

## A NOTE ON THE MODIGLIANI-HOHN PRODUCTION SMOOTHING MODEL\*†

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### 1. Introduction

Modigliani and Hohn [10] have formulated a production planning and inventory control model that appears relevant to an important class of non-stochastic decision-making problems.<sup>1</sup> In their own words:

We consider the problem of scheduling the production of a given commodity  $x$  over  $T$  equal, successive periods of time in such a way as (1) to meet initially known requirements  $s_1, s_2, \dots, s_T$  in these periods while (2) incurring the lowest possible cost. [10, p. 46]

It is the purpose of this note to suggest: (1) that the Modigliani Hohn problem may be studied in terms of linear programming as well as through the calculus model originally employed; (2) that by working through the stock-flow relationships, it is possible to effect certain conceptual and computational improvements over Edward Bowman's original method for converting this type of problem into the "transportation" form [1]; and (3) that the linear programming version is especially suitable for tracing out the cost implications of stabilizing the work force at alternative levels. Like the "caterer problem" [7] and the "warehousing problem" [4], this linear programming model represents another instance in which every basis is pure triangular and contains no elements but zero, +1, and -1.

### 2. Statement of the problem

The Modigliani-Hohn problem is one of planning production rates and inventory levels for a single product over time so as to minimize the overall sum of inventory costs plus the production costs incurred within each period. Inventory costs are taken to be proportional to the levels of inventory carried over from one period to the next, and production costs to depend upon the rate of production—but not upon changes in that rate. Since it is further assumed that *marginal* production costs are monotone increasing, there is an economic balance problem between the costs of carrying inventory forward from slack periods versus the costs of high levels of production during the peaks of demand.

In converting the Modigliani-Hohn problem to linear programming form, all

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<sup>1</sup> Robert Dorfman has called my attention to the fact that the Modigliani-Hohn problem is quite similar to one posed by Erich Schneider in 1938, and solved by Børge Jessen in that year [11].

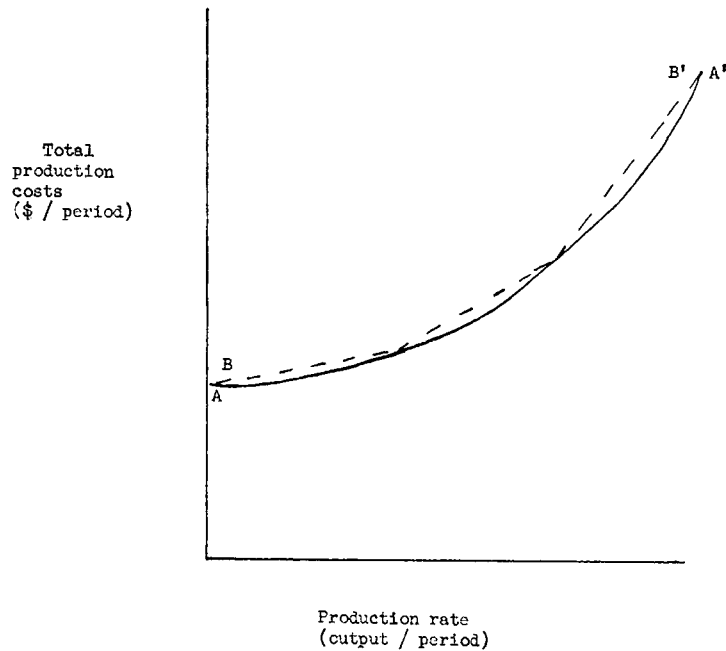


FIG. 1

that needs to be done is to interpret the production rate cost function, not as differentiable (the solid curve  $AA'$ , Figure 1), but rather as piecewise linear ( $BB'$ , the dotted curve). Corresponding to the calculus assumption of increasing marginal costs along  $AA'$ , we take the slope of each successive linear segment along  $BB'$  to be greater than that of the preceding one. Each of the segments on  $BB'$  may be given an economic interpretation in terms of the labor premiums paid for successive shifts within a given plant—night work, overtime, Saturdays, Sundays and holidays, etc.—and for this reason we shall refer to the individual segments as though they corresponded to shifts 1, 2,  $\dots$ ,  $J$ .<sup>2</sup> Within any one shift as so defined, marginal production costs remain constant, but in going from one shift to the next higher cost category, the marginal costs increase discontinuously.

