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Comparative Statics in Linear Programming and the Giffen Paradox [1]

As Professor Samuelson has pointed out [1] predictions about the directions in which variables tend to change in response to changes in the data can be made for Linear Programming programs just as for ordinary maximization problems and without going into the technicalities of linear programming theory. The purpose of this paper is to show how these predictions can be generalized by use of the Minimax (or Lagrange multiplier) Theorem of Linear Programming [2].

The point of such qualitative analysis of comparative statics is of course that explicit representations of the solutions of a linear programming problem as functions of the parameters of the problem are not known. Every change in the data would therefore call for a new analysis of the problem.[2]

The present generalization of Samuelson's inequality, which applies to changes in profitabilities, consists in admitting variations of the technology matrix (section 1). This reintroduces the possibility of a Giffen paradox (ruled out in Samuelson's case), of which an example is given (section 2).

1. Consider a linear programming problem :

$$\max_x b'x$$

subject to

$$Ax \leqq c \tag{1}$$

$$x \geqq 0 \tag{2}$$

where $x = (x_k)$ represents the (unknown) activity levels, $b = (b_k)$ the vector of profitabilities, $c = (c_n)$ the vector of resource capacities and $A = (a_{nk})$ the technology matrix. We also have a vector $\lambda \geqq 0$ of Lagrange multipliers, representing the marginal productivity (efficiency price) of the resource quantities c. (For fuller explanation cf. [3]). Let now some arbitrary changes :

$$\begin{aligned} A &\longrightarrow A + \delta A \\ b &\longrightarrow b + \delta b \\ c &\longrightarrow c + \delta c \end{aligned}$$

in the data be imposed, such that the new problem still has a solution, giving rise to changes

$$\begin{aligned} x &\longrightarrow x + \delta x \\ \lambda &\longrightarrow \lambda + \delta\lambda \end{aligned}$$

in the activity levels and efficiency prices, respectively.

[1] Thanks are due to Professors Dorfman, Harberger, and Koopmans for many helpful comments.

[2] Of course the general form of the dependence of the solution vector on the data is known. The solution vector is a step function of the profitabilities (the objective function), a piecewise linear function of the capacity vector (*i.e.*, of the requirements) and a piecewise continuous broken linear function of the coefficients of the technology matrix (the input-output matrix). All this follows from the fact that any optimizing activity levels are a solution of some subset of equations among the constraints.

The minimax theorem of linear programming states the following. Let:

$$\phi(u, v) = b'u + v'(c - Au) \tag{3}$$

Then if $u = x$ is a solution to the linear programming problem and λ is the vector of Lagrange multipliers :

$$\phi(x,\lambda) = \operatorname*{Max}_{u \geqq 0} \phi(u,\lambda) = \operatorname*{Min}_{v \geqq 0} \phi(x,v). \tag{4}$$

Denote the function (4) associated with the changed problem by ϕ_δ. (4) implies that:

$$\phi(x,\lambda) - \phi(x + \delta x,\lambda) \geqq 0 \tag{5}$$

$$\phi(x,\lambda) - \phi(x,\lambda + \delta\lambda) \leqq 0 \tag{6}$$

$$\phi_\delta(x + \delta x,\lambda + \delta\lambda) - \phi_\delta(x,\lambda + \delta\lambda) \geqq 0 \tag{7}$$

$$\phi_\delta(x + \delta x,\lambda + \delta\lambda) - \phi_\delta(x + \delta x,\lambda) \leqq 0 \tag{8}$$

Consider the expression:

$$(5) + (7) - (6) - (8) \geqq 0$$

Eliminating and rearranging terms one obtains:

$$(\delta b' - \lambda'\delta A)\cdot\delta x - \delta\lambda'(\delta c - \delta A\cdot x) \geqq 0 \tag{9}$$

If $\delta A = 0$, $\delta c = 0$, this inequality is Samuelson's Le Chatelier principle :

$$\delta b'\,\delta x \geqq 0 \tag{10}$$

as applied to Linear Programming ([1], p. 42, equation (5). The inversion of the $\geqq$ sign results from the fact that there a minimum problem is considered.)

(10) says that activity levels tend to move in the same direction as the generating changes in profitabilities. In particular in the case of a change in a single profitability $\delta b_j \neq 0$, $\delta b_{j'} = 0$ if $j' \neq j$

$$\delta b_j\,\delta x_j \geqq 0. \tag{11}$$

By means of the duality principle a corresponding inequality follows for the efficiency prices.

$$\delta c'\,\delta\lambda \leqq 0 \tag{12}$$

stating that efficiency prices tend to move in a direction opposite to the changes in capacities. Under simultaneous changes in profitabilities and capacities, we have:

$$\delta b'\,\delta x - \delta c'\,\delta\lambda \geqq 0. \tag{13}$$

This says that at least one of the two sets of variables must behave in the expected manner ; unless activities respond in the proper way to the profitability changes, prices must adjust in the proper way to changes in availability.

The generalization obtained by (9) consists in the inclusion of changes in the technological coefficients, δA. The result is as follows. Provided the " gross change " δb in profitability is corrected for a simultaneous change in technology in terms of the old prices, and provided the gross change in capacities is adjusted for technological changes in terms of the old activity levels x, then the same tendencies prevail as in the case (13) of mere changes in profitabilities and capacities.

