Liquidity Preference as Behavior Towards Risk[1]

One of the basic functional relationships in the Keynesian model of the economy is the liquidity preference schedule, an inverse relationship between the demand for cash balances and the rate of interest. This aggregative function must be derived from some assumptions regarding the behavior of the decision-making units of the economy, and those assumptions are the concern of this paper. Nearly two decades of drawing downward-sloping liquidity preference curves in textbooks and on classroom blackboards should not blind us to the basic implausibility of the behavior they describe. Why should anyone hold the non-interest bearing obligations of the government instead of its interest bearing obligations ? The apparent irrationality of holding cash is the same, moreover, whether the interest rate is 6 %, 3 % or ½ of 1 %. What needs to be explained is not only the existence of a demand for cash when its yield is less than the yield on alternative assets but an inverse relationship between the aggregate demand for cash and the size of this differential in yields.[2]

1. *Transactions balances and investment balances.*

Two kinds of reasons for holding cash are usually distinguished : transactions reasons and investment reasons.

1.1 *Transactions balances : size and composition.* No economic unit—firm or household or government—enjoys perfect synchronization between the seasonal patterns of its flow of receipts and its flow of expenditures. The discrepancies give rise to balances which accumulate temporarily, and are used up later in the year when expenditures catch up. Or, to put the same phenomenon the other way, the discrepancies give rise to the need for balances to meet seasonal excesses of expenditures over receipts. These balances are *transactions balances*. The aggregate requirement of the economy for such balances depends on the institutional arrangements that determine the degree of synchronization between individual receipts and expenditures. Given these institutions, the need for transactions balances is roughly proportionate to the aggregate volume of transactions.

The obvious importance of these institutional determinants of the demand for transactions balances has led to the general opinion that other possible determinants, including

[1] I am grateful to Challis Hall, Arthur Okun, Walter Salant, and Leroy Wehrle for helpful comments on earlier drafts of this paper.

[2] " . . . . in a world involving no transaction friction and no uncertainty, there would be no reason for a spread between the yield on any two assets, and hence there would be no difference in the yield on money and on securities . . . in such a world securities themselves would circulate as money and be acceptable in transactions ; demand bank deposits would bear interest, just as they often did in this country in the period of the twenties." Paul A. Samuelson, *Foundations of Economic Analysis* (Cambridge : Harvard University Press, 1947), p. 123. The section, pp. 122-124, from which the passage is quoted makes it clear that liquidity preference must be regarded as an explanation of the existence and level not of the interest rate but of the differential between the yield on money and the yields on other assets.

interest rates, are negligible.[1] This may be true of the size of transactions balances, but the composition of transactions balances is another matter. Cash is by no means the only asset in which transactions balances may be held. Many transactors have large enough balances so that holding part of them in earning assets, rather than in cash, is a relevant possibility. Even though these holdings are always for short periods, the interest earnings may be worth the cost and inconvenience of the financial transactions involved. Elsewhere[2] I have shown that, for such transactors, the proportion of cash in transactions balances varies inversely with the rate of interest ; consequently this source of interest-elasticity in the demand for cash will not be further discussed here.

1.2 *Investment balances and portfolio decisions.* In contrast to transactions balances, the investment balances of an economic unit are those that will survive all the expected seasonal excesses of cumulative expenditures over cumulative receipts during the year ahead. They are balances which will not have to be turned into cash within the year. Consequently the cost of financial transactions—converting other assets into cash and vice versa—does not operate to encourage the holding of investment balances in cash.[3] If cash is to have any part in the composition of investment balances, it must be because of expectations or fears of loss on other assets. It is here, in what Keynes called the speculative motives of investors, that the explanation of liquidity preference and of the interest-elasticity of the demand for cash has been sought.

The alternatives to cash considered, both in this paper and in prior discussions of the subject, in examining the speculative motive for holding cash are assets that differ from cash only in having a variable market yield. They are obligations to pay stated cash amounts at future dates, with no risk of default. They are, like cash, subject to changes in real value due to fluctuations in the price level. In a broader perspective, all these assets, including cash, are merely minor variants of the same species, a species we may call monetary assets—marketable, fixed in money value, free of default risk. The differences of members of this species from each other are negligible compared to their differences from the vast variety of other assets in which wealth may be invested : corporate stocks, real estate, unincorporated business and professional practice, etc. The theory of liquidity preference does not concern the choices investors make between the whole species of monetary assets, on the one hand, and other broad classes of assets, on the other.[4] Those choices are the concern of other branches of economic theory, in particular theories of investment and of consumption. Liquidity preference theory takes as given the choices determining how much wealth is to be invested in monetary assets and concerns itself with the allocation of these amounts among cash and alternative monetary assets.

[1] The traditional theory of the velocity of money has, however, probably exaggerated the invariance of the institutions determining the extent of lack of synchronization between individual receipts and expenditures. It is no doubt true that such institutions as the degree of vertical integration of production and the periodicity of wage, salary, dividend, and tax payments are slow to change. But other relevant arrangements can be adjusted in response to money rates. For example, there is a good deal of flexibility in the promptness and regularity with which bills are rendered and settled.

[2] " The Interest Elasticity of the Transactions Demand for Cash," *Review of Economics and Statistics*, vol. 38 (August 1956), pp. 241-247.

[3] Costs of financial transactions have the effect of deterring changes from the existing portfolio, whatever its composition ; they may thus operate against the holding of cash as easily as for it. Because of these costs, the *status quo* may be optimal even when a different composition of assets would be preferred if the investor were starting over again.

[4] For an attempt by the author to apply to this wider choice some of the same theoretical tools that are here used to analyze choices among the narrow class of monetary assets, see " A Dynamic Aggregative Model ", *Journal of Political Economy*, vol. 63 (April 1955), pp. 103-115.

Why should any investment balances be held in cash, in preference to other monetary assets ? We shall distinguish two possible sources of liquidity preference, while recognizing that they are not mutually exclusive. The first is inelasticity of expectations of future interest rates. The second is uncertainty about the future of interest rates. These two sources of liquidity preference will be examined in turn.

2. *Inelasticity of interest rate expectations.*

2.1 *Some simplifying assumptions.* To simplify the problem, assume that there is only one monetary asset other than cash, namely consols. The current yield of consols is r per " year ". \$1 invested in consols today will purchase an income of \$$r$ per " year " in perpetuity. The yield of cash is assumed to be zero ; however, this is not essential, as it is the current and expected differentials of consols over cash that matter. An investor with a given total balance must decide what proportion of this balance to hold in cash, A_1, and what proportion in consols, A_2. This decision is assumed to fix the portfolio for a full " year ".[1]

2.2 *Fixed expectations of future rate.* At the end of the year, the investor expects the rate on consols to be r_e. This expectation is assumed, for the present, to be held with certainty and to be independent of the current rate r. The investor may therefore expect with certainty that every dollar invested in consols today will earn over the year ahead not only the interest \$$r$, but also a capital gain or loss g :

$$g = \frac{r}{r_e} - 1 \tag{2.1}$$

For this investor, the division of his balance into proportions A_1 of cash and A_2 of consols is a simple all-or-nothing choice. If the current rate is such that $r + g$ is greater than zero, then he will put everything in consols. But if $r + g$ is less than zero, he will put everything in cash. These conditions can be expressed in terms of a critical level of the current rate r_c, where :

$$r_c = \frac{r_e}{1 + r_e} \tag{2.2}$$

At current rates above r_c, everything goes into consols ; but for r less than r_c, everything goes into cash.

2.3 *Sticky and certain interest rate expectations.* So far the investor's expected interest-rate r_e has been assumed to be completely independent of the current rate r. This assumption can be modified so long as some independence of the expected rate from the current rate is maintained. In Figure 2.1, for example, r_e is shown as a function of r, namely $\varphi(r)$. Correspondingly $\frac{r_e}{1 + r_e}$ is a function of r. As shown in the figure, this function $\frac{\varphi}{1 + \varphi}$ has only one intersection with the 45° line, and at this intersection its slope $\frac{\varphi'}{(1 + \varphi)^2}$ is less than one. If these conditions are met, the intersection determines a critical rate r_c such that if r exceeds r_c the investor holds no cash, while if r is less than r_c he holds no consols.

[2] As noted above, it is the costs of financial transactions that impart inertia to portfolio composition. Every reconsideration of the portfolio involves the investor in expenditure of time and effort as well as of money. The frequency with which it is worth while to review the portfolio will obviously vary with the investor and will depend on the size of his portfolio and on his situation with respect to costs of obtaining information and engaging in financial transactions. Thus the relevant " year " ahead for which portfolio decisions are made is not the same for all investors. Moreover, even if a decision is made with a view to fixing a portfolio for a given period of time, a portfolio is never so irrevocably frozen that there are no conceivable events during the period which would induce the investor to reconsider. The fact that this possibility is always open must influence the investor's decision. The fiction of a fixed investment period used in this paper is, therefore, not a wholly satisfactory way of taking account of the inertia in portfolio composition due to the costs of transactions and of decision making.

