

## AN ELEMENTARY ANALYSIS OF THE LEONTIEF SYSTEM<sup>1</sup>

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The standard theorems on the Leontief System are proved in a new and elementary way. These concern the existence of a unique nonnegative solution, duality properties, efficiency and profitability, and the generalization to allow substitution of the inputs to an industry.

### 1. INTRODUCTION

I HAVE three objectives. First, I will give a direct, elementary proof of the more useful necessary and sufficient conditions that a Leontief system possesses a unique nonnegative solution. Then I will prove, with elementary means, the efficiency theorem of activity analysis in the appropriate form for this system. Finally, I will give a short proof of the Samuelson substitution theorem for generalized Leontief models.

Let  $B$  represent a square matrix and let  $x$  and  $y$  be column vectors. The equation  $Bx = y$  represents a Leontief system when the following conditions are met:

$$(1) \quad b_{ij} \leq 0 \quad (i \neq j), \quad y_i \geq 0.$$

$$(2) \quad \sum_{i=1}^n b_{ij} \geq 0.$$

The  $b_{ij}$  for  $i \neq j$  are the quantities of other goods needed to produce  $b_{jj}$  units of the  $j$ th good.  $x_j$  is the activity rate in the  $j$ th industry, and  $y_j$  is the net output of the  $j$ th good. The interest in nonnegative solutions arises from the fact that negative rates of activity are not meaningful. In the latter part of the discussion where the system is treated as a linear activities model, it will prove appropriate to drop requirement (2) of "dominant" outputs.<sup>2</sup> This will leave the question of "workability" entirely open.<sup>3</sup>

### 2. EXISTENCE OF A SOLUTION

In dealing with the question of a nonnegative solution, we will need a lemma due to Arrow [1, p. 159]. Its proof is immediate.

LEMMA 1: If  $Bx \geq 0$  implies  $x \geq 0$ , then  $B$  is nonsingular, where  $B$  is any matrix.

$Bx = 0$  implies  $B(-x) = 0$ . Then by the hypothesis, it must be true that  $x = 0$ , and  $B$  is nonsingular.

We may now state a lemma which leads directly to the well-known necessary and sufficient conditions for a unique nonnegative solution. Let us refer to a

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<sup>2</sup> See Morgenstern [5].

<sup>3</sup> See Chipman [2].

matrix  $B$  as positive if  $Bx \geq 0$  implies  $x \geq 0$ . Let  $B$  now define a Leontief system.

LEMMA 2: A necessary and sufficient condition for  $B$  to be positive is that every principal minor of  $B$  have at least one column sum greater than zero.

Suppose the condition is met, but for some  $y$  with  $y_i \geq 0$ ,  $Bx = y$ , and  $x_i < 0$  for some  $i$ . By identical rearrangements of rows and columns  $Bx = y$  may be written  $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$ , where  $x^1 < 0$  and  $x^2 \geq 0$ . This gives  $B_1x^1 + B_2x^2 = y^1$ . Since off-diagonal elements are not positive,  $B_2x^2 \leq 0$ . Therefore,  $B_1x^1 \geq 0$ . By the hypothesis,  $p'B_1 \geq 0$ ,<sup>4</sup> where  $p' = (1, \dots, 1)$ . Thus,  $p'B_1x^1 < 0$ , since  $x^1 < 0$ . But this implies  $B_1x^1$  not  $\geq 0$ . Since we have reached a contradiction,  $x^1 < 0$  is excluded and  $B$  is positive.

On the other hand, suppose that  $B$  is positive but there is a principal minor whose column sums are all equal to zero. Without loss of generality we may suppose that this minor  $B_1$  lies in the upper left-hand corner of  $B$ . Since the column sums of  $B$  are not negative, the elements in the columns below  $B_1$  must be zero. We may write  $B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix}$ . Since  $p'B_1 = 0$ ,  $B_1$  is singular. Thus there is a nonzero vector  $x^1$  with  $B_1x^1 = 0$ . Consider  $Bx$ , where  $x = (x^1 \ 0)$ . Since  $Bx = 0$ ,  $B$  is singular, and by Lemma 1,  $B$  is not positive.