In order to simplify subsequent work, a few changes have been made in the notation used by Modigliani and Hohn. Ours is as follows:

(a) *unknowns*

$x_{jt}$  = output during the  $j$ th shift within time period  $t$ .

$$(j = 1, 2, \dots, J)$$

$$(t = 1, 2, \dots, T)$$

<sup>2</sup> The rise in marginal production costs along curve  $BB'$  need not have anything to do with pay premiums, but might result from the diversion of a given material (say, heating oil) away from successively more valuable alternate uses. See Charnes, Cooper, and Symonds [3]. The identical form for  $BB'$  would still remain applicable.

- $y_{jt}$  = excess of upper bound over actual output during the  $j$ th shift within time period  $t$  ( $j = 1, 2, \dots, J$ )  
( $t = 1, 2, \dots, T$ )
- $h_t$  = inventory on hand at end of period  $t$  ( $t = 1, 2, \dots, T-1$ )
- $H_t$  = inventory on hand at beginning of period  $t$  ( $t = 2, 3, \dots, T$ )
- (b) *constants*
- $s_t$  = sales requirement occurring within period  $t$  ( $t = 1, 2, \dots, T$ )
- $a_{jt}$  = upper bound on output during the  $j$ th shift within period  $t$ .  
( $j = 1, 2, \dots, J$ )  
( $t = 1, 2, \dots, T$ )
- $H_1$  = inventory available at beginning of period 1.  
 $h_T$  = inventory requirement for end of period  $T$ .
- (c) *cost coefficients*
- $\alpha$  = cost of bringing one unit of inventory forward from one period to the next.
- $c_j$  = marginal production cost within the  $j$ th shift. ( $j = 1, 2, \dots, J$ )

(For  $j = 2, 3, \dots, J, c_j > c_{j-1}$ .)

The linear programming problem may be phrased as one of choosing values for the unknowns in such a way as to minimize expression (2.1), subject to meeting conditions (2.2)–(2.5). The minimand represents the sum of the inventory costs plus production costs during each shift within each time period:

$$(2.1) \quad \alpha \sum_{t=1}^{T-1} h_t + \sum_{j=1}^J c_j \sum_{t=1}^T x_{jt}$$

Condition (2.2) states that within each period, the production plus net inventory change equals the sales requirement for that period:

$$(2.2) \quad \sum_j x_{jt} + H_t - h_t = s_t \quad (t = 1, 2, \dots, T)$$

Condition (2.3) indicates the upper bound on output within each shift for each time period:

$$(2.3) \quad x_{jt} + y_{jt} = a_{jt} \quad (j = 1, 2, \dots, J) \\ (t = 1, 2, \dots, T)$$

And condition (2.4) indicates the identity between the inventory carried over from the end of one period and that on hand at the beginning of the next:<sup>3</sup>

$$(2.4) \quad h_t = H_{t+1} \quad (t = 1, 2, \dots, T-1)$$

<sup>3</sup> The “costless” variables  $H_t$ , along with the identity conditions (2.4), are really superfluous. They are introduced here only to facilitate setting up the linear programming problem in the format of the “transportation” problem.

TABLE 1  
*Array of 22 Variables and 14 Constants in "Transportation" Form*

period $t$ .....	1			2			3			Inventory carried over		Requirements
	1	2	3	1	2	3	1	2	3	period 1 to 2	period 2 to 3	
shift $j$ .....	$x_{11}$	$x_{21}$	$x_{31}$	$x_{12}$	$x_{22}$	$x_{32}$	$x_{13}$	$x_{23}$	$x_{33}$	$-h_1$ $+H_2$	$-h_2$ $+H_3$	$s_1 - H_1$ $s_2$ $s_3 + h_3$
	$y_{11}$	$y_{21}$	$y_{31}$	$y_{12}$	$y_{22}$	$y_{32}$	$y_{13}$	$y_{23}$	$y_{33}$			
upper limits on output, $a_{jt}$	$a_{11}$	$a_{21}$	$a_{31}$	$a_{12}$	$a_{22}$	$a_{32}$	$a_{13}$	$a_{23}$	$a_{33}$	0	0	

Finally, there are the usual non-negativity restrictions on all unknowns:

$$(2.5) \quad \begin{aligned} x_{jt} &\geq 0 && \left[ \begin{array}{l} j = 1, 2, \dots, J \\ t = 1, 2, \dots, T \end{array} \right] \\ y_{jt} &\geq 0 \\ h_t &\geq 0 && (t = 1, 2, \dots, T - 1) \end{aligned}$$