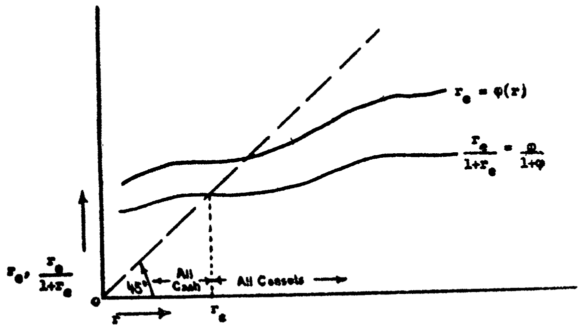

FIGURE 2.1

Stickiness in the Relation between Expected and Current Interest Rate.

2.4 *Differences of opinion and the aggregate demand for cash.* According to this model, the relationship of the individual's investment demand for cash to the current rate of interest would be the discontinuous step function shown by the heavy vertical lines $LMNW$ in Figure 2.2 How then do we get the familiar Keynesian liquidity preference function, a smooth, continuous inverse relationship between the demand for cash and the rate of interest ? For the economy as a whole, such a relationship can be derived from individual behaviour of the sort depicted in Figure 2.2 by assuming that inaividual investors diffei in their critical rates r_c. Such an aggregate relationship is shown in Figure 2.3.

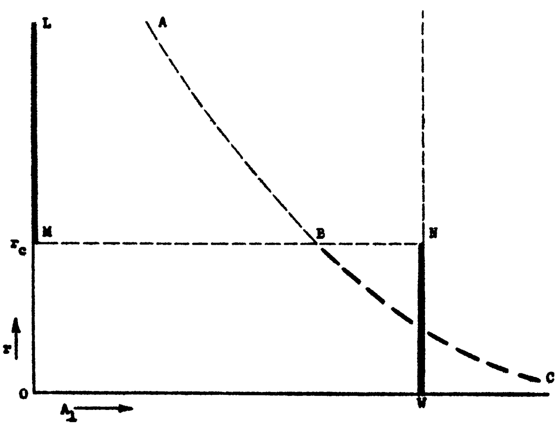

FIGURE 2.2

Individual Demand for Cash Assuming Certain but Inelastic Interest Rate Expectations

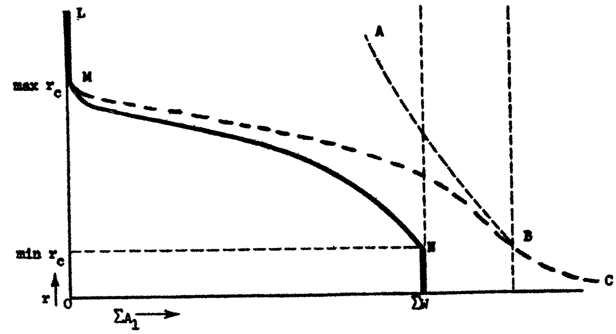

FIGURE 2.3

Aggregate Demand for Cash Assuming Differences Among Individuals in Interest Rate Expectations.

At actual rates above the maximum of individual critical rates the aggregate demand for cash is zero, while at rates below the minimum critical rate it is equal to the total investment balances for the whole economy. Between these two extremes the demand for cash varies inversely with the rate of interest r. Such a relationship is shown as $LMN\Sigma W$ in Figure 2.3. The demand for cash at r is the total of investment balances controlled by investors whose critical rates r_c exceed r. Strictly speaking, the curve is a step function ; but, if the number of investors is large, it can be approximated by a smooth curve. Its shape depends on the distribution of dollars of investment balances by the critical rate of the investor controlling them ; the shape of the curve in Figure 2.3 follows from a unimodal distribution.

2.5 *Capital gains or losses and open market operations.* In the foregoing analysis the size of investment balances has been taken as independent of the current rate on consols r. This is not the case if there are already consols outstanding. Their value will depend inversely on the current rate of interest. Depending on the relation of the current rate to the previously fixed coupon on consols, owners of consols will receive capital gains or losses. Thus the investment balances of an individual owner of consols would not be constant at W but would depend on r in a manner illustrated by the curve ABC in Figure 2.2.[1] Similarly, the investment balances for the whole economy would follow a curve like ABC in Figure 2.3, instead of being constant at ΣW. The demand for cash would then be described by $LMBC$ in both figures. Correspondingly the demand for consols at any interest rate would be described by the horizontal distance between $LMBC$ and ABC. The value of consols goes to infinity as the rate of interest approaches zero ; for this reason, the curve BC may never reach the horizontal axis. The size of investment balances would be bounded if the monetary assets other than cash consisted of bonds with definite maturities rather than consols.

According to this theory, a curve like $LMBC$ depicts the terms on which a central bank can engage in open-market operations, given the claims for future payments outstanding in the form of bonds or consols. The curve tells what the quantity of cash must be in order for the central bank to establish a particular interest rate. However, the curve will

[1] The size of their investment balances, held in cash and consols may not vary by the full amount of these changes in wealth ; some part of the changes may be reflected in holdings of assets other than monetary assets. But presumably the size of investment balances will reflect at least in part these capital gains and losses.

be shifted by open market operations themselves, since they will change the volume of outstanding bonds or consols. For example, to establish the rate at or below *min* r_c, the central bank would have to buy all outstanding bonds or consols. The size of the community's investment balances would then be independent of the rate of interest ; it would be represented by a vertical line through, or to the right of, *B*, rather than the curve *ABC*. Thus the new relation between cash and interest would be a curve lying above *LMB*, of the same general contour as $LMN\Sigma W$.

2.6 *Keynesian theory and its critics.* I believe the theory of liquidity preference I have just presented is essentially the original Keynesian explanation. The *General Theory* suggests a number of possible theoretical explanations, supported and enriched by the experience and insight of the author. But the explanation to which Keynes gave the greatest emphasis is the notion of a " normal " long-term rate, to which investors expect the rate of interest to return. When he refers to uncertainty in the market, he appears to mean disagreement among investors concerning the future of the rate rather than subjective doubt in the mind of an individual investor.[1] Thus Kaldor's correction of Keynes is more verbal than substantive when he says, " It is . . . not so much the *uncertainty* concerning future interest rates as the *inelasticity* of interest expectations which is responsible for Mr. Keynes' ' liquidity preference function,' . . ."[2]

Keynes' use of this explanation of liquidity preference as a part of his theory of underemployment equilibrium was the target of important criticism by Leontief and Fellner. Leontief argued that liquidity preference must necessarily be zero *in equilibrium*, regardless of the rate of interest. Divergence between the current and expected interest rate is bound to vanish as investors learn from experience ; no matter how low an interest rate may be, it can be accepted as " normal " if it persists long enough. This criticism was a part of Leontief's general methodological criticism of Keynes, that unemployment was not a feature of equilibrium, subject to analysis by tools of static theory, but a phenomenon of disequilibrium requiring analysis by dynamic theory.[3] Fellner makes a similar criticism of the logical appropriateness of Keynes' explanation of liquidity preference for the purposes of his theory of underemployment equilibrium. Why, he asks, are interest rates the only variables to which inelastic expectations attach ? Why don't wealth owners and others regard pre-depression price levels as " normal " levels to which prices will return ? If they did, consumption and investment demand would respond to reductions in money wages and prices, no matter how strong and how elastic the liquidity preference of investors.[4]

These criticisms raise the question whether it is possible to dispense with the assumption of stickiness in interest rate expectations without losing the implication that Keynesian

[1] J. M. Keynes, *The General Theory of Employment, Interest, and Money* (New York : Harcourt Brace, 1936), Chapters 13 and 15, especially pp. 168-172 and 201-203. One quotation from p. 172 will illustrate the point : " It is interesting that the stability of the system and its sensitiveness to changes in the quantity of money should be so dependent on the existence of a *variety* of opinion about what is uncertain. Best of all that we should know the future. But if not, then, if we are to control the activity of the economic system by changing the quantity of money, it is important that opinions should differ."

[2] N. Kaldor, " Speculation and Economic Stability," *Review of Economic Studies*, vol. 7 (1939), p. 15.

[3] W. Leontief, " Postulates : Keynes' General Theory and the Classicists ", Chapter XIX in S. Harris, editor, *The New Economics* (New York : Knopf, 1947), pp. 232-242. Section 6, pp. 238-239, contains the specific criticism of Keynes' liquidity preference theory.