We may derive other useful necessary and sufficient conditions very easily from those of Lemma 2.

THEOREM 1: Let  $B$  satisfy conditions (1) and (2). The following are equivalent:

- (a) Any Leontief system defined by  $B$  has a unique nonnegative solution.
- (b) Each principal minor of  $B$  has at least one column sum positive.
- (c)  $B$  is nonsingular.<sup>5</sup>
- (d) If  $B$  is completely decomposed, each indecomposable minor on the diagonal has a positive column sum.<sup>6</sup>
- (e) It is possible to perform identical permutations of rows and columns of  $B$  so that  $\sum_{i=1}^j b_{ij} > 0$  for all  $j$ .<sup>7</sup>

PROOF: Lemma 2 states the equivalence of (a) and (b). (a) obviously implies (c). But (c) also implies (b) as we see from the second part of the proof of Lemma 2. Therefore, (c) implies (a), and (a), (b), and (c) are equivalent. (b) immediately implies (d) since the minors on the diagonal are principal minors. Conversely, if there is a principal minor whose column sums all equal zero, it may occupy the place of  $B_1$  in the proof of Lemma 2. If  $B_1$  is then completely decomposed,<sup>8</sup> the indecomposable minor occupying the upper left-hand

<sup>4</sup>  $x \geq \bar{x}$  means  $x_i \geq \bar{x}_i$  for all  $i$  and  $x_i > \bar{x}_i$  for some  $i$ .

<sup>5</sup> This condition may be found in Chipman [2] and Wong [9].

<sup>6</sup> More or less explicitly given by Solow [8] and Woodbury [10].

<sup>7</sup> See G. B. Price [6], and in the present context Wong [9].

<sup>8</sup> A matrix  $A$  is decomposable if it can be put into the form  $\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$  through identical

corner will have its column sums equal to zero, since permuting columns and rows does not change the column sums. Thus (d) also implies (b).

It remains to be shown that (b) is equivalent to (e). This may be proved by considering what will prevent the completion of such an arrangement of the matrix when one begins with the  $n$ th column and proceeds backward. Suppose the  $k$ th column is in order. The  $(k-1)$ st column is available, without disturbing the arrangements already made, if one of the first  $k-1$  columns as they now appear has the sum of its first  $k-1$  elements greater than zero. But these elements form a principal minor, and thus (b) implies that the condition is met. This shows that (b) implies (e).

On the other hand, if (b) does not hold, that is, if there is a principal minor with all its columns summing to zero, (e) cannot hold either. For every such minor contains one column whose elements lie on and above the diagonal. Moreover, all the permutations which can be performed leave one of its columns with all its elements on and above the diagonal of  $B$ . But this column will violate (e). Thus, (e) implies (b). This completes the proof of the theorem.

### 3. THE DUALITY THEOREM

The form of condition (2) is a result of the normalization (to be defined shortly) of the Leontief matrix. This also explains the special role of the vector  $(1, \dots, 1)$  in the proof. An equivalent condition without normalization is

$$(2') \quad p'B \geq 0 \quad \text{for some } p > 0.$$

$p$  may be interpreted as a price vector. Then for actual economies such a  $p$  is provided by the prevailing prices unless some industry realizes a loss on its current transactions. However, for a technology which is not in use, the existence of  $p$  allowing (2') to be satisfied will be an open question. If (2') can be satisfied, we may rephrase Lemma 2 as

LEMMA 2': Let  $p$  and  $B$  satisfy (2'). A necessary and sufficient condition for  $B$  to be positive is that  $p^k B_k \geq 0$  for  $B_k$  an arbitrary principal minor of  $B$ .

