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A Generalized Comparative Statics in Linear Programming

1. In the preceding note, Beckmann shows how the conclusion given by Samuelson[1] that $\delta b' \, \delta x \geqslant 0$, where the vector of activities x changes as a result of a change of the vector of revenues b, can be generalized to include changes in the vector of factor availabilities c and in the technology matrix A. These results are arrived at solely by consideration of known relationships among the vectors in question, not by any explicit consideration of the restraints applying to the individual elements of these vectors. It is the purpose of the present note to show the further considerable sharpening of these results which may be obtained by taking into account the specific restraints on the individual variables. We shall find, in fact, (*a*) that the two quantities in Beckmann's expression (9) are *individually* subject to a corresponding restraint ; (*b*) that the restraint applies not merely to the vector products shown but to each term in the vector products ; and (*c*) that a simple rule separates those cases in which the equality sign holds in these expressions from those cases in which the inequality sign holds.

In section 2 the assertions (*a*) and (*b*) will be simultaneously demonstrated ; in sections 3 and 4 the consequent sharpening of Beckmann's result will be applied to the special case he considers in his section 2 of a change in a single technological coefficient ; in section 5 some further consequences of (*a*) are examined ; in section 6 the rule mentioned in (*c*) is set out and demonstrated in detail ; and in section 7 the rule (*c*) is illustrated using the example from Marshall given by Beckmann in the preceding note.

2. Consider an activity analysis problem $b'x = \max\limits_{x}$ subject to $Ax \leqslant c$, $x \geqslant 0$, and consider some arbitrary changes in the parameters $A \longrightarrow A + \delta A$, $c \longrightarrow c + \delta c$, $b \longrightarrow b + \delta b$, giving rise to the corresponding changes in the solutions for activity levels and efficiency prices $x \longrightarrow x + \delta x$ and $\lambda \longrightarrow \lambda + \delta\lambda$. To simplify the presentation of this section, we shall define a vector of *net* revenues p, where the typical term of this vector is :

$$P_k = b_k - \sum_n \lambda_n a_{kn} \leqslant 0.$$

and where the change in net revenue of the kth activity, as a result of the changes in parameters, is :

$$\begin{aligned} \delta P_k &= \delta(b_k - \sum_n \lambda_n a_{kn}) \\ &= \delta b_k - \sum_n \delta\lambda_n \cdot a_{kn} - \sum_n \lambda_n \, \delta a_{kn}. \end{aligned}$$

Now the efficiency price theorem states that all activities for which net revenue is negative have zero activity levels ; that is, if, for any k, $P_k < 0$ then $x_k = 0$, and if $P_k + \delta P_k < 0$ then $x_k + \delta x_k = 0$. Hence it is evident that if $x_k > 0$ and $x_k + \delta x_k > 0$, then $\delta P_k = 0$, because then $P_k = 0 = P_k + \delta P_k$. Similarly we know that if $P_k < 0$ and $P_k + \delta P_k < 0$, then $\delta x_k = 0 = x_k$.

Hence if *both* $\delta x_k \neq 0$ *and* $\delta P_k \neq 0$ for any k, it must be true for that k either that $x_k = 0$ and $\delta x_k > 0$ or that $x_k + \delta x_k = 0$ and $\delta x_k < 0$. In the first of these two cases

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[1] Samuelson, Paul A., Comparative Statics in the Logic of Economic Maximising, *Rev. of Economic Studies*, Vol. XIV, 1, (1946-47), pp. 41-43.

$(1 + \delta)\, P_k = 0$ and it follows that $\delta P_k > 0$: in the second case $P_k = 0$ and $\delta P_k < 0$. Hence it must be true for *every* k that :

$$\delta P_k \; \delta x_k = \delta(b_k - \sum_n \lambda_n \, a_{kn}) \; \delta x_k \geqslant 0. \tag{1}$$

A symmetrical argument, using the restraint $c \geqslant Ax$ [also $(1 + \delta)c \geqslant (1 + \delta)\,(Ax)$] and the relation :

$$\text{“ if } c_n \left\{ \begin{matrix} > \\ = \end{matrix} \right\} \sum_k a_{kn} \; x_k \text{ then } \lambda_n \quad \left\{ \begin{matrix} = \\ \geqslant \end{matrix} \right\} 0 \text{ ”},$$

can be used to prove that :

$$\delta\lambda_n \delta(c_n - \sum_k a_{kn} \, x_k) \leqslant 0 \tag{2}$$

for every n. Obviously it follows from these two results that :

$$\delta(b' - \lambda' A) \; \delta x \geqslant 0 \tag{1*}$$

$$\text{and} \qquad \delta\lambda' \delta(c - Ax) \leqslant 0. \tag{2*}$$

These inequalities are a step beyond (9) of Beckmann's paper, as they are stronger than that result, and imply it.