[4] W. Fellner, *Monetary Policies and Full Employment* (Berkeley : University of California Press, 1946), p. 149.

theory drew from it. Can the inverse relationship of demand for cash to the rate of interest be based on a different set of assumptions about the behaviour of individual investors ? This question is the subject of the next part of the paper.

3. *Uncertainty, risk aversion, and liquidity preference.*

3.1 *The locus of opportunity for risk and expected return.* Suppose that an investor is not certain of the future rate of interest on consols ; investment in consols then involves a risk of capital gain or loss. The higher the proportion of his investment balance that he holds in consols, the more risk the investor assumes. At the same time, increasing the proportion in consols also increases his expected return. In the upper half of Figure 3.1, the vertical axis represents expected return and the horizontal axis risk. A line such as OC_1 pictures the fact that the investor can expect more return if he assumes more risk. In the lower half of Figure 3.1, the left-hand vertical axis measures the proportion invested in consols. A line like OB shows risk as proportional to the share of the total balance held in consols.

The concepts of expected return and risk must be given more precisio...

The individual investor of the previous section was assumed to have, for any current rate of interest, a definite expectation of the capital gain or loss g (defined in expression (2.1) above) he would obtain by investing one dollar in consols. Now he will be assumed instead to be uncertain about g but to base his actions on his estimate of its probability distribution. This probability distribution, it will be assumed, has an expected value of zero and is independent of the level of r, the current rate on consols. Thus the investor considers a doubling of the rate just as likely when rate is 5% as when it is 2%, and a halving of the rate just as likely when it is 1% as when it is 6%.

A portfolio consists of a proportion A_1 of cash and A_2 of consols, where A_1 and A_2 add up to 1. We shall assume that A_1 and A_2 do not depend on the absolute size of the initial investment balance in dollars. Negative values of A_1 and A_2 are excluded by definition ; only the government and the banking system can issue cash and government consols. The return on a portfolio R is :

$$R = A_2 (r + g) \qquad 0 \leqslant A_2 \leqslant 1 \tag{3.1}$$

Since g is a random variable with expected value zero, the expected return on the portfolio is :

$$E(R) = \mu_R = A_2 r. \tag{3.2}$$

The risk attached to a portfolio is to be measured by the standard deviation of R, σ_R. The standard deviation is a measure of the dispersion of possible returns around the mean value μ_R. A high standard deviation means, speaking roughly, high probability of large deviations from μ_R, both positive and negative. A low standard deviation means low probability of large deviations from μ_R; in the extreme case, a zero standard deviation would indicate certainty of receiving the return μ_R. Thus a high-σ_R portfolio offers the investor the chance of large capital gains at the price of equivalent chances of large capital losses. A low-σ_R portfolio protects the investor from capital loss, and likewise gives him little prospect of unusual gains. Although it is intuitively clear that the risk of a portfolio is to be identified with the dispersion of possible returns, the standard deviation is neither the sole measure of dispersion nor the obviously most relevant measure. The case for the standard deviation will be further discussed in section 3.3 below.

The standard deviation of R depends on the standard deviation of g, σ_g, and on the amount invested in consols :

$$\sigma_R = A_2 \sigma_g \qquad 0 \leqslant A_2 \leqslant 1. \tag{3.3}$$

Thus the proportion the investor holds in consols A_2 determines both his expected return μ_R and his risk σ_R. The terms on which the investor can obtain greater expected return at the expense of assuming more risk can be derived from (3.2) and (3.3) :

$$\mu_R = \frac{r}{\sigma_g} \sigma_R \qquad 0 \leqslant \sigma_R \leqslant \sigma_g \tag{3.4}$$

Such an *opportunity locus* is shown as line OC_1 (for $r = r_1$) in Figure 3.1. The slope of the line is $\frac{r_1}{\sigma_g}$. For a higher interest rate r_2, the opportunity locus would be OC_2; and for r_3, a still higher rate, it would be OC_3. The relationship (3.3) between risk and investment in consols is shown as line OB in the lower half of the Figure. Cash holding $A_1(= 1 - A_2)$ can also be read off the diagram on the right-hand vertical axis.

3.2 *Loci of indifference between combinations of risk and expected return.* The investor is assumed to have preferences between expected return μ_R and risk σ_R that can be represented by a field of indifference curves. The investor is indifferent between all pairs (μ_R, σ_R) that lie on a curve such as I_1 in Figure 3.1. Points on I_2 are preferred to those on I_1; for given risk, an investor always prefers a greater to a smaller expectation of return. Conceivably, for some investors, *risk-lovers*, these indifference curves have negative slopes. Such individuals are willing to accept lower expected return in order to have the chance of unusually high capital gains afforded by high values of σ_R. *Risk-averters*, on the other

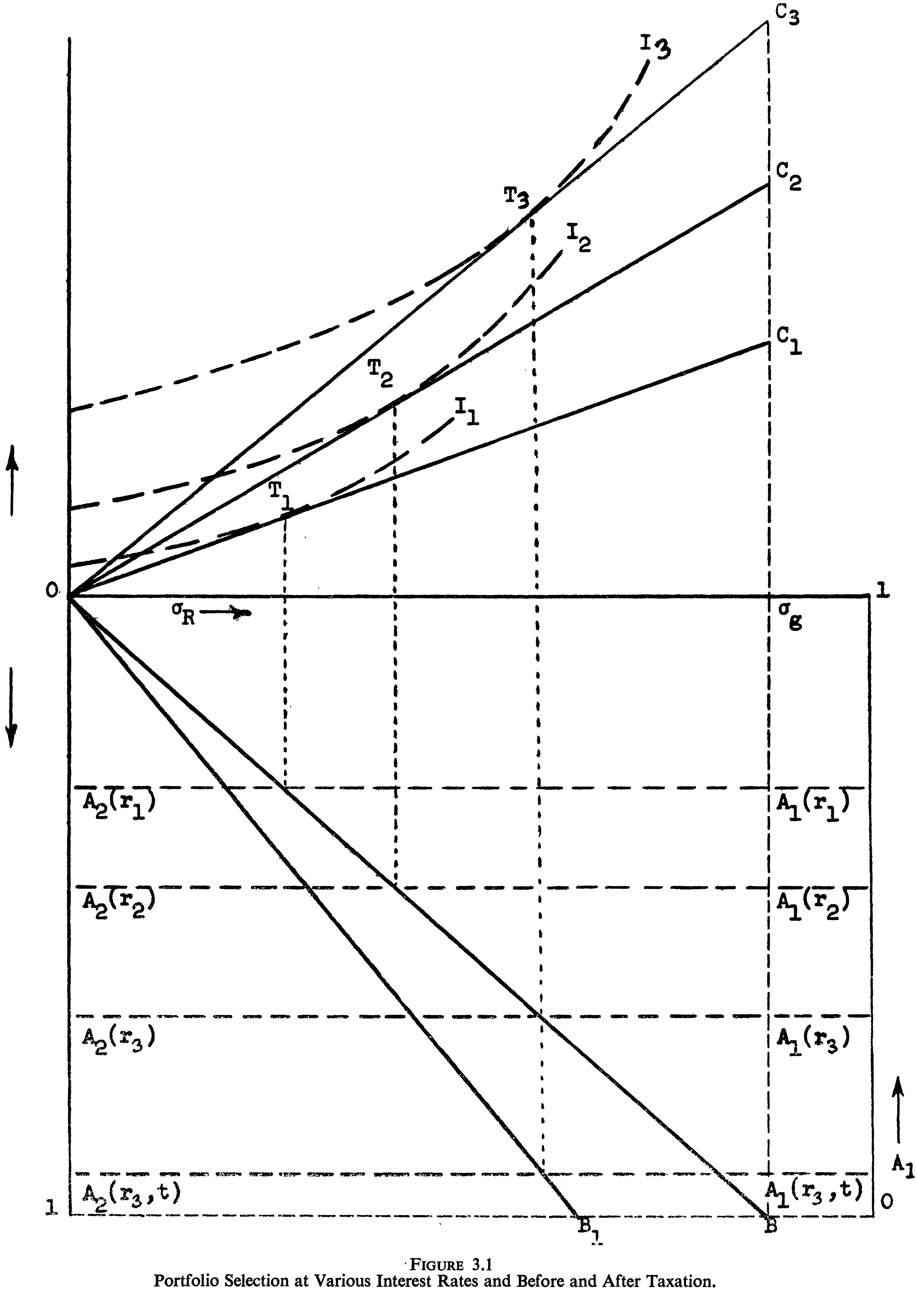

FIGURE 3.1
Portfolio Selection at Various Interest Rates and Before and After Taxation.

hand, will not be satisfied to accept more risk unless they can also expect greater expected return. Their indifference curves will be positively sloped. Two kinds of risk-averters need to be distinguished. The first type, who may be called *diversifiers* for reasons that will become clear below, have indifference curves that are concave upward, like those in Figure 3.1. The second type, who may be called *plungers*, have indifference curves that are upward sloping, but either linear or convex upward.