$p^k$  is the subvector of prices for goods appearing in  $B_k$ . The proof of Lemma 2' is entirely analogous to that of Lemma 2 except that  $p$  need not equal  $(1, \dots, 1)$ . The theorem can be similarly rephrased and the proof is as before. Of course, the proofs need not be repeated anyway, since (2') is obviously equivalent to (2), and then the lemmas are equivalent. The normalization is accomplished through multiplying the  $i$ th row of  $B$  by  $p_i$  when (2') is satisfied for  $p$ . Leontief, in addition, divides each column by its resulting diagonal element. He is assuming, however, that this element is positive. This is true if (2') is met unless a trivial column of zeros is allowed in  $B$ .

Now we may note that condition (2') and Lemma 2' may equally well be

permutations of rows and columns. It is completely decomposed if it is put by this means into the form  $\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ & \ddots & \vdots \\ 0 & & A_{nn} \end{bmatrix}$  and  $A_{11}, \dots, A_{nn}$  are indecomposable.

applied to  $B'$ , the transpose of  $B$ . In particular, they imply that if we have an  $x > 0$  such that  $x'B' > 0$ , then  $B'$  is positive. In other words,  $B'p = \pi$  has a unique nonnegative solution for every  $\pi \geq 0$ . On the other hand, when  $B$  is positive such an  $x$  does exist. For the fact that only the diagonal elements can be positive implies that  $Bx > 0$  is not possible unless  $x > 0$ . Similarly  $B'$  positive implies  $p > 0$  when  $B'p > 0$ . Thus the circle is complete, and we have justified

**THEOREM 2:** *If  $B$  is a square matrix whose off-diagonal elements are not positive, the following are equivalent:*

- (a) *There is a  $p > 0$  such that  $p'B > 0$ .*
- (b)  *$B$  is positive.*
- (c) *There is an  $x > 0$  such that  $Bx > 0$ .*
- (d)  *$B'$  is positive.*

It is a corollary of Theorem 2 that if it is possible to produce something of all goods together, output can be produced in any proportions.

#### 4. THE THEOREM ON EFFICIENCY

We may use the results now obtained to prove the efficiency theorem of activity analysis for the Leontief model. In the complete statement of the open Leontief model it is recognized that each industry requires labor input. Let the columns of  $B$  represent the inputs and outputs of goods which accompany an input of one unit of labor in the respective industries. Then the open Leontief model may be written, where  $B$  is a square matrix subject to condition (1),

$$(3) \quad \begin{aligned} Bx &= y, & x &\geq 0. \\ \sum x_j &\leq c, & c &> 0. \end{aligned}$$

$c$  is the available labor supply. An output vector  $y$  is said to be efficient if  $y + \Delta y$ , where  $\Delta y \geq 0$ , is inconsistent with (3). It is apparent that the labor supply must be fully used. For if  $\sum x_j = c - u$ ,  $u > 0$ ,  $y + \frac{u}{c-u}y$  would be possible.

Also it is no lack of generality to take the price of labor to be unity. Then the efficiency theorem of activity analysis<sup>9</sup> can be written, for this model, as

**THEOREM 3:** *An output  $y$  of the Leontief model (3) is efficient if and only if the labor supply is fully used and there exists a price vector  $p > 0$  such that*

$$p'B - 1 \leq 0$$

and  $\sum_{i=1}^n p_i b_{ij} - 1 = 0$  when  $x_j > 0$ .

**PROOF:** Suppose there is such a  $p$ . Then  $\Delta y \geq 0$  implies  $p'\Delta y = p'B\Delta x > 0$ . But  $p'B \leq 1$ . Therefore,  $p'\Delta y \leq \sum \Delta x_j$ . But  $\sum \Delta x_j \leq 0$ , since the labor supply is fully used. Thus  $\Delta y \geq 0$  is impossible.

<sup>9</sup> The general theorem may be found in Koopmans [3, p. 82]. An elementary proof of the theorem for the Graham model of world trade is in [4].

