target survives all other guns, then Flood's problem becomes one of selecting values for the x_{ij} so as to minimize‡

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‡ Note that if the ith gun is assigned to the jth target with a probability of x_{ij}, the probability of surviving this gun is $(1-x_{ij})+x_{ij}\,(1-p_{ij})=(1-p_{ij}\,x_{ij})$.

$$\sum_{j=1}^{j=n} a_j \prod_{i=1}^{i=m} (1-p_{ij}\,x_{ij}), \tag{1}$$

subject to

$$\sum_{j=1}^{j=n} x_{ij}=1 \qquad (i=1, \cdots, m) \tag{2}$$

and

$$x_{ij} \geqq 0. \tag{3}$$

No reader of this JOURNAL will need to be reminded that Flood's model is intentionally a highly simplified one. The typical military problem is one in which target assignments are to be made sequentially—not simultaneously as assumed here. The typical military problem is also one in which it is extremely difficult to make any meaningful estimates of the individual kill probabilities p_{ij} or of the target values a_j. However, if one is willing to assume that these difficulties can be overcome, and if one is also willing to accept the idea that the *expected* value of the survivors is all that matters, then it seems ungracious to cavil at the relatively minor issues—e.g., the idea that a gun assigned to one point has no chance at all of scoring a hit on other targets nearby, or the idea that the assignment probabilities x_{ij} may lie in the interval between zero and one. What will concern us here is much less the realism than the mathematical structure of Flood's problem.

Minimand (1) is clearly of a nonlinear nature, and I know of no way to obtain an exact solution to the stated problem. What is proposed is to make the following not-so-heroic approximations: (I) If a gun has a nonzero probability of destroying the jth target, the individual kill probability is p_j—a value which is identical with that for all other guns which can be brought to bear upon the particular target. And (II), the approximate survival probabilities are to coincide with the original ones for integral assignments—i.e., for x_{ij} values of zero and one. If these two shortcuts are legitimate, expression (1) becomes equivalent to the following: Minimize

$$\sum_{j=1}^{j=n} a_j\,(1-p_j)^{y_j}, \tag{1.A}$$

where y_j represents the total number of guns assigned to the jth target.

In its new form, the minimand still appears distinctly nonlinear, but at this point the idea can be invoked that DANTZIG, CHARNES, and LEMKE have suggested for dealing with 'separable convex functions.'[1, 3] The individual terms $(1-p_j)^{y_j}$ are each convex decreasing functions of y_j, and there is no difficulty in providing a linear-programming analogue to such functions. (Convexity results from the fact that $1>p_j>0$.) All that needs to be done is to replace y_j with a sum of individual terms y_{kj}, to impose upper bounds of unity upon these individual terms, and to label them in decreasing sequence of their 'marginal productivity.'

Since p_j represents the probability of destroying the jth target with a single gun, $1>p_j>0$. Now if k guns are assigned against this target, the probability of survival is $(1-p_j)^k$. Hence the 'marginal productivity' of the kth gun (or reduction in total survival probability attributable to that gun) is

$$[(1-p_j)^{k-1}-(1-p_j)^k]=p_j\,(1-p_j)^{k-1}.$$

Clearly the kth term in such a series is smaller than the $k-1$st term. Hence in a minimizing solution, $y_{kj}\leqq y_{k-1,j}$. The process of minimization ensures that the components y_{kj} will be assigned positive values in ascending order of their index k, so that the objective function (1.B) will provide a close approximation to (1.A). The model now looks as follows: Minimize

$$\sum_{j=1}^{j=n} a_j[1-\sum_{k=1}^{k=m} p_j\,(1-p_j)^{k-1}\,y_{kj}], \qquad (1.\mathrm{B})$$

subject to

$$\sum_{i=1}^{i=m} x_{ij}-\sum_{k=1}^{k=m} y_{kj}=0, \qquad (j=1,\cdots,n) \quad (4)$$