3.3 *Indifference curves as loci of constant expected utility of wealth.* The reader who is willing to accept the indifference fields that have just been introduced into the analysis may skip to section 3.4 without losing the main thread of the argument. But these indifference curves need some explanation and defence. Indifference curves between μ_R and σ_R do not necessarily exist. It is a simplification to assume that the investor chooses among the alternative probability distributions of R available to him on the basis of only two parameters of those distributions. Even if this simplification is accepted, the mean and standard deviation may not be the pair of parameters that concern the investor.

3.3.1 One justification for the use of indifference curves between μ_R and σ_R would be that the investor evaluates the future of consols only in terms of some two-parameter family of probability distributions of g. For example, the investor might think in terms of a range of equally likely gains or losses, centered on zero. Or he might think in terms that can be approximated by a normal distribution. Whatever two-parameter family is assumed—uniform, normal, or some other—the whole probability distribution is determined as soon as the mean and standard deviation are specified. Hence the investor's choice among probability distributions can be analyzed by μ_R-σ_R indifference curves ; any other pair of independent parameters could serve equally well.

If the investor's probability distributions are assumed to belong to some two-parameter family, the shape of his indifference curves can be inferred from the general characteristics of his utility-of-return function. This function will be assumed to relate utility to R, the percentage growth in the investment balance by the end of the period. This way of formulating the utility function makes the investor's indifference map, and therefore his choices of proportions of cash and consols, independent of the absolute amount of his initial balance.

On certain postulates, it can be shown that an individual's choice among probability distributions can be described as the maximization of the expected value of a utility function.[1] The ranking of probability distributions with respect to the expected value of utility will not be changed if the scale on which utility is measured is altered either by the addition of a constant or by multiplication by a positive constant. Consequently we are free to choose arbitrarily the zero and unit of measurement of the utility function $U(R)$ as follows : $U(0) = 0$; $U(-1) = -1$.

[1] See Von Neumann, J. and Morgenstern, O., *Theory of Games and Economic Behavior*, 3rd Edition (Princeton : Princeton University Press, 1953), pp. 15-30, pp. 617-632 ; Herstein, I. N. and Milnor, J., " An Axiomatic Approach to Measurable Utility ", *Econometrica*, vol. 23 (April 1953), pp. 291-297 ; Marschak, J., " Rational Behavior, Uncertain Prospects, and Measurable Utility ", *Econometrica*, vol. 18 (April 1950), pp. 111-141 ; Friedman, M. and Savage, L. J., " The Utility Analysis of Choices Involving Risk ", *Journal of Political Economy*, vol. 56 (August 1948), pp. 279-304, and " The Expected Utility Hypothesis and the Measurability of Utility ", *Journal of Political Economy*, vol. 60 (December 1952), pp. 463-474. For a treatment which also provides an axiomatic basis for the subjective probability estimates here assumed, see Savage, L. J., *The Foundations of Statistics* (New York : Wiley, 1954).

Suppose that the probability distribution of R can be described by a two-parameter density function $f(R; \mu_R, \sigma_R)$. Then the expected value of utility is :

$$E[U(R)] = \int_{-\infty}^{\infty} U(R) f(R; \mu_R, \sigma_R)\, dR. \tag{3.5}$$

Let $z = \dfrac{R - \mu_R}{\sigma_R}$.

$$E[U(R)] = E(\mu_R, \sigma_R) = \int_{-\infty}^{\infty} U(\mu_R + \sigma_R z) f(z; 0, 1)\, dz. \tag{3.6}$$

An indifference curve is a locus of points (μ_R, σ_R) along which expected utility is constant. We may find the slope of such a locus by differentiating (3.6) with respect to σ_R:

$$0 = \int_{-\infty}^{\infty} U'(\mu_R + \sigma_R z) \left[\frac{d\mu_R}{d\sigma_R} + z\right] f(z; 0, 1)\, dz.$$

3.7)

$$\frac{d\mu_R}{d\sigma_R} = - \frac{\displaystyle\int_{-\infty}^{\infty} z\, U'(R) f(z; 0, 1)\, dz}{\displaystyle\int_{-\infty}^{\infty} U'(R) f(z; 0, 1)\, dz}$$

$U'(R)$, the marginal utility of return, is assumed to be everywhere non-negative. If it is also a decreasing function of R, then the slope of the indifference locus must be positive; an investor with such a utility function is a risk-averter. If it is an increasing function of R, the slope will be negative ; this kind of utility function characterizes a risk-lover.

Similarly, the curvature of the indifference loci is related to the shape of the utility function. Suppose that (μ_R, σ_R) and (μ_R', σ_R') are on the same indifference locus, so that $E(\mu_R, \sigma_R) = E(\mu_R, \sigma_R)$. Is $\left(\dfrac{\mu_R + \mu_R'}{2}, \dfrac{\sigma_R + \sigma_R'}{2}\right)$ on the same locus, or on a higher or a lower one ? In the case of declining marginal utility we know that for every z :

$$\tfrac{1}{2} U(\mu_R + \sigma_R z) + \tfrac{1}{2} U(\mu_R' + \sigma_R' z) < U\left(\frac{\mu_R + \mu_R'}{2} + \frac{\sigma_R + \sigma_R'}{2} z\right)$$

Consequently $E\left(\dfrac{\mu_R + \mu_R'}{2}, \dfrac{\sigma_R + \sigma_R'}{2}\right)$ is greater than $E(\mu_R, \sigma_R)$ or $E(\mu_R', \sigma_R')$, and

$\left(\frac{\mu_R + \mu_R'}{2}, \frac{\sigma_R + \sigma_R'}{2}\right)$, which lies on a line between (μ_R, σ_R) and (μ_R', σ_R'), is on a higher locus than those points. Thus it is shown that a risk-averter's indifference curve is necessarily concave upwards, provided it is derived in this manner from a two-parameter family of probability distributions and declining marginal utility of return. All risk-averters are diversifiers ; plungers do not exist. The same kind of argument shows that a risk-lover's indifference curve is concave downwards.

3.3.2 In the absence of restrictions on the subjective probability distributions of the investor, the parameters of the distribution relevant to his choice can be sought in parametric restrictions on his utility-of-return function. Two parameters of the utility function are determined by the choice of the utility scale. If specification of the utility function requires no additional parameters, one parameter of the probability distribution summarizes all the information relevant for the investor's choice. For example, if the utility function is linear $[U(R) = R]$, then the expected value of utility is simply the expected value of R, and maximizing expected utility leads to the same behaviour as maximizing return in a world of certainty. If, however, one additional parameter is needed to specify the utility function, then two parameters of the probability distribution will be relevant to the choice ; and so on. Which parameters of the distribution are relevant depends on the form of the utility function.

Focus on the mean and standard deviation of return can be justified on the assumption that the utility function is quadratic. Following our conventions as to utility scale, the quadratic function would be :

(3.8) $$U(R) = (1 + b)R + bR^2$$

Here $0 < b < 1$ for a risk-lover, and $-1 < b < 0$ for a risk-averter. However (3.8) cannot describe the utility function for the whole range of R, because marginal utility cannot be negative. The function given in (3.8) can apply only for :

$$(1 + b) + 2bR \geqq 0 ;$$

that is, for :

(3.9) $$R \geqq -\left(\frac{1+b}{2b}\right) \quad (b > 0) \quad \text{(Risk-lover)}$$

$$R \leqq -\left(\frac{1+b}{2b}\right) \quad (b < 0) \quad \text{(Risk-averter).}$$

In order to use (3.8), therefore, we must exclude from the range of possibility values of R outside the limits (3.9). At the maximum investment in consols ($A_2 = 1$), $R = r + g$. A risk-averter must be assumed therefore, to restrict the range of capital gains g to which he attaches non-zero probability so that, for the highest rate of interest r to be considered :

(3.10) $$r + g \leqq -\left(\frac{1+b}{2b}\right).$$

The corresponding limitation for a risk-lover is that, for the lowest interest rate r to be considered :

(3.11) $$r + g \geqq -\left(\frac{1+b}{2b}\right).$$

Given the utility function (3.8), we can investigate the slope and curvature of the indifference curves it implies. The probability density function for R, $f(R)$, is restricted by the limit (3.10) or (3.11) ; but otherwise no restriction on its shape is assumed.

$$(3.12) \qquad E\,[U\,(R)\,] = \int_{-\infty}^{\infty} U\,(R)\,f\,(R)\,d\,R = (1 + b)\,\mu_R + b\,(\sigma_R^2 + \mu_R^2).$$

Holding $E\,[U\,(R)\,]$ constant and differentiating with respect to σ_R to obtain the slope of au indifference curve, we have :

$$(3.13) \qquad \frac{d\,\mu_R}{d\,\sigma_R} = \frac{\sigma_R}{-\dfrac{1+b}{2b} - \mu_R}$$

For a risk-averter, $-\dfrac{1+b}{2b}$ is positive and is the upper limit for R, according to (3.9) ; $-\dfrac{1+b}{2b}$ is necessarily larger than μ_R. Therefore the slope of an indifference locus is positive. For a risk-lover, on the other hand, the corresponding argument shows that the slope is negative.