On the other hand, let  $y$  be efficient. Let  $B_k x^k = y^k$  be the reduced system of industries actually in use. Then (2') applied to  $B'_k$  and  $x^k$  is satisfied. If the condition in Lemma 2' is also satisfied for  $B'_k$  and  $x^k$ ,  $B'_k$  is positive and  $p^k B_k = 1$  can be solved for  $p^k > 0$ . Then, by setting components of  $p$  for industries not in use low enough, it is possible to realize  $\sum p_i b_{ij} < 1$  for  $x_j = 0$ . For this  $p$ , the conditions of Theorem 3 are met.

Suppose, however, that the condition of Lemma 2' is not satisfied for  $B_k$  and  $x^k$ . Then for some principal minor  $B_h$  of  $B_k$ ,  $B_h x^h = 0$ . If  $x^h$  is set equal to zero, no output is lost. However, labor is released which can be used to increase the outputs of other industries. Then  $y$  would not be efficient. Thus the condition Lemma 2' is met.

Incidentally, this discussion shows that a Leontief model can be inefficient only through having a set of industries which make no contribution to the rest of the economy.

### 5. THE GENERALIZED MODEL

The Leontief model (3) is more general than might be supposed at first sight. Indeed, it is applicable even where alternative processes are available for the production of each good, so long as labor is the only unproduced factor.<sup>10</sup> This result is a consequence of Theorem 3.

Suppose there is for the  $j$ th good a set  $\mathfrak{B}_j$  of basic input-output vectors  $\bar{b}^j$ , which are taken to be normalized on labor in the sense that one unit of labor is needed in conjunction with the inputs of the vector to produce its output. We assume that  $\bar{b}_i^j \leq 0$  for  $i \neq j$ . Let us also choose the unit of labor so that the total supply is equal to one. Let  $\mathfrak{B}_j$  be the set of vectors  $b^j = \sum t_k \bar{b}^{jk}$ , for  $\sum t_k = 1$  and  $t_k > 0$ . Thus  $\mathfrak{B}_j$  is the set of convex combinations of basic input-output vectors. Any  $b^j$  in  $\mathfrak{B}_j$  is also a possible input-output combination for the  $j$ th industry and uses one unit of labor, since  $\sum t_j = 1$ . The set of net output vectors  $Y$  which can be attained within the resource limitations can now be seen to be the set of all  $y \geq 0$  such that  $y = \sum_{j=1}^n x_j b^j$ , where  $b^j$  lies in  $\mathfrak{B}_j$  for each  $j$  and  $\sum_{j=1}^n x_j \leq 1$ ,  $x_j \geq 0$ . If we suppose the  $\mathfrak{B}_j$  to be closed and bounded from above, the set  $Y$  is closed and bounded. I shall refer to this system as a generalized Leontief model.

Let  $\bar{y}$  be an efficient output vector, if one exists (it does, of course). Then  $\bar{y} = B\bar{x}$  for a Leontief matrix  $B$  with its  $j$ th column equal to some  $b^j$  lying in  $\mathfrak{B}_j$ . We may reduce the system to the set of industries for which  $\bar{x}_j > 0$ . Then by Theorem 3 there is a price vector  $\bar{p} > 0$  such that  $\bar{p}'B = (1, \dots, 1)$ . This implies by Theorem 2 that  $B$  is positive and  $Bx = y$  may be solved for an  $x \geq 0$  for any  $y \geq 0$ . Also  $\bar{p}'Bx = \sum x_i \leq 1$  for  $y$  attainable by use of  $B$ , and equality must hold if  $y$  is efficient. Consequently,  $\bar{p}'y \leq 1$  implies, for  $y \geq 0$ , that  $y$  lies in the set  $Y$  and can be attained by use of  $B$ . Since  $\bar{y}$  is efficient, we have in particular  $\bar{p}'\bar{y} = 1$ .