Remark :

For a quadratic programming problem with the maximand

$$x'\,Qx + b'x \qquad Q \text{ negative semi-definite}$$

we obtain similarly:

$$(2x'\,\delta Q + \delta b' - \lambda'\delta A)\,\delta x - \delta\lambda'(\delta c - \delta A\cdot x) \geqq -\,\delta x'\,\delta Q\delta x \tag{9*}$$

Only if δQ itself (or its converse — δQ) is negative semi-definite, making the right hand side non-negative, does (9*) (or its alternative) make a definite statement about the direction of change of x and λ. The interpretation of (9*) corresponds then to that given for (9).

2. As an example consider the case in which only one technological coefficient (say a_{ij}) is changed and all profitabilities and capacities remain constant. From (9) we obtain $-\lambda_i\, \delta a_{ij}\, \delta x_j + \delta\lambda_i\, \delta a_{ij}\, x_j \geqq 0$ or

$$\delta a_{ij}\,(x_j\, \delta\lambda_i - \lambda_i \delta x_j) \geqq 0 \tag{14}$$

If either $\lambda_i = 0$ or $x_j = 0$ this inequality is implied in the previously imposed restrictions $x_j \geqq 0$ and $\lambda_i \geqq 0$. Let therefore $\lambda_i > 0$, $x_j > 0$. In the case of technological progress, $\delta a_{ij} < 0$, we now conclude that :

$$\frac{\delta x_j}{x_j} \geqq \frac{\delta\lambda_i}{\lambda_i} \tag{15}$$

Suppose that a technological coefficient associated with a " scarce " input and an " active " activity is decreased and that all other constants remain unchanged. Then the induced relative change in the level of the activity that is directly involved, is not exceeded by the relative change in the efficiency price (the marginal productivity) of the input directly involved. In particular a decrease of the activity level calls for at least as large a decrease (in relative size) of the efficiency price.

That an activity level may actually decline in response to a " favorable " technological change is indicated by the fact that Marshall's paradigm of the Giffen paradox [4] may be put into the form of a linear programming problem[1].

Marshall's example runs as follows :

> " I believe that people in Holland travel by canal boat instead of railway sometimes on account of its cheapness. Suppose a man was in a hurry to travel 150 kilos. [sic]. He had two florins for it and no more. The fare by boat was one cent a kilo, by third class train two cents. So he decided to go 100 kilos by boat and fifty by train : total cost two florins. On arriving at the boat he found the charge had been raised to 1¼ cents per kilo. ' So then I will travel 133⅓ kilos (or as near as may be) by boat, I can't afford more than 16⅔ kilos by train' " ([5], p. 441).

Let x_1, x_2 denote the distance travelled by train and boat respectively, and let a_1, a_2 be the fare per kilometer. If c_1 denotes the funds available and c_2 the distance to be covered, and if b_1, b_2 are the times required per mile covered of the two means of transportation, the object is to minimize :

$$b_1x_1 + b_2x_2$$

under the conditions:

$$\begin{aligned} a_1x_1 + a_2x_2 &\leqq c_1 \\ x_1 + x_2 &\geqq c_2 \\ x_1, \quad x_2 &\geqq 0 \end{aligned}$$

In standard form:

$$\begin{array}{ll} \text{Max} & -b_1x_1 - b_2x_2 \\ \text{subject to} & a_1x_1 + a_2x_2 \leqq c_1 \\ & -x_1 - x_2 \leqq -c_2 \\ & x_1,\, x_2 \geqq 0 \end{array}$$

[1] Prof. H. S. Houthakker first drew my attention to this ancient example of a linear programming problem.

If, as in Marshall's example $\delta a_2 > 0$ the inequality (15) is reversed :

(16) $$\frac{\delta x_2}{x_2} \leqq \frac{\delta \lambda_1}{\lambda_1}$$

Here λ_1 is the efficiency price (or Lagrange Multiplier) associated with the first inequality. Since an efficiency price represents the marginal productivity of the factor it is associated with, λ_1 expresses the marginal utility (in terms of travel time) of the money sum c_1. The assertion of (16) is therefore, that the distance travelled by boat can increase as a result of a raise of the boat fare only if the marginal utility of money has been increased even more in proportion. This points directly at the source of the Giffen paradox, the income effect of the price change.

1. Samuelson, P. A., Comparative Statics and the Logic of Economic Maximizing, *Rev. of Ec. Studies*, Vol. XIV, 1, (1946-7) pp. 41-43.

2. Kuhn, H. and A. W. Tucker, Nonlinear Programming. *Second Berkeley Symposium*, 1951, pp. 481-492.

3. Koopmans, T. C., Analysis of Production as an Efficient Combination of Activities, *Activity Analysis of Production and Allocation* (T. C. Koopmans ed.) John Wiley and Sons, New York, 1951, pp. 33-97.

4. Stigler, G. J., *The Theory of Price*. Revised edition. Macmillan, New York, 1954, pp. 43, 44.

5. Letter from A. Marshall to F. Y. Edgeworth, 22. IV. 09, Pigou, A. (ed.) *Memorials of Alfred Marshall*, London 1925, pp. 439-442.

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