$$\sum_{j=1}^{j=n} x_{ij}=1, \qquad (i=1,\cdots,m) \quad (5)$$

$$x_{ij}\geqq 0, \qquad (6)$$

$$1\geqq y_{kj}\geqq 0. \qquad (7)$$

Interestingly enough, the model is now of the same form as the 'transportation' problem. This means that in any optimal solution each of the x_{ij} and y_{kj} variables must take on the value of either zero or unity.[2] Not only does the integral nature of such solutions mean that (1.A) and (1.B) will coincide at all relevant points. This fact also eliminates any embarrassing questions about the physical interpretation of a solution that requires a gun to be assigned against each of two targets with a 50-50 probability. Fractional assignment values are automatically excluded.

A NUMERICAL ILLUSTRATION

To DEMONSTRATE the equivalence between this and the 'transportation' problem,* it seems easiest to make use of a numerical example. Hereafter we shall confine the discussion to the case of $m=n=4$ (four guns and four targets). Further, to illustrate the case of zero kill probabilities, we shall assume that gun 1 is incapable of destroying target 3, and that gun 2 cannot hit target 1. The array of variables now appears as in Table I. In this 'transportation' tableau, the column sums correspond to equations (4), the first four row sums to equations (5), and the restrictions on y_{kj}, to the inequalities (7). Both the x_{ij} and the y_{kj} are understood to be nonnegative.

All that remains is to state the 'cost' coefficients. From the minimand (1.B), we observe that those coefficients associated with the x_{ij} variables are all zero, and that those connected with the y_{kj} depend upon only two

* It is true that this is a 'transportation' problem which also involves upper bounds upon the individual y_{kj} variables. But see Dantzig[3] for a proof that such upper bounds do not alter the essential nature of things.

parameters for each target j: the kill probability p_j and the unit worth a_j. Table II contains a set of assumed numerical values for these parameters, and then the corresponding unit cost coefficients.

The optimal solution for this problem can be read off by inspection: Assign gun 1 to target 2 and all the others to target 3. In terms of our

TABLE I

	Target j				Availabilities
	1	2	3	4	
Gun 1	x_{11}	x_{12}	—	x_{14}	1
Gun 2	—	x_{22}	x_{23}	x_{24}	1
Gun 3	x_{31}	x_{32}	x_{33}	x_{34}	1
Gun 4	x_{41}	x_{42}	x_{43}	x_{44}	1
Guns of effectiveness 1	$-y_{11} \geqq -1$	$-y_{12} \geqq -1$	$-y_{13} \geqq -1$	$-y_{14} \geqq -1$	
Guns of effectiveness 2	$-y_{21} \geqq -1$	$-y_{22} \geqq -1$	$-y_{23} \geqq -1$	$-y_{24} \geqq -1$	
Guns of effectiveness 3	$-y_{31} \geqq -1$	$-y_{32} \geqq -1$	$-y_{33} \geqq -1$	$-y_{34} \geqq -1$	
Guns of effectiveness 4	—	$-y_{42} \geqq -1$	—	$-y_{44} \geqq -1$	
Requirements	0	0	0	0	

unknowns, $x_{12}=x_{23}=x_{33}=x_{43}=y_{12}=y_{13}=y_{23}=y_{33}=1$. All remaining variables are set at zero.

This assignment pattern looks peculiar only if one's attention is confined to the target value coefficients, a_j. Target 2 (with the lowest value of a_j) is attacked in preference to targets 1 and 4. The paradox is easily

TABLE II

CALCULATION OF THE y_{kj} 'COST' COEFFICIENTS

Index k					
	Target j	1	2	3	4
	Kill probability, p_j	0.2	0.9	0.5	0.4
	Target value, a_j	100	40	400	60
1	$-a_j p_j (1-p_j)^0$	−20	−36	−200	−24
2	$-a_j p_j (1-p_j)^1$	−16	−3.6	−100	−14.4
3	$-a_j p_j (1-p_j)^2$	−12.8	−0.36	−50	−8.64
4	$-a_j p_j (1-p_j)^3$	—	−0.036	—	−5.184

cleared up, however, by referring to the kill probabilities, p_j. The fact that p_1 and p_4 are so low makes it worthwhile to concentrate on the other targets, and to leave these two alone. Drawing a reckless generalization from this single example, we conclude that if the kill probability for a particular target is low, either the target should not be attacked at all, or else a considerable effort should be expended against it. An intermediate policy is unlikely to minimize the total expected value of the survivors.

