Cowles Foundation Paper 109

EXTRACT FROM NAVAL RESEARCH LOGISTICS QUARTERLY,

VOLUME 3, No. 4, December, 1956

A BOUND ON THE USE OF INEFFICIENT INDIVISIBLE UNITS

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The authors discuss a problem of resource allocation in which the usual economic assumption of perfect divisibility of factors does not apply.

The existing body of economic theory on resource allocation rests rather critically on the assumption of perfect divisibility of factors. Whenever the factors involved are indivisible and are to be combined in small numbers, the principles of marginal analysis offer no guidance, and entirely new considerations seem to be called for. To obtain an idea of the peculiar difficulties that may then be faced, we propose to study the following simple but perhaps not unrepresentative problem.

Suppose that a certain number of passengers is to be transported on a given route for which planes of two types are available. Type 1 planes have larger capacity, s_1, but smaller operating cost, c_1, per seat. Provided the total operating cost of the type 2 planes is smaller,

$$s_2 c_2 < s_1 c_1 , \tag{1}$$

the cheapest transportation for a certain number of passengers may call for the use of one or more of the less-efficient type 2 planes. In this paper we obtain a bound on the number of type 2 planes that should ever be used. This bound is of interest in itself for the question of the maximal feasible investment in small planes. But its main usefulness is in reducing the number of trial and error solutions that need be examined in determining the optimal combination of planes for transporting a given number of passengers. Computational problems proper, however, will not be considered here.

Suppose the optimal solution calls for the use of n type 2 planes, with possibly some number of type 1 planes. It then follows that ns_2 is not divisible by s_1. This is so because if $ns_2 = ks_1$ (k an integer), then the passengers in the n type 2 planes could be transported more economically in k type 1 planes.

*Manuscript received April 27, 1956.

Let

$$\frac{ns_2}{s_1} = m + \frac{r}{s_1}, \tag{2}$$

where m is an integer and $1 \leq r \leq s_1 - 1$.

We know that the number of passengers in the n type 2 planes can not be transported more economically with the use of one or more type 1 planes. In particular, there would be no saving in using (m + 1) type 1 planes. Hence

$$n s_2 c_2 \leq (m + 1) s_1 c_1 .$$

Then by use of (2) we have

$$ns_2 c_2 \leq \left(\frac{ns_2}{s_1} - \frac{r}{s_1} + 1\right) s_1 c_1$$

$$\leq \left(\frac{ns_2}{s_1} - \frac{1}{s_1} + 1\right) s_1 c_1$$

$$\leq ns_2 c_1 - c_1 + s_1 c_1 ;$$

and finally,

$$n \leq \frac{s_1 - 1}{s_2} \cdot \frac{c_1}{c_2 - c_1} . \tag{3}$$

Similarly, there would be no saving in using m type 1 planes to transport ms_1 passengers and using type 2 planes to transport the remaining passengers. Since the number of remaining passengers is at most $(ns_2 - ms_1)$, the number of type 2 planes needed is not greater than

$$\frac{ns_2 - ms_1}{s_2} + \frac{s_2 - 1}{s_2} .$$

It follows that

$$ns_2 c_2 \leq ms_1 c_1 + \left(\frac{ns_2 - ms_1}{s_2} + \frac{s_2 - 1}{s_2}\right) s_2 c_2 ,$$

$$\leq ms_1 c_1 + ns_2 c_2 - ms_1 c_2 + (s_2 - 1) c_2 .$$

We then get

$$ms_1 \leq \frac{(s_2 - 1)\, c_2}{c_2 - c_1}.$$

By use of (2) we have

$$ns_2 - r \leq \frac{(s_2 - 1)\, c_2}{c_2 - c_1},$$

or

$$n \leq \frac{s_2 - 1}{s_2} \cdot \frac{c_2}{c_2 - c_1} + \frac{r}{s_2},$$

$$\leq \frac{s_2 - 1}{s_2} \cdot \frac{c_2}{c_2 - c_1} + \frac{s_1 - 1}{s_2}.$$

Then

(4) $$n < \frac{c_2}{c_2 - c_1} + \frac{s_1 - 1}{s_2}.$$

Combining (3) and (4), we have

(5) $$n \leq \mathrm{Min}\left(\frac{s_1 - 1}{s_2} \cdot \frac{c_1}{c_2 - c_1},\ \frac{s_1 - 1}{s_2} + \frac{c_2}{c_2 - c_1}\right).$$