**3. Equivalence to the transportation model**

To show that the problem just defined is equivalent to a transportation model<sup>4</sup> it suffices to illustrate the array for  $J = T = 3$ . (See Table 1.) In this instance, 22 variables altogether are present:  $x_{11}, x_{21}, \dots, x_{33}; y_{11}, y_{21}, \dots, y_{33}; h_1, h_2, H_2$ , and  $H_3$ . The remaining cells of the  $4 \times 11$  array are empty. There are 14 restrictions on the row and column totals of Table 1: the three row totals corresponding to equation group (2.2), the first nine column totals corresponding to equation group (2.3), and the last two column totals to group (2.4). No explicit restriction is placed upon the sum of the row containing the "slack variables",  $y_{jt}$ .

In the case of a "transportation" array, Dantzig has proved the following theorem: At each iteration of the simplex method, it will be possible to find either a row or a column containing exactly one basis variable. Solving for this variable and then working with the reduced array formed by deleting the cor-

<sup>4</sup> A "transportation model" represents that special case of a linear programming system in which the complete set of restraint conditions may be written:

$$\begin{aligned} \sum_{j=1}^n a_{ij}x_{ij} &= Q_i && (i = 1, \dots, m) \\ \sum_{i=1}^m a_{ij}x_{ij} &= R_j && (j = 1, \dots, n) \\ x_{ij} &\geq 0 && (\text{all } i, j) \end{aligned}$$

Without loss of generality, it may be assumed that each of the coefficients  $a_{ij}$  equals either +1, -1, or zero.

TABLE 2

*Constants and Cost Coefficients for a Hypothetical 3-Shift, 3 Time Period Smoothing Problem*

Constants (unit of output)	Cost coefficients (\$10/unit of output)
$s_1 - H_1 = 80$	$c_1 = 10$
$s_2 = 160$	$c_2 = 15$
$s_3 + h_3 = 240$	$c_3 = 20$
$a_{jt} = 100$ (all $j, t$ )	$\alpha = 4$

responding row or column, it is proved by induction that every basis must be strictly triangular<sup>5</sup>. Furthermore, since the only coefficients entering into the equations of the initial linear programming problem are zero, +1, and -1, the only arithmetic operations required for a solution consist of addition and subtraction. [Dantzig, 5, p. 365].

#### 4. An illustrative example

Suppose that the constants and the cost coefficients for a 3-shift, 3 time period problem are as given in Table 2. For these data, Table 3 then contains the minimum cost solution as computed by the "transportation" method. Values for the dual variables  $u_t$ , the "shadow prices" associated with each of the sales requirement equations, are also entered in Table 3. In this, as in any optimal solution to this class of linear programming problems, the following conditions are necessary, and may be employed to facilitate computations:<sup>6</sup>

$$(4.1) \quad u_{t+1} \leq u_t + \alpha \quad (t = 1, 2, \dots, T - 1)$$

$$(4.2) \quad (u_t + \alpha - u_{t+1})(h_t) = 0 \quad (t = 1, 2, \dots, T - 1)^7$$

In words, condition (4.1) says that in an optimal solution the marginal cost of meeting the requirements in one period can be no greater than the marginal cost for the preceding period plus the carry-over charge,  $\alpha$ . Equations (4.2) say that if the amount of carry-over is positive—i.e.,  $h_t > 0$ —then the difference in marginal costs will be exactly equal to the carrying charge. And if the difference is less than the carrying charge, then the amount of carry-over,  $h_t$ , will equal zero. Conditions (4.3)–(4.5) are self-explanatory:

For each shift  $j$ ,  $c_j > c_{j-1}$ .

<sup>5</sup> A "triangular" basis is a non-singular square matrix for which it is possible to permute the rows and columns in such a way that all elements lying above the main diagonal are zero. If the square matrix  $A$  is a triangular one, and if one wishes to solve for  $X$  in the matrix equation  $AX = Q$  (where  $X$  and  $Q$  are column vectors, with components  $x_i$  and  $q_i$  respectively), then it is possible to calculate values for each of the unknown  $x_i$  recursively solely from the knowledge of  $x_1, \dots, x_{i-1}$ , and without taking  $x_{i+1}, x_{i+2}, \dots$  into account.