This generalization does not seem like an unduly reckless one. Note that if, in an optimal solution, the 'imputed value' of a gun exceeds $a_j p_j$, the jth target should not be attacked at all, and that if this value falls below $a_j p_j (1-p_j)^{k-1}$, at least k guns should be assigned against the target. For $p_j=0$, the gap between these two critical values vanishes, and as p_j increases, the gap constantly widens. In other words, for $k>1$, and for $1>p_j>0$:

$$(d/dp_j)[a_j p_j - a_j p_j (1-p_j)^{k-1}]>0.$$

To an economist, this example will be quite reminiscent of the problem faced by a discriminating monopolist selling in two isolated markets—one with a downward-sloping marginal revenue curve and the other with a horizontal one. Either the monopolist should sell nothing at all in the second market, or else he should dump his entire surplus there.

A GENERALIZATION

George Dantzig, in a letter to the author, has proposed an ingenious generalization of the allocation method that has just been described. Since his proposal is free from the assumption made here about uniformity of all nonzero kill probabilities, it is a considerably more powerful one. His suggestion is to replace minimand (1) with the following: Minimize

$$\sum_{j=1}^{j=n} a_j \prod_{i=1}^{i=m} (1-p_{ij})^{x_{ij}}. \qquad (1.\mathrm{C})$$

Dantzig points out that the functions (1) and (1.C) take on identical values whenever the x_{ij} are integral, and that they should not be too dissimilar for fractional x_{ij}. There is, of course, no assurance that the set of x_{ij} which minimizes (1.C) will also minimize (1). The only guarantee that can be made is that since

$$(1-p_{ij} x_{ij}) \geqq (1-p_{ij})^{x_{ij}} \qquad (0 \leqq p_{ij} \leqq 1;\ 0 \leqq x_{ij} \leqq 1)$$

the minimum value of (1.C) represents a lower bound upon that of (1).*

The problem of minimizing (1.C) subject to constraint (2) and (3) is still not a linear-programming problem, but it may be converted into that form by defining new variables y_j as follows:

$$-y_j = \sum_{i=1}^{i=m} x_{ij} \log_e(1-p_{ij}). \qquad (8)$$

Then (1.C) may be written: Minimize

$$\sum_{j=1}^{j=n} a_j e^{-y_j}. \qquad (1.\mathrm{D})$$

* The lower the value of the kill probabilities p_{ij}, the closer will be the approximation that is provided by (1.C). Even for $x_{ij}=p_{ij}=0.5$, the approximation is not bad. In that case $(1-p_{ij} x_{ij})=0.750 \approx (1-p_{ij})^{x_{ij}}=0.707$.

Since e^{-y_j} is convex, the new minimand may be approximated by a convex broken-line function, and the argument proceeds as previously in the conversion of (1.A) to (1.B). The problem has been reduced to the 'transportation' form with upper bounds upon the individual components of y_j. The only major difference between the two models is that the simple column sum equation (4) is replaced by the more general linear equation (8). Numerical solutions to this class of problems are not difficult to obtain. The computing layout is now identical with that encountered by FERGUSON AND DANTZIG in their model of aircraft allocation to routes with uncertain demands.[4] Aside from some increase in numerical effort, the only objection to Dantzig's suggestion is, as he points out, the fact that the resulting linear-programming solution cannot be guaranteed to be free from fractional values for the x_{ij} variables. Offhand, it is not easy to judge whether or not the gains in realism would be outweighed by the difficulties of physical interpretation.

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