For brevity, write

$$\lambda = \frac{s_1 - 1}{s_2}, \quad \rho = \frac{c_1}{c_2}.$$

Then (5) becomes

(6) $$n \leq \mathrm{Min}\left(\frac{\rho}{1 - \rho}\,\lambda,\quad \frac{1}{1 - \rho} + \lambda\right).$$

From (1) we have a lower bound on ρ, namely

(7) $$\rho > \frac{s_2}{s_1}.$$

The first term of the right hand side of (6) is the smaller one when $\rho < \frac{1}{2} + \frac{1}{2\lambda}$, and the second term is the smaller one for larger values of ρ. The second term increases with ρ and becomes $(s_1 - 1)$ when $\rho = 1 - \frac{1}{\lambda(s_2 - 1)}$. Since $(s_1 - 1)$ is an obvious bound for n, (6) should not be used for larger values of ρ. Also, since n is an integer, we can omit any fractional part of the number obtained for its bound. Letting [x] stand for the largest integer not exceeding x, we can summarize the above by

$$
(8) \qquad
\begin{aligned}
n &\le \left[\frac{\rho}{1-\rho}\lambda\right], & \frac{s_2}{s_1} < \rho \le \frac{1}{2} + \frac{1}{2\lambda}, \\
&\le \left[\frac{1}{1-\rho} + \lambda\right], & \frac{1}{2} + \frac{1}{2\lambda} \le \rho \le 1 - \frac{1}{\lambda(s_2 - 1)}, \\
&\le s_1 - 1, & \rho > 1 - \frac{1}{\lambda(s_2 - 1)}.
\end{aligned}
$$

The bounds (8) increase with both ρ (the ratio of costs) and λ (the approximate ratio of sizes), as must be expected for the true maximum of n.

As an example, suppose the capacities of DC-6's and DC-3's on first class flights are 58 and 36 passengers, respectively. Then

$$\lambda = \frac{19}{12}, \quad \frac{s_2}{s_1} = \frac{18}{29}, \quad \frac{1}{2} + \frac{1}{2\lambda} = \frac{31}{38}, \quad 1 - \frac{1}{\lambda(s_2 - 1)} = \frac{653}{665}.$$

The bounds on the number of DC-3's which need be considered for possible use, for various cost ratios, are given below.

ρ	$\frac{12}{19}$	$\frac{13}{19}$	$\frac{14}{19}$	$\frac{15}{19}$	$\frac{31}{38}$	$\frac{16}{19}$	$\frac{17}{19}$	$\frac{18}{19}$	$\frac{653}{665}$
Bound for n	2	3	4	5	7	7	11	20	57

The problem of finding the optimal number of type 1 and type 2 planes to transport a given number of passengers can be formulated as a discrete linear programming problem.

Let

x_1 = number of type 1 planes,

x_2 = number of type 2 planes,

p = number of passengers to be transported.

Then, we seek integral values of x_1 and x_2 which minimize

$$x_1 s_1 c_1 + x_2 s_2 c_2 , \tag{9}$$

subject to the constraints,

$$x_1 s_1 + x_2 s_2 \geq p \tag{10}$$

$$x_i \geq 0 . \tag{11}$$

Denote the optimal solution by x_1^0, x_2^0. Comparing its cost with that of using

$$\begin{aligned} x_1 &= x_1^0 + \left[\frac{x_2^0 s_2 + s_1 - 1}{s_1}\right] \\ x_2 &= 0 , \end{aligned} \tag{12}$$

one can obtain inequality (3) with x_2^0 for n.

Also, letting

$$\begin{aligned} x_1 &= x_1^0 + \left[\frac{x_2^0 s_2}{s_1}\right] \\ x_2 &= \left[\frac{p - x_1 s_1 + s_2 - 1}{s_2}\right], \end{aligned} \tag{13}$$

one can get inequality (4).