Differentiating (3.13) leads to the same conclusions regarding curvature as the alternative approach of section 3.3.1, namely that a risk-averter is necessarily a diversifier.

$$(3.14) \qquad \frac{d^2\,\mu_R}{d\,\sigma_R} = \frac{1 + \left(\dfrac{d\,\mu_R{}^2}{d\,\sigma_R}\right)}{\left(-\dfrac{1+b}{2b} - \mu_R\right)^2}$$

For a risk-averter, the second derivative is positive and the indifference locus is concave upwards ; for a risk-lover, it is concave downwards.

3.4 *Effects of changes in the rate of interest.* In section 3.3 two alternative rationalizations of the indifference curves introduced in section 3.2 have been presented. Both rationalizations assume that the investor (1) estimates subjective probability distributions of capital gain or loss in holding consols, (2) evaluates his prospective increase in wealth in terms of a cardinal utility function, (3) ranks alternative prospects according to the expected value of utility. The rationalization of section 3.3.1 derives the indifference curves by restricting the subjective probability distributions to a two-parameter family. The rationalization of section 3.3.2 derives the indifference curves by assuming the utility function to be quadratic within the relevant range. On either rationalization, a risk-averter's indifference curves must be concave upwards, characteristic of the diversifiers of section 3.2, and those of a risk-lover concave downwards. If the category defined as *plungers* in 3.2 exists at all, their indifference curves must be determined by some process other than those described in 3.3.

The opportunity locus for the investor is described in 3.1 and summarized in equation (3.4). The investor decides the amount to invest in consols so as to reach the highest indifference curve permitted by his opportunity-locus. This maximization may be one of three kinds :

I. Tangency between an indifference curve and the opportunity locus, as illustrated by points T_1, T_2, and T_3 in Figure 3.1. A regular maximum of this kind can occur only for a risk-averter, and will lead to diversification. Both A_1, cash holding, and A_2, consol holding, will be positive. They too are shown in Figure 3.1, in the bottom half of the diagram, where, for example, $A_1\,(r_1)$ and $A_2\,(r_1)$ depict the cash and consol holdings corresponding to point T_1.

II. A corner maximum at the point $\mu_R = r$, $\sigma_R = \sigma_g$, as illustrated in Figure 3.2.

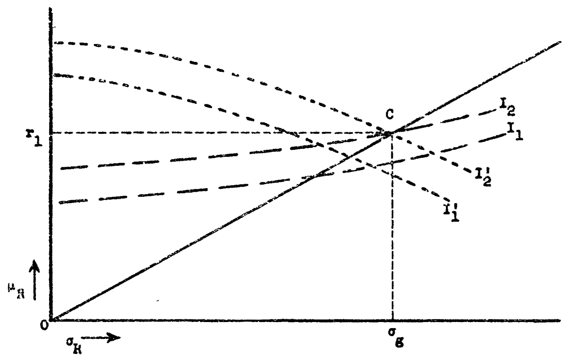

FIGURE 3.2

" Risk-lovers " and " Diversifiers " : Optimum Portfolio at Maximum Risk and Expected Return.

In Figure 3.2 the opportunity locus is the ray OC, and point C represents the highest expected return and risk obtainable by the investor i.e. the expected return and risk from holding his entire balance in consols. A utility maximum at C can occur either for a risk-averter or for a ıisk-lover. I_1 and I_2 represent indifference curves of a diversifier ; I_2 passes through C and has a lower slope, both at C and everywhere to the left of C, than the opportunity locus. I_1' and I_2' represent the indifference curves of a risk-lover, for whom it is clear that C is always the optimum position. Similarly, a plunger may, if his indifference curves stand with respect to his opportunity locus as in Figure 3.3 (OC_2) plunge his entire balance in consols.

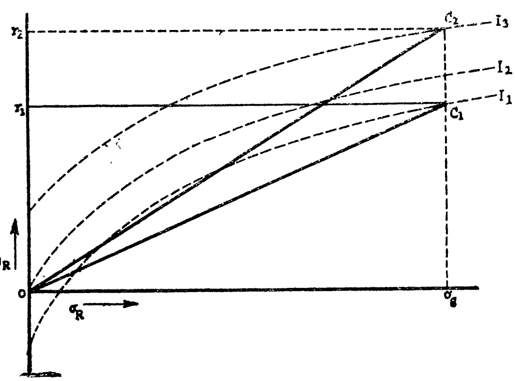

FIGURE 3.3

" Plungers "—Optimum Portfolio at Minimum or Maximum Risk and Expected Return.

III. A corner maximum at the origin, where the entire balance is held in cash. For a plunger, this case is illustrated in Figure 3.3 (OC_1). Conceivably it could also occur foı a diversifier, if the slope of his indifference curve at the origin exceeded the slope of the opportunity locus. However, case III is entirely excluded for investors whose indifference curves represent the constant-expected-utility loci of section 3.3. Such investors, we have already noted, cannot be plungers. Furthermore, the slope of all constant-expected-utility loci at $\sigma_R = 0$ must be zero, as can be seen from (3.7) and (3.13).

We can now examine the consequences of a change in the interest rate r, holding constant the investor's estimate of the risk of capital gain or loss. An increase in the interest rate will rotate the opportunity locus OC to the left. How will this affect the investor's holdings of cash and consols ? We must consider separately the three cases.

I. In Figure 3.1, OC_1, OC_2, and OC_3 represent opportunity loci for successively higher rates of interest. The indifference curves I_1, I_2, and I_3 are drawn so that the points of tangency T_1, T_2, and T_3, correspond to successively higher holdings of consols A_2. In this diagram, the investor's demand for cash depends inversely on the interest rate.

This relationship is, of course, in the direction liquidity preference theory has taught us to expect, but it is not the only possible direction of relationship. It is quite possible to draw indifference curves so that the point of tangency moves left as the opportunity locus is rotated counter-clockwise. The ambiguity is a familiar one in the theory of choice, and reflects the ubiquitous conflict between income and substitution effects. An increase in the rate of interest is an incentive to take more risk ; so far as the substitution effect is concerned, it means a shift from security to yield. But an increase in the rate of interest also has an income effect, for it gives the opportunity to enjoy more security along with more yield. The ambiguity is analogous to the doubt concerning the effect of a change in the interest rate on saving ; the substitution effect argues for a positive relationship, the income effect for an inverse relationship.

However, if the indifference curves are regarded as loci of constant expected utility, as derived in section 3.3, part of this ambiguity can be resolved. We have already observed that these loci all have zero slopes at $\sigma_R = 0$. As the interest rate r rises from zero, so also will consul holding A_2. At higher interest rates, however, the inverse relationship may occur.

This reversal of direction can, however, virtually excluded in the case of the quadratic utility function (section 3.3.2). The condition for a maximum is that the slope of an indifference locus as given by (3.13) equal the slope of the opportunity locus (3.4).

$$(3.15) \qquad \frac{r}{\sigma_g} = \frac{A_2\,\sigma_g}{-\dfrac{1+b}{2b} - A_2\,r} \; ; \quad A_2 = \frac{r}{r^2+\sigma_g^2}\left(-\frac{1+b}{2b}\right)$$

Equation (3.15) expresses A_2 as a function of r, and differentiating gives :

$$(3.16) \qquad \frac{dA_2}{dr} = \frac{\sigma_g^2 - r^2}{(\sigma_g^2 + r^2)^2}\left(-\frac{1+b}{2b}\right); \quad \frac{r}{A_2}\,\frac{dA_2}{dr} = \frac{\sigma_g^2 - r^2}{\sigma_g^2 + r^2}$$

Thus the share of consols in the portfolio increases with the interest rate for r less than σ_g. Moreover, if r exceeds σ_g, a tangency maximum cannot occur unless r also exceeds g_{max}, the largest capital gain the investor conceives possible (see 3.10).[1] The demand for consols is less elastic at high interest rates than at low, but the elasticity is not likely to become negative.

II and III. A change in the interest rate cannot cause a risk-lover to alter his position, which is already the point of maximum risk and expected yield. Conceivably a " diversifier " might move from a corner maximum to a regular interior maximum in response either to a rise in the interest rate or to a fall. A " plunger " might find his position altered by an increase in the interest rate, as from r_1 to r_2 in Figure 3.3 ; this would lead him to shift his entire balance from cash to consols.