<sup>6</sup> Note that the conditions of (4.1) correspond to the Modigliani-Hohn calculus inequalities (3.10) [10, p. 50].

<sup>7</sup> In the terminology of Modigliani and Hohn, if  $u_t + \alpha > u_{t+1}$ , period  $t$  and  $t + 1$  belong to two different "intervals." "Small" changes in  $s_{t+1}, s_{t+2}, \dots, s_T + h_T$  would not alter the optimal solution for periods  $1, 2, \dots, t$ .

TABLE 3  
Optimal Solution to Hypothetical Smoothing Problem

time period $t$	shift $j$			$H_t$	$h_t$	Requirements	$u_t$ (\$10 <sup>3</sup> )
	1	2	3				
1	100	0	0	—	20	80	11
2	100	80	0	20	40	160	15
3	100	100	0	40	—	240	19
$a_{jt}$	100	100	100				
$c_j$ (\$10 <sup>3</sup> )	10	15	20	Inventory + labor costs = $\alpha \sum_{t=1}^{T-1} h_t + \sum_{j=1}^J c_j \sum_{t=1}^T x_{jt} = \$5,940,000.$			

$$(4.3) \quad \text{Hence if } x_{jt} > 0, \text{ then } x_{j-1,t} = a_{j-1,t} \quad (j = 2, 3, \dots, J)$$

$$\text{Similarly, if } x_{jt} < a_{jt}, \text{ then } x_{j+1,t} = 0 \quad (j = 1, \dots, J-1)$$

If, for any given shift  $j$  and period  $t$ ,

$$(4.4) \quad a_{jt} > x_{jt} > 0, \text{ then } u_t = c_j. \text{ If } x_{jt} = a_{jt}, \text{ then } u_t \geq c_j.$$

$$\text{And if } x_{jt} = 0, \text{ then } u_t \leq c_j. \quad (\text{all } j, t)$$

(4.5) The "costless" activities associated with the variables  $H_2, H_3, \dots, H_T$  will all be contained in an optimal basis—although some of them may take on the value of zero.

One further remark on computational aspects: Equations (2.3) simply represent upper bounds upon the levels of the  $x_{jt}$  variables. It is well-known [2, 6] that a linear programming model including (a) upper bounds on individual variables in addition to (b) other linear inequality constraints may be solved with little more effort than is required for the solution of a system involving the constraints (b) alone. By exploiting this fact, a considerable reduction may be effected in the amount of work required to solve the production smoothing model under discussion here. In the particular three-period example with which we have been dealing, this means that the "essential" part of the system consists of the three equations (2.2) and the two in (2.4). From the viewpoint of numerical analysis, the nine upper bounds represented by equations (2.3) are "inessential," and so the work involved is something of the order of a 5-equation "transportation" system, rather than that of a 14-equation one. By contrast, Bowman's original form for this model would have required a 12-equation system—nine for the availability constraints and three for the delivery requirements. [1] In general, the reduction is from a transportation tableau with  $JT$  columns and  $T$  rows to one containing essentially only  $T-1$  columns and  $T$  rows.

Not only does the stock-flow formulation suggested in this paper have computational advantages over a model phrased entirely in terms of flows, but it also

permits upper bounds to be imposed upon the magnitudes of certain stocks. Both in the case of the "warehousing" model [4] and the Little-Koopmans hydroelectric model [8, 9], capacity limits upon the amount of inventory (or water) accumulation are relevant. Since these limits may be introduced directly into the production smoothing model by placing upper bounds upon the "stock" variables  $h_t$  as well as upon the "flow" variables  $x_{jt}$ , this new complexity adds little to the difficulty of solving the dynamic allocation problem.

**5. Work stabilization policies**

The model with which we have been dealing includes no explicit costs for altering the rate of production or the size of the work force from one period to the next. Indeed, in any concrete application, it may be extremely difficult for a firm to place dollar cost estimates on the loss of employee and community good will that results from deliberate fluctuations in employment. As an alternative to making arbitrary estimates of these "rate of change" costs, it will frequently