The linear programming formulation permits the following geometric interpretation of the allocation problem. In Figure 1, combinations of numbers of the two types of planes are represented by points of a quadratic lattice having integral coordinates. All combinations having a capacity for at least p passengers are located on and above a line, to be called an isocapacity, whose slope is $- s_1/s_2$ and whose intercept on the x_i axis is p/s_i. The family of isocosts (lines of constant cost) consists of lines of greater slope, $- c_1 s_1/c_2 s_2$, than the isocapacities. The optimum combination lies on that isocost nearest to the origin which passes through a lattice point to the right of or on the isocapacity line for

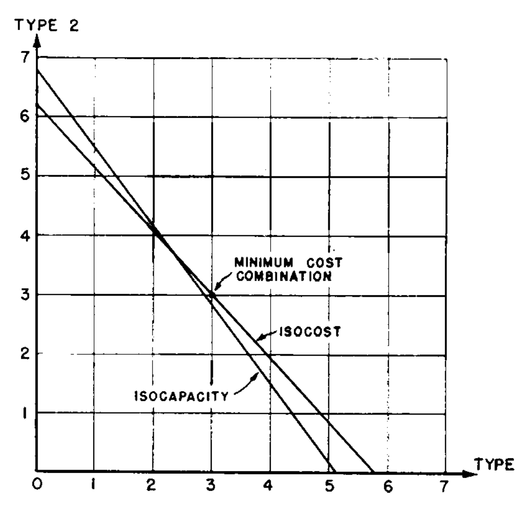

Figure 1

p passengers. This means that the interior of the triangle, formed by the isocost on the right, the isocapacity on the left, and the x_1 axis at the bottom, should not contain any lattice points.

If there are more than two unit sizes of the indivisible factor, the inequalities (8) may still be applied to yield a bound on the input of any unit in terms of the sizes and costs of larger units. Let the units be labelled in order of decreasing size and (hence) of increasing average costs,

$$s_1 > s_2 > \cdots\cdots > s_n ,$$

$$c_1 < c_2 < \cdots\cdots < c_n ,$$

with

$$s_1 c_1 > s_2 c_2 > \cdots\cdots > s_n c_n .$$

Define

$$\lambda_{ij} = \frac{s_i - 1}{s_j} , \qquad \rho_{ij} = \frac{c_i}{c_j},$$

then (8) becomes

$$n_j \le \underset{i < j}{\mathrm{Min}} \ \mathrm{Min}\left(\left[\frac{\rho_{ij}}{1 - \rho_{ij}} \lambda_{ij} \right] , \left[\frac{1}{1 - \rho_{ij}} + \lambda_{ij} \right] , \ s_i - 1 \right) . \tag{14}$$

The constraint $i < j$ is essential, for we must have $\rho_{ij} < 1$.

Every cost minimization problem suggests an associated maximization problem of similar structure. (This should not be confused with the duality principle of linear programming). For example, in the airplane problem one can ask for a bound on the number of type 2 planes that should ever be used to transport the maximum number of passengers having available a fixed budget. This companion piece will be treated by specific reference to a more practical revenue maximization problem. A firm with a limited capital budget considers the purchase of two types of equipment having unit costs of q_1 and q_2 and yielding returns of r_1 and r_2 per dollar, respectively, where $q_1 > q_2$ and $r_1 > r_2$. (Here q_i and r_i correspond respectively to $s_i c_i$ and $1/c_i$ in the airplane problem). Then the best combination that the firm can purchase depends on the exact amount of capital available. But we may ask again for the largest number of the less-efficient production units that should ever be bought.

Let n be the maximal number of type 2 production units that should ever be bought. To determine n, we need consider only the case that the budget limit q equals nq_2. Comparing the revenue from n production units of type 2 with that from the use of only type 1 units, we have

$$nq_2r_2 \geq \left[\frac{nq_2}{q_1}\right] q_1r_1 ,$$

$$nq_2r_2 \geq \left(\frac{nq_2}{q_1} - \frac{q_1 - 1}{q_1}\right) q_1r_1 ,$$

$$nq_2r_2 \geq nq_2r_1 - (q_1 - 1)\, r_1 ,$$

(15)
$$n \leq \frac{q_1 - 1}{q_2} \frac{r_1}{r_1 - r_2} ,$$

which is analogous to (3).