3. As a special case consider the change of a single technological coefficient $\delta a_{j_0 m_0} < 0$, where all other parameters remain constant. That is, let $\delta b = \delta c = 0$, and let every element of δA except $\delta a_{j_0 m_0}$ be zero. If we substitute these values into equations (1), taking due account of the definition of δP_m given prior to equations (1), we directly obtain the following equations :

$$(3) \qquad \left\{ \begin{array}{ll} - \sum_n \delta\lambda_n \; a_{kn} \; \delta x_k \geqslant 0 & \qquad \text{for all } k \neq j_0 \\ -\lambda_{m_0} \; \delta a_{j_0 m_0} \; \delta x_{j_0} - \sum_n \delta\lambda_n \; a_{j_0 n} \; \delta x_{j_0} \geqslant 0 & \end{array} \right.$$

and from (2) we similarly obtain :

$$(4) \qquad \left\{ \begin{array}{ll} - \sum_k \delta\lambda_n \; a_{kn} \; \delta x_k \leqslant 0 & \qquad \text{for all } n \neq m_0 \\ -\delta\lambda_{m_0} \; \delta a_{j_0 m_0} \; x_{j_0} - \sum_k \delta\lambda_{m_0} \; a_{k m_0} \; \delta x_k \leqslant 0. & \end{array} \right.$$

If we subtract the sum over n of the equations (4) from the sum over k of the equations (3) we obtain :

$$\delta a_{j_0 m_0} \; (\delta\lambda_{m_0} \; x_{j_0} - \lambda_{m_0} \; \delta x_{j_0}) \geqslant 0 \tag{5}$$

If x_{j_0}, $\lambda_{m_0} \neq 0$, this gives $\dfrac{\delta x_{j_0}}{x_{j_0}} \geqslant \dfrac{\delta\lambda_{m_0}}{\lambda_{m_0}}$ (which implies either $\delta\lambda_{m_0} \leqslant 0$ or $\delta x_{j_0} \geqslant 0$ or both) as shown by Beckmann in his section 2. But the result may be sharpened : after summing (3) if we add the term $\delta\lambda_{m_0} \; \delta a_{j_0 m_0} \; x_{j_0}$ to both sides of the resulting inequality and rearrange we obtain :

$$\begin{aligned} & \delta\lambda_{m_0} \; \delta a_{j_0 m_0} \; x_{j_0} - \lambda_{m_0} \; \delta a_{j_0 m_0} \; \delta x_{j_0} \geqslant \\ & \delta x_{m_0} \; \delta a_{j_0 m_0} \, x_{j_0} + \sum_k \sum_n \delta\lambda_n \, a_{kn} \, \delta x_k. \end{aligned} \tag{6}$$

Further, by summing (4) we learn that the righthand term of this inequality is non-negative. That is, the expression in (5) is not merely non-negative, but is not less than the negative of the sum of (4).

4. The interpretation of (6) in economic terms is facilitated if we write it as follows :

$$\begin{aligned} & (\delta\lambda_{m_0} \; \delta a_{j_0 m_0} \; x_{j_0} + \delta\lambda_{m_0} \; a_{j_0 m_0} \; \delta x_{j_0}) - (\lambda_{m_0} \; \delta a_{j_0 m_0} \; \delta x_{j_0} + \delta\lambda_{m_0} \; a_{j_0 m_0} \; \delta x_{j_0}) \\ & \geqslant \delta\lambda_{m_0} \; \delta a_{j_0 m_0} \; x_{j_0} + \sum_k \sum_n \delta\lambda_n \, a_{kn} \, \delta x_k \; (\geqslant 0). \end{aligned}$$

The expression to the left of the first inequality now represents the tendency for the efficiency price of a factor and its utilization to move together, minus the tendency for the activity level and its " costs " (evaluated using the efficiency price) to change in opposite directions, where the stated changes in utilization and cost refer only to that part of these changes which are attributable to the activity and the factor, respectively, which are directly involved in the technological change. The expression (6) then tells us that the indicated sum of this particular set of tendencies must not be less than the sum of the tendencies of efficiency prices and utilizations of factors (including those not directly affected by the technological change) to move together for every factor (if they move at all).