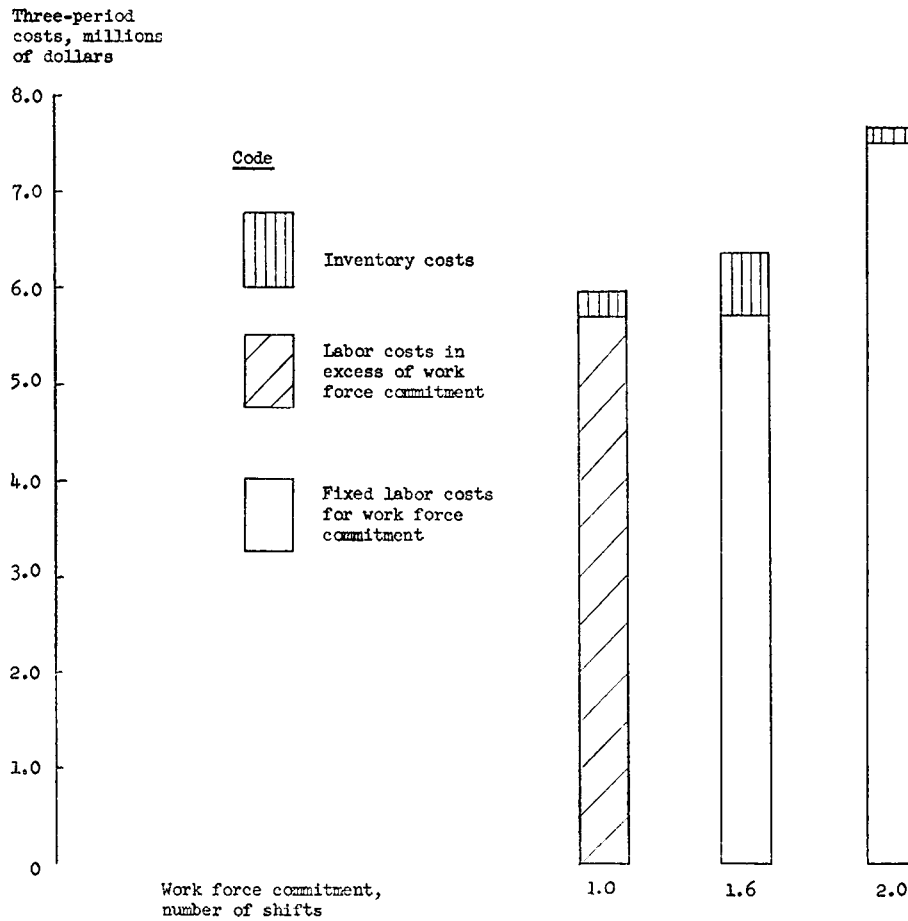


FIG. 2

be desirable to calculate several alternative inventory and production plans--each corresponding to a different assumption as to the "work force commitment," i.e., the level below which employment will not be reduced during any of the time periods covered by the production plan.

In our hypothetical example, the optimum solution shown in Table 3 already provides a minimum of one-shift employment during each of the three periods. The bar chart of Figure 2 compares the cost of this plan with those of two further linear programming solutions--one that assumes a work force commitment of 1.6 shifts and one of 2.0.<sup>8</sup> Interestingly enough, the total of inventory plus labor costs increases very little from a 1.0 to a 1.6 shift commitment--from \$5.94 millions to 6.34. In going from the 1.6 to the 2.0 shift level, however, some of the employees must remain idle, and a substantial cost increase occurs. The total goes up to \$7.66 millions. When presented with an explicit choice of this sort, few managements would hesitate to choose the 1.6 shift policy--despite their initial reluctance to place a dollar value on the worth of stabilizing employment.

In order to obtain a linear programming solution for the cases of 1.6 and 2.0 shifts, only a few changes need be made in the constants and cost coefficients shown in Tables 2 and 3:

Work force commitment, number of shifts.....	none (Tables 2 and 3)	1.6	2.0
$a_{1t}$	100	160	0
$a_{2t}$	100	40	200
$c_1$	10	0	0
$c_2$	15	15	0

These alterations consist simply of recognizing that the effect of a work force stabilization policy is to make the marginal labor costs zero for output rates that fall short of using up the fixed labor commitment, but that the marginal costs increase discontinuously at production rates in excess of this level. Here it is apparent that a linear programming idealization approaches economic reality much more closely than a calculus model which assumes that the marginal cost curve is a continuous one. Both from the viewpoint of realism in problem formulation and from that of ease in numerical computations, the linear programming version seems to be as useable as its calculus counterpart.

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<sup>8</sup> Note that 1.6 shifts yield 480 units of output during the three periods--exactly the cumulated production requirements.

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