[1] For this statement and its proof, I am greatly indebted to my colleague Arthur Okun. The proof is as follows :

If $r^2 \geqq \sigma_g^2$, then by (3.15) and (3.10) :

$$1 \geqq A_2 \geqq \frac{r}{2r^2}\left(-\frac{1+b}{2b}\right) \geqq \frac{1}{2r}\,(r + g_{max}).$$

From the two extremes of this series of inequalities it follows that $2r \geqq r + g_{max}$ or $r \geqq g_{max}$. Professor Okun also points out that this condition is incompatible with a tangency maximum if the distribution of g is symmetrical. For then $r \geqq g_{max}$ would imply $r + g_{min} \geqq 0$. There would be no possibility of net loss on consols and thus no reason to hold any cash.

3.5 *Effects of changes in risk.* Investor's estimates σ_g of the risk of holding monetary assets other than cash, " consols," are subjective. But they are undoubtedly affected by market experience, and they are also subject to influence by measures of monetary and fiscal policy. By actions and words, the central bank can influence investors' estimates of the variability of interest rates ; its influence on these estimates of risk may be as important in accomplishing or preventing changes in the rate as open-market operations and other direct interventions in the market. Tax rates, and differences in tax treatment of capital gains, losses, and interest earnings, affect in calculable ways the investor's risks and expected returns. For these reasons it is worth while to examine the effects of a change in an investor's estimate of risk on his allocation between cash and consols.

In Figure 3.4, T_1 and A_2 (r_1, σ_g) represent the initial position of an investor, at interest rate r_1 and risk σ_g. OC_1 is the opportunity locus (3.4), and OB_1 is the risk-consols relationship (3.3). If the investor now cuts his estimate of risk in half, to $\frac{\sigma_g}{2}$, the opportunity locus will double in slope, from OC_1 to OC_2, and the investor will shift to point T_2. The risk-consols relationship will have also doubled in slope, from OB_1 to OB_2. Consequently point T_2 corresponds to an investment in consols of $A_2\left(r_1, \frac{\sigma_g}{2}\right)$. This same point T_2 would have been reached if the interest rate had doubled while the investor's risk estimate σ_g remained unchanged. But in that case, since the risk-consols relationship would remain at OB_1, the corresponding investment in consols would have been only half as large, i.e., A_2 $(2r_1, \sigma_g)$. In general, the following relationship exists between the elasticity of the

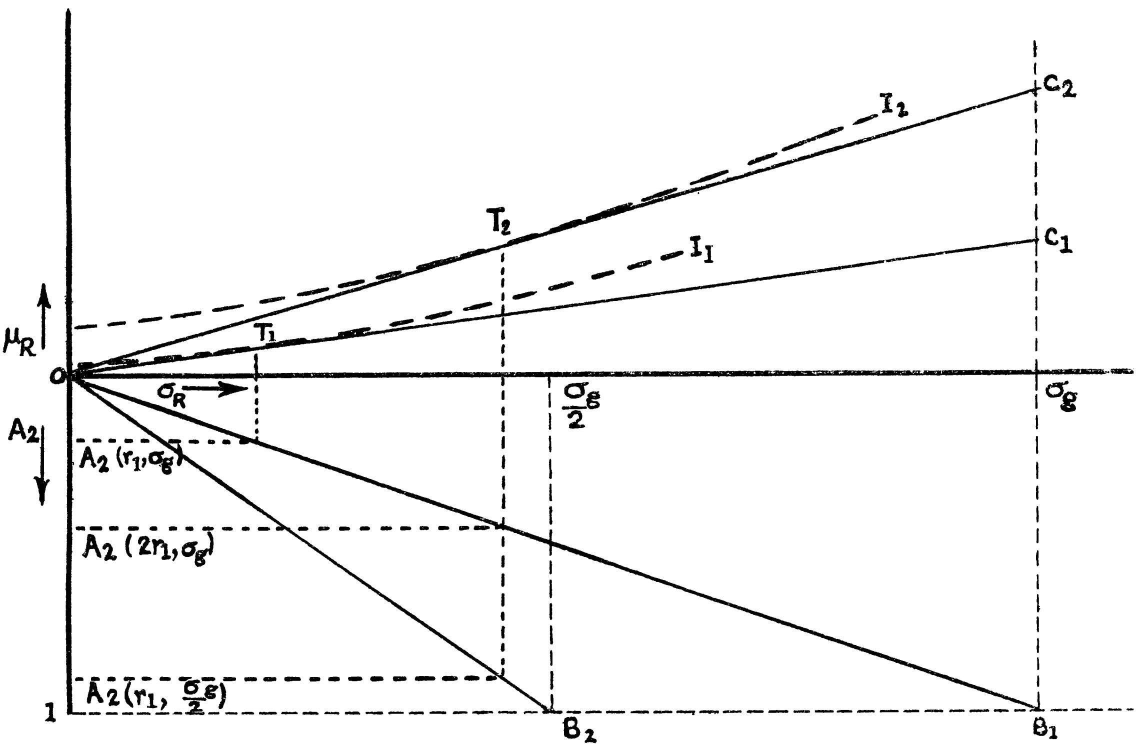

FIGURE 3.4
Comparison of effects of changes in interest rate (r) and in " risk " (σ_g) on holding of consols.

demand for consols with respect to risk and its elasticity with resepct to the interest rate :

$$(3.17) \qquad \frac{\sigma_g}{A_2}\frac{dA_2}{d\sigma_g} = -\frac{r}{A_2}\frac{dA_2}{dr} - 1.$$

The implications of this relationship for analysis of effects of taxation may be noted in passing, with the help of Figure 3.4. Suppose that the initial position of the investor is T_2 and A_2 $(2r_1, \sigma_g)$. A tax of 50% is now levied on interest income and capital gains alike, with complete loss offset porvisions. The result of the tax is to reduce the expected net return per dollar of consols from $2r_1$ to r_1 and to reduce the risk to the investor per dollar of consols from σ_g to $\sigma_g/2$. The opportunity locus will remain at OC_2, and the investor will still wish to obtain the combination of risk and expected return depicted by T_2. To obtain this combination, however, he must now double his holding of consols, to $A_2(r_1, \sigma_g/2)$; the tax shifts the risk-consols line from OB_1 to OB_2. A tax of this kind, therefore, would reduce the demand for cash at any market rate of interest, shifting the investor's liquidity preference schedule in the manner shown in Figure 3.5. A tax on interest income only,

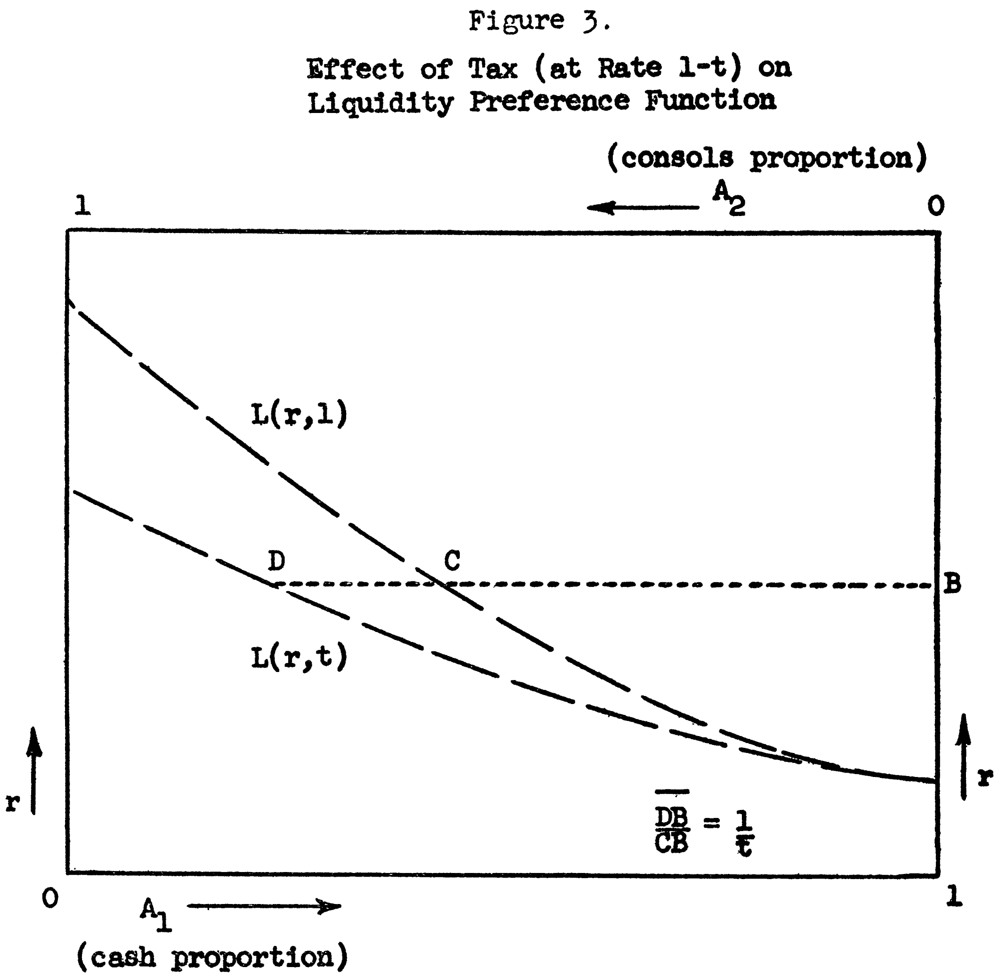

FIGURE 3.5
Effect of Tax (at Rate 1—*t*) on Liquidity Preference Function.

with no tax on capital gains and no offset privileges for capital losses, would have quite different effects. If the Treasury began to split the interest income of the investor in Figure 3.4 but not to share the risk, the investor would move from his initial position, T_2 and A_2 $(2r_1, \sigma_g)$; to T_1 and A_2 (r_1, σ_g). His demand for cash at a given market rate of interest would be increased and his liquidity preference curve shifted to the right.