Let  $y^i$  be the vector with  $y_k^i = 0$ , for  $k \neq i$ , which is maximal in the attain-

<sup>10</sup> This fact was discovered by Paul Samuelson [7]. It was given a general proof by Arrow [1].

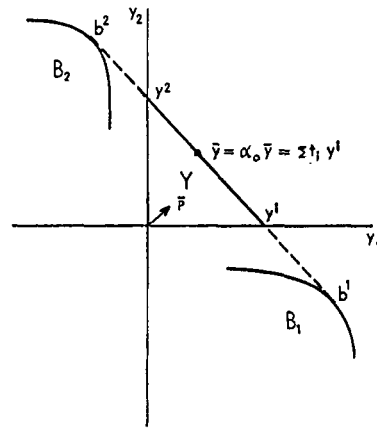


FIGURE 1

able set; that is,  $y^i$  is an element of  $Y$  but  $\alpha y^i$  is not an element of  $Y$  for  $\alpha > 1$ . Since  $y^i$  is maximal in  $Y$ ,  $\bar{p}'y^i \geq 1$  for each  $i$ . Otherwise, the vector  $\alpha y^i$  which satisfies  $\bar{p}'(\alpha y^i) = 1$ , and, therefore, lies in  $Y$ , would have  $\alpha > 1$ , contrary to the definition of  $y^i$ . Now consider the vector  $\alpha_0 \bar{y} = \sum t_i y^i$  where  $\sum t_i = 1$ ,  $t_i \geq 0$ . Since the  $y^i$  lie along the positive coordinate axes and  $\bar{y} \geq 0$ , this representation of  $\alpha_0 \bar{y}$  is clearly possible for the appropriate value of  $\alpha_0$ . Then,

$$\bar{p}'(\alpha_0 \bar{y}) = \sum t_i \bar{p}'y^i \geq 1.$$

But  $\bar{p}'\bar{y} = 1$ . Thus,  $\alpha_0 \geq 1$ . Therefore,  $\alpha_0 = 1$ , since  $\bar{y}$  is efficient, and  $\bar{p}'y^i = 1$  if the  $i$ th industry is in use. Moreover, this equation determines  $\bar{p}_i$  for such an  $i$ . Hence  $\bar{p}$  is independent of  $\bar{y}$  when this group of industries is used. But the argument may be repeated for any efficient  $y$ . Furthermore, the maximal set of industries which can be used together efficiently clearly contains all industries that can be used efficiently at all. This means that  $\bar{p}$  is uniquely defined in a system including every industry that can contribute to the economy. Other industries may be ignored.

In the reduced space of goods which can be produced, we may conclude that the efficiency of  $y$  is equivalent to  $\bar{p}'y = 1$ . This means that  $y = Bx$  for some  $x$  such that  $\sum x_i = 1$ , if  $B$  is derived from a  $\bar{y}$  which includes a positive output of each good. The Leontief model (3) defined by this  $B$  is equivalent to the generalized Leontief model defined by the  $\mathfrak{B}_j$  in the sense that they possess the same efficient outputs (indeed, the same attainable outputs). We have proved

**THEOREM 4:** *The set of efficient outputs of any generalized Leontief model may be attained by a Leontief model (3) defined by  $B$  where the  $j$ -th column of  $B$  lies in  $\mathfrak{B}_j$ .*

#### 6. EFFICIENCY AND UNIQUENESS

It may be worth noting explicitly that  $x$  is determined uniquely in the simple Leontief model if, and only if,  $y$  is efficient. The necessity is immediate from

Theorem 3. But the argument shows also that if  $x$  is not unique,  $y$  is not efficient, since there are industries in use which make no contribution to the net output. Thus efficiency and uniqueness are equivalent in the simple model.

In the generalized model there may be more than one way of choosing  $B$ . However,  $\bar{p}$  is independent of this choice, and  $x$  is determined uniquely given  $B$  and a  $y$  which is efficient.

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