5. This result may be generalised in terms of the matrix equations (1*) and (2*) :

$$\delta(b' - \lambda' A)\ \delta x \geqslant 0 \geqslant \delta\lambda'\ \delta(c - Ax).$$

Distributing the operators and rearranging, we obtain :

$$(7) \qquad \delta b'\ \delta x + \delta\lambda'\ \delta A.\ x - \lambda'\ \delta A\ \delta x \geqslant \delta\lambda'\ \delta(Ax) \geqslant \delta\lambda'\ \delta c. \qquad (8)$$

We may now consider some interesting special cases. Consider, for example, $\delta b = \delta c = 0$, i.e., where the only changes are in technology, this gives :

$$\delta\lambda\ \delta A \cdot x - \lambda'\ \delta A\ \delta x \geqslant \delta\lambda'\ \delta(Ax) \geqslant 0, \qquad (6^*)$$

which is the generalization of (6). Its economic interpretation is the same as that just given for (6). Another interesting case is $\delta A = \delta c = 0$, i.e., when the only changes are in the activity revenues. Here we obtain :

$$\delta b'\ \delta x \geqslant \delta\lambda'\ A\ \delta x \geqslant 0 \qquad (9)$$

which is a sharpened form of the inequality obtained by Samuelson.

It should be noted that there are no term by term inequalities behind (7), (8), (6*), and (9) in the same sense that there are such inequalities behind (1*) and (2*), namely those given in (1) and (2). By (1) and (2) we know that for every k and every n respectively, the indicated inequality holds for the kth term of the product of the two vectors $\delta p'$ and δx on the left hand side of (1*) and for the nth term of the product of the two corresponding vectors on the left hand side of (2*). No such statement can be made about the inequalities (7), (8), (6*), and (9) because the rearrangement of terms necessary to obtain them has obliterated the term by term relations.

6. It is of interest to note when it may be expected that the strict inequalities hold, and when not. We have already seen that if there is no change in the set of free factors we may write $\delta\lambda'\ \delta(Ax) = \delta\lambda'\ \delta c$. (In fact, we may do so also if any factor which ceases to be or becomes free is *only just* free before or after the parameter change, resp., that is if $c_k = A_k x$ even though $\lambda_k = 0$ or if $(1 + \delta)c_k = (1 + \delta)\ (A_k x)$ even though $(1 + \delta)\lambda_k = 0$.) Similarly it can be shown that if there is no change in the set of activities, or if any activity which ceases or begins goes to or from the borderline case of inactivity, then :

$$\delta b'\ \delta x = \delta(\lambda' A)\ \delta x.$$

Hence we conclude that if for any factor c_k which ceases to be or becomes free the relation $c_k = \sum_j a_{kj}\, x_j$ holds for all k both before and after the change in parameters, and if for any activity x_j which ceases or begins the relation $b_j = \sum_k \lambda_k\, a_{kj}$ holds for all j both before and after the change in parameters, then the relations (7) and (8) may be written :

$$(7^*) \qquad \delta b'\ \delta x + \delta\lambda' \delta A \cdot x - \lambda'\ \delta A\ \delta x = \delta\lambda'\ \delta(Ax) = \delta\lambda' \delta c. \qquad (8^*)$$

The special cases (6*) and (9) then become :

$$\delta\lambda'\ \delta A \cdot x - \lambda'\ \delta A\ \delta x = \delta\lambda'\ \delta(Ax) = 0 \qquad (6^{**})$$

$$\text{and } \delta b'\ \delta x = \delta\lambda' A\ \delta x = 0 \qquad (9^*)$$

Thus we know that if only one technical coefficient, and no other parameter, changes and if the consequent adjustment of the system stays within the above limits, then $\frac{\delta x_{j0}}{x_{j0}} = \frac{\delta \lambda_{m0}}{\lambda_{m0}}$; and we know that if the only change is in one activity revenue, and if the consequent adjustment of the system stays within the above limits, then the activity level of the activity whose revenue changes *will not change.*

Conversely, we may conclude that the necessary and sufficient condition that inequalities hold in (8) and (7), respectively, is that the adjustments attendant upon the changes in parameters must include at least one change in the set of free factors and at least one change in the set of zero activity levels such that the factor in question is interior to the set of free factors when free and the activity level in question is interior to the set of zero activity levels when zero.

7. We may now say something more on the economic meaning of our results. The expression ($b' - \lambda' A$) is a vector of net profitabilities of activities—the difference, in each case, between revenue and cost, where the latter is evaluated at the " efficiency " prices of the productive factors. Each element in this vector is necessarily zero or negative ; and when it is negative, the activity level of the corresponding activity is zero. On the face of it, (1*) says that whenever there are changes in the parameters b and A, the resultant changes in λ and x are such that each activity which has become more profitable (the net profitability has become less negative, or has become zero) as a result of all these changes, will have increased its activity level if this level changes at all, and conversely ; and likewise, if either one has changed in the negative direction, so has the other. This seems reasonable enough : the more profitable activities have increased their output, the less profitable ones have decreased it, insofar as they have changed at all. Similarly, (2*) says that each productive factor whose unemployed residue (always positive or zero) has increased will have suffered a fall in price, and conversely ; and each factor whose residue has decreased will have enjoyed a rise in price, and conversely ; but in each case a definite change in the one variable does not exclude a zero change in the other.