3.6 *Multiple alternatives to cash.* So far it has been assumed that there is only one alternative to cash, and A_2 has represented the share of the investor's balance held in that asset, "consols". The argument is not essentially changed, however, if A_2 is taken to be the aggregate share invested in a variety of non-cash assets, e.g. bonds and other debt instruments differing in maturity, debtor, and other features. The return R and the risk σ_g on "consols" will then represent the average return and risk on a composite of these assets.

Suppose that there are m assets other than cash, and let x_i $(i = 1, 2, \ldots . m)$ be the amount invested in the ith of these assets. All x_i are non-negative, and $\sum_{i=l}^{m} x_i = A_2 \leqq 1$. Let r_i be the expected yield, and let g_i be the capital gain or loss, per dollar invested in the ith asset. We assume $E(g_i) = 0$ for all i. Let v_{ij} be the variance or covariance of g_i and g_j as estimated by the investor.

(3.18) $$v_{ij} = E(g_i \, g_j) \quad (i, j, = 1, 2, \ldots m)$$

The over-all expected return is :

(3.19) $$\mu_R = A_2 r = \sum_{i=1}^{m} x_i \, r_i$$

The over-all variance of return is :

(3.20) $$\sigma_R^2 = A_2^2 \, \sigma_g^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i \, x_j \, v_{ij}.$$

A set of points x_i for which $\sum_{i=1}^{m} x_i \, r_i$ is constant may be defined as a *constant-return locus*. A constant-return locus is linear in the x_i. For two assets x_1 and x_2, two loci are illustrated in Figure 3.6. One locus of combinations of x_1 and x_2 that give the same expected return μ_R is the line from $\frac{\mu_R}{r_2}$ to $\frac{\mu_R}{r_1}$, through C ; another locus, for a higher constant, μ_R', is the parallel line from $\frac{\mu_R'}{r_2}$ to $\frac{\mu_R'}{r_1}$, through C'.

A set of points x_i for which σ_R^2 is constant may be defined as a *constant-risk locus.* These loci are ellipsoidal. For two assets x_1 and x_2, such a locus is illustrated by the quarter-ellipse from $\frac{\sigma_R}{\sqrt{v_{22}}}$ to $\frac{\sigma_R}{\sqrt{v_{11}}}$, through point C. The equation of such an ellipse is ;

$$x_1^2 \, v_{11} + 2 \, x_1 \, x_2 \, v_{12} + x_2^2 \, v_{22} = \sigma_R^2 = \text{constant}.$$

Another such locus, for a higher risk level, σ_R', is the quarter-ellipse from $\frac{\sigma_R'}{\sqrt{v_{22}}}$ to $\frac{\sigma_R'}{\sqrt{v_{11}}}$ through point C'.

From Figure 3.7, it is clear that C and C' exemplify *dominant* combinations of x_1 and x_2. If the investor is incurring a risk of σa, somewhere on the ellipse through C, he will

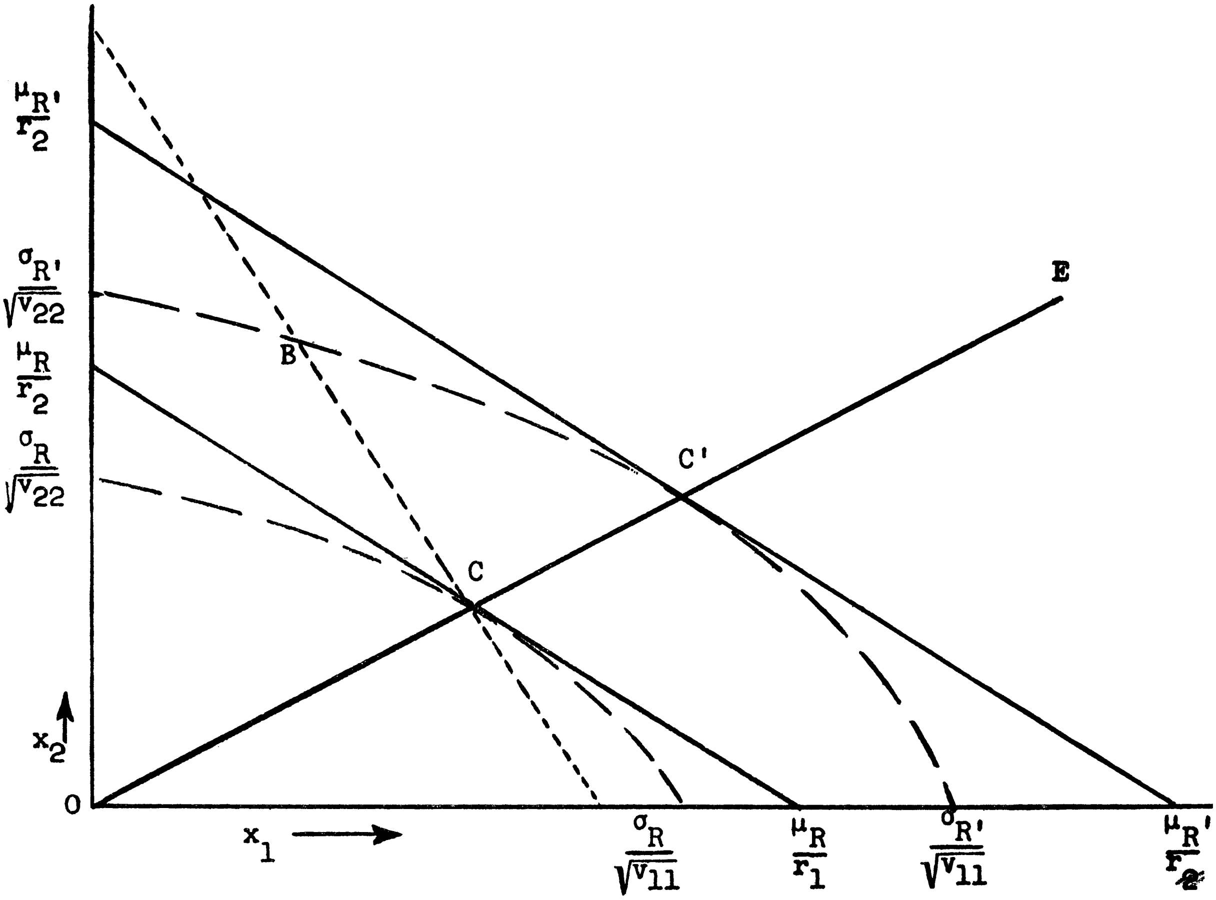

FIGURE 3.6
Dominant Combinations of Two Assets.

to have the highest possible expectation of return available to him at that level of risk. The highest available expected return is represented by the constant-expected-return line tangent to the ellipse at C. Similarly C' is a dominant point : it would not be possible to obtain a higher expected return than at C' without incurring additional risk, or to diminish risk without sacrificing expected return.