Next compare the revenue from n production units of type 2 with that from a combination of m units of type 1 and as many units of type 2 as the remaining budget permits. We have

$$mq_1r_1 + \left[\frac{nq_2 - mq_1}{q_2}\right] q_2r_2 \leq nq_2r_2 .$$

Since

$$\left[\frac{nq_2 - mq_1}{q_2}\right] \geq \frac{nq_2 - mq_1}{q_2} - \frac{q_2 - 1}{q_2} ,$$

it follows that

$$mq_1r_1 + nq_2r_2 - mq_1r_2 - (q_2 - 1)\, r_2 \leq nq_2r_2 ,$$

and then

(16)
$$m \leq \frac{q_2 - 1}{q_1} \frac{r_2}{r_1 - r_2} .$$

Now let m take on its largest value compatible with a given $q = nq_2$. Then

$$nq_2 \leq (m + 1)\, q_1 - 1 ,$$

and from (16) we get

$$nq_2 \leq \left(\frac{q_2 - 1}{q_1} \frac{r_2}{r_1 - r_2} + 1\right) q_1 - 1 ,$$

$$n \leq \frac{q_2 - 1}{q_2} \frac{r_2}{r_1 - r_2} + \frac{q_1 - 1}{q_2} .$$

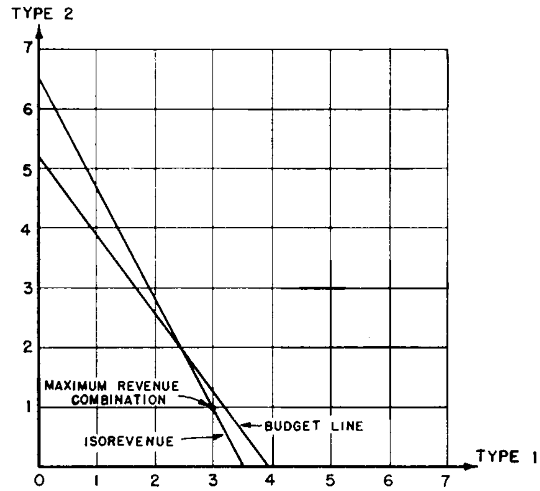

Figure 2

Finally,

$$n < \frac{r_2}{r_1 - r_2} + \frac{q_1 - 1}{q_2}, \tag{17}$$

which corresponds to (4). As in the airplane problem, here also there is an obvious upper bound of $(q_1 - 1)$ for n.

The geometric interpretation of the solution is similar to the one in the airplane problem. In Figure 2 all combinations which cost less than the fixed budget are to the left of the budget line. We let isorevenue lines denote combinations of constant revenue. Then the optimal combination lies on that isorevenue line which passes through a lattice point to the left of or on the budget line and leaves no lattice point in the interior of the triangle bordered by the isorevenue line on the left, the budget line on the right, and the x_1 axis at the bottom.

If there are more than two types of units, we let

$$\phi_{ij} = \frac{q_i - 1}{q_j}, \quad \psi_{ij} = \frac{r_i}{r_j},$$

and a bound on the number of units of the j-th type that should ever be purchased is given by

$$n_j \leq \underset{i<j}{\text{Min}}\ \text{Min}\left(\left[\frac{\psi_{ij}}{\psi_{ij} - 1}\ \phi_{ij}\right], \left[\frac{1}{\psi_{ij} - 1} + \phi_{ij}\right], q_i - 1\right), \tag{18}$$

which corresponds to (14).

One important difference between the results in these two problems is that in the maximization problem the bound given by (15) should be ignored because it is never less than the one in (17). Hence (18) can be replaced by

$$n_j \leq \underset{i<j}{\text{Min}}\ \text{Min}\left(\left[\frac{1}{\psi_{ij} - 1} + \phi_{ij}\right], q_i - 1\right). \tag{19}$$