In evaluating these results, we must note the following points : Whenever an element of the vector ($b' - \lambda' A$) is changing, the corresponding x is almost always not ; and vice versa. If the former is changing, then the activity level in question is zero either before or after the change, and will be zero both before and after the change if net profitability is not zero either before or after. To take an example of a case when *both* are changing, suppose that net profitability rises from a negative value to zero while the activity level rises from zero to a positive value, for a particular activity; the conclusion that both must have changed in the same (algebraic) direction is a perfectly straightforward result of the assumption, basic to linear programming, that negatively profitable activities have zero activity levels. The interpretation of (2) and (2*) is exactly analogous.

8. The results just arrived at are very helpfully illustrated by Marshall's train-boat example,[1] used by Beckmann in the preceding note. First it is necessary to note that the restraints applying to the dual of the form of the problem given by Beckmann are :

$$-b_1 = \lambda_1 a_1 - \lambda_2$$
$$-b_2 = \lambda_1 a_2 - \lambda_2$$

whenever both kinds of transport are used ; this gives :

$$\lambda_1 = \frac{b_1 - b_2}{a_2 - a_1}$$

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[1] Letter from A. Marshall to F. Y. Edgworth, 22. IV. 09. Pigou, A. C. (ed.) *Memorials of Alfred Marshall*, London 1925, pp. 439-442.

and

$$\frac{\delta\lambda_1}{\delta a_2} = \frac{b_2 - b_1}{(a_2 - a_1)^2} = -\frac{\delta\lambda_1}{\delta a_1} = \frac{\lambda_1}{a_1 - a_2}$$

Further, from the form of the problem given by Beckmann we have :

$$x_1 = \frac{c_1 - a_2c_2}{a_2 - a_1}$$

$$x_2 = \frac{c_1 - a_1c_2}{a_1 - a_2}$$

and

$$\frac{\delta x_1}{\delta a_1} = \frac{c_1 - a_2c_2}{(a_1 - a_2)^2} = \frac{x_1}{a_2 - a_1}$$

$$\frac{\delta x_2}{\delta a_2} = \frac{c_1 - a_1c_2}{(a_1 - a_2)^2} = \frac{x_2}{a_1 - a_2}$$

whenever both kinds of transport are used. Hence for $a_2 > 0$ it can be seen that :

$$\frac{\delta x_2}{x_2} = \frac{\delta\lambda_1}{\lambda_1} = \frac{\delta a_2}{a_1 - a_2} > 0\,;$$

similarly if $a_1 > 0$:

$$\frac{\delta x_1}{x_1} = \frac{\delta\lambda_1}{\lambda_1} = \frac{\delta a_1}{a_2 - a_1} < 0.$$

That is, the asserted equality does in fact hold both for a rise in the boat fare and for a rise in the train fare.

Carrying on with Beckmann's interpretation, we note that a rise in the boat fare increases λ_1 (the marginal product of money in terms of negative travel time) because a narrowing of the gap between the train fare and the boat fare increases the number of miles for which a given increment of money makes possible a switch from boat to train travel ; similarly, a rise in the train fare *decreases* λ_1 because it widens the gap between the train and boat fares. This brings into clearer light the seeming paradox that a rise in the cheaper fare increases the amount of the cheaper form of travel, while a rise in the higher fare decreases the amount of the more expensive form of travel, and that all this is consistent with equiproportionate changes in λ_1 and that form of travel whose fare changes.

When the boat fare passes the point at which taking the boat the whole way uses up the traveller's two florins, leaving no margin for any train travel, then the first of the equalities determining λ_1 and λ_2 becomes an inequality, and we have :

$$\frac{\delta x_2}{x_2} < \frac{\delta\lambda_1}{\lambda_1}\,;$$

in fact we may have $\delta x_2 < 0$ for $\delta a_2 > 0$, supposing either that the traveller will ride the boat as far as his two florins will take him and walk the rest of the way, or that he will give up the trip entirely. If he persists in taking the trip the analysis continues to apply ; and for any *further* increases in the boat fare (beyond the fare at which the traveller has quit using the train) it is once again true that :

$$\frac{\delta x_2}{x_2} = \frac{\delta\lambda_1}{\lambda_1}$$

since the set of activities is stable again. In fact, equality will hold even if either end point of a boat fare increase coincides with the critical fare at which train travel ceases ; the strict inequality will apply only when a fare increase overlaps this critical fare, causing the traveller both to stop train travel and to start walking.

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