In general, a dominant combination of assets is defined as a set x_i which minimizes σ_R^2 for μ_R constant :

(3.21) $$\sum_i \left(\sum_j v_{ij}\, x_j\right) x_i - \lambda \left(\sum_i r_i x_i - \mu_R\right) = \min$$

where λ is a Lagrange multiplier. The conditions for the minimum are that the x_i satisfy the constraint (3.19) and the following set of m simultaneous linear equations, written in matrix notation :

(3.22) $$[v_{ij}]\ [x_i] = [\lambda\, r_i].$$

All dominant sets lie on a ray from the origin. That is, if $[x_i^{(0)}]$ and $[x_i^{(1)}]$ are dominant sets, then there is some non-negative scalar κ such that $[x_i^{(1)}] = [\kappa\, x_i^{(0)}]$. By definition of a dominant set, there is some $\lambda^{(0)}$ such that :

$$[v_{ij}] \ [x_i^{(0)}] = [\lambda^{(0)} r_i],$$

and some $\lambda^{(1)}$ such that :

$$[v_{ij}] \ [x_i^{(1)}] = [\lambda^{(1)} r_i].$$

Take $\kappa = \frac{\lambda^{(1)}}{\lambda^{(0)}}$. Then :

$$[v_{ij}] \ [\kappa x_i^{(0)}] = [\kappa\lambda^{(0)} r_i] = [\lambda^{(1)} r_i] = [v_{ij}] \ [x_i^{(1)}].$$

At the same time, $\sum_i r_i x_i^{(0)} = \mu_R^{(0)}$ and $\sum_i r_i x_i^{(1)} = \mu_R^{(1)}$.

Hence, $\mu_R^{(1)} = \kappa\mu_R^{(0)}$. Conversely, every set on this ray is a dominant set. If $[x_i^{(0)}]$ is a dominant set, then so is $[\kappa x_i^{(0)}]$ for any non-negative constant κ. This is easily proved. If $[x_i^{(0)}]$ satisfies (3.19) and (3.22) for $\mu_R^{(0)}$ and $\lambda^{(0)}$, then $[\kappa x_i^{(0)}]$ satisfies (3.19) and (3.22) for $\lambda^{(\kappa)} = \kappa\lambda^{(0)}$ and $\mu_R^{(\kappa)} = \kappa\mu_R^{(0)}$. In the two dimensional case pictured in Figure 3.6, the dominant pairs lie along the ray $OCC'E$.

There will be some point on the ray (say E in Figure 3.6) at which the investor's holdings of non-cash assets will exhaust his investment balance ($\sum_i x_i = 1$) and leave nothing for cash holding. Short of that point the balance will be divided among cash and non-cash assets in proportion to the distances along the ray ; in Figure 3.6 at point C for example, $\frac{OC}{OE}$ of the balance would be non-cash, and $\frac{CE}{OE}$ cash. But the convenient fact that has just been proved is that the proportionate composition of the non-cash assets is independent of their aggregate share of the investment balance. This fact makes it possible to describe the investor's decisions as if there were a single non-cash asset, a composite formed by combining the multitude of actual non-cash assets in fixed proportions.

Corresponding to every point on the ray of dominant sets is an expected retrun μ_R and risk σ_R ; these pairs (μ_R, σ_R) are the opportunity locus of sections 3.1 and 3.4. By means of (3.22), the opportunity locus can be expressed in terms of the expected return and variances and covariances of the non-cash assets : Let :

$$[V_{ij}] = [V_{ij}]^{-1}.$$

Then :

(3.23) $$\mu_R = \lambda \sum_i \sum_j r_i r_j V_{ij}$$

(3.24) $$\sigma_R^2 = \lambda^2 \sum_i \sum_j r_i r_j V_{ij}.$$

Thus the opportunity locus is the line :

(3.25) $$\mu_R = \sigma_R \sqrt{\sum_i \sum_j r_i r_j V_{ij}} = \sigma_R \frac{r}{\sigma_g}$$

This analysis is applicable only so long as cash is assumed to be a riskless asset. In the absence of a residual riskless asset, the investor has no reason to confine his choices to the ray of dominant sets. This may be easily verified in the two-asset case. Using Figure 3.6 for a different purpose now, suppose that the entire investment balance must be divided between x_1 and x_2. The point (x_1, x_2) must fall on the line $x_1 + x_2 = 1$, represented by the line through BC in the diagram. The investor will not necessarily choose point C. At point B, for example, he would obtain a higher expected yield as well as a higher risk ; he may prefer B to C. His opportunity locus represents the pairs

(μ_R, σ_R) along the line through $BC, (x_1 + x_2 = 1)$, rather than along the ray OC, and is a hyperbola rather than a line. It is still possible to analyze portfolio choices by the apparatus of (μ_R, σ_R) indifference and opportunity loci, but such analysis is beyond the scope of the present paper.[1]

It is for this reason that the present analysis has been deliberately limited, as stated in section 1.2, to choices among monetary assets. Among these assets cash is relatively riskless, even though in the wider context of portfolio selection, the risk of changes in purchasing power, which all monetary assets share, may be relevant to many investors. Breaking down the portfolio selection problem into stages at different levels of aggregation —allocation first among, and then within, asset categories—seems to be a permissible and perhaps even indispensable simplification both for the theorist and for the investor himself.

4. *Implications of the analysis for liquidity preference theory.*

The theory of risk-avoiding behaviour has been shown to provide a basis for liquidity preference and for an inverse relationship between the demand for cash and the rate of interest. This theory does not depend on inelasticity of expectations of future interest rates, but can proceed from the assumption that the expected value of capital gain or loss from holding interest-bearing assets is always zero. In this respect, it is a logically more satisfactory foundation for liquidity preference than the Keynesian theory described in section 2. Moreover, it has the empirical advantage of explaining diversification—the same individual holds both cash and " consols "—while the Keynesian theory implies that each investor will hold only one asset.

The risk aversion theory of liquidity preference mitigates the major logical objection to which, according to the argument of section 2.6, the Keynesian theory is vulnerable. But it cannot completely meet Leontief's position that in a strict stationary equilibrium liquidity preference must be zero unless cash and consols bear equal rates. By their very nature consols and, to a lesser degree, all time obligations contain a potential for capital gain or loss that cash and other demand obligations lack. Presumably, however, there is some length of experience of constancy in the interest rate that would teach the most stubbornly timid investor to ignore that potential. In a pure stationary state, it could be argued, the interest rate on consols would have been the same for so long that investors would unanimously estimate σ_g to be zero. So stationary a state is of very little interest. Fortunately the usefulness of comparative statics does not appear to be confined to comparisons of states each of which would take a generation or more to achieve. As compared to the Keynesian theory of liquidity preference, the risk aversion theory widens the applicability of comparative statics in aggregative analysis ; this is all that need be claimed for it.

The theory, however, is somewhat ambiguous concerning the direction of relationship between the rate of interest and the demand for cash. For low interest rates, the theory implies a negative elasticity of demand for cash with respect to the interest rate, an elasticity that becomes larger and larger in absolute value as the rate approaches zero. This implication, of course, is in accord with the usual assumptions about liquidity preference. But

[1] A forthcoming book by Harry Markowitz, *Techniques of Portfolio Selection*, will treat the general problem of finding dominant sets and computing the corresponding opportunity locus, for sets of securities all of which involve risk. Markowitz's main interest is prescription of rules of rational behaviour for investors ; the main concern of this paper is the implications for economic theory, mainly comparative statics, that can be derived from assuming that investors do in fact follow such rules. For the general nature of Markowitz's approach, see his article, " Portfolio Seclection ", *Journal of Finance*, Vol. VII, No. 1 (March 1952), pp. 77-91.

for high interest rates, and especially for individuals whose estimates σ_g of the risk of capital gain oı loss on " consols " are low, the demand for cash may be an increasing, rather than a decreasing, function of the interest rate. However, the force of this reversal of direction is diluted by recognition, as in section 2.5, that the size of investment balances is not independent of the current rate of interest r. In section 3.4 we have considered the proportionate allocation between cash and " consols " on the assumption that it is independent of the size of the balance. An increase in the rate of interest may lead an investor to desire to shift towards cash. But to the extent that the increase in interest also reduces the value of the investor's consol holdings, it automatically gratifies this desire, at least in part.

The assumption that investors expect on balance no change in the rate of interest has been adopted for the theoretical reasons explained in section 2.6 rather than for reasons of realism. Clearly investors do form expectations of changes in interest rates and differ from each other in their expectations. For the purposes of dynamic theory and of analysis of specific market situations, the theories of sections 2 and 3 are complementary rather than competitive. The formal apparatus of section 3 will serve just as well for a non-zero expected capital gain or loss as for a zero expected value of g. Stickiness of interest rate expectations would mean that the expected value of g is a function of the rate of interest r, going down when r goes down and rising when r goes up. In addition to the rotation of the opportunity locus due to a change in r itself, there would be a further rotation in the same direction due to the accompanying change in the expected capital gain or loss. At low interest rates expectation of capital loss may push the opportunity locus into the negative quadrant, so that the optimal position is clearly no consols, all cash. At the other extreme, expectation of capital gain at high interest rates would increase sharply the slope of the opportunity locus and the frequency of no cash, all consols positions, like that of Figure 3.3. The stickier the investor's expectations, the more sensitive his demand for cash will be to changes in the rate of interest.

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