

AN INVENTORY POLICY FOR A CASE OF LAGGED DELIVERY*

M. BECKMAN AND R. MUTH

Center for Advanced Study in the Behavioral Sciences and Johns Hopkins University

1. Introduction

In this paper we study the question which inventory policy is optimal in the presence of long delivery lags under otherwise simple conditions. The data on which the solution depends are the average demand per day (say), the carrying cost per unit and day, the fixed and per unit costs of ordering, the penalty per day of shortage, and the delivery lag. It will be shown that under certain conditions, mentioned presently, the optimal inventory policy is of the following simple type: order a fixed amount whenever the level of stock plus outstanding orders reaches a certain level. Formulas for the determination of the order size and of the reordering point will be derived.

The economy of holding inventories rests essentially on two circumstances: the presence of a fixed cost in ordering or manufacturing a quantity of a commodity, and the existence of lags between orders and receipts, or the beginning and the completion of a manufacturing process.

Of these two factors the first has received most attention in formal models of optimal inventory policy. To our knowledge the literature does not contain any treatment, at once rigorous and practical, of optimal inventory policy under uncertainty for the case of substantial delivery lags. Assume for the moment that decisions can be taken only at the beginning of each time interval of unit length (normally a day). The equation system proposed by Dvoretzsky, Kiefer and Wolfowitz [2, pp. 211–213] for the case of fixed lags of T time units are of a high order of complexity, except in the particular case $T = 1$. If T is very large, say of the order of 100 or even 10 time units, this method ceases to be useful. A different approach appears to be called for. This paper explores the inventory problem with long lags for the simplest case involving uncertainty of demand.

2. Case of No Lag

To fix ideas we shall formulate and treat the problem first without delivery lags. The terms to be used throughout are as indicated in the diagram on page 146.

Demand (of customers) is described by a Poisson process. This means: the probability of more than one unit being demanded at a time is zero; for a small period the probability of a demand for one unit is proportional to the length of time with a proportionality factor independent of time. It is well known [3, pp. 364–367] that under these conditions the time intervals between successive demands obey a negative-exponential distribution and that the number of units

* Research undertaken by the Cowles Commission for Research in Economics under contract Nonr-358(01), NR 047-006 with the Office of Naval Research. We are indebted to J. Marschak for many helpful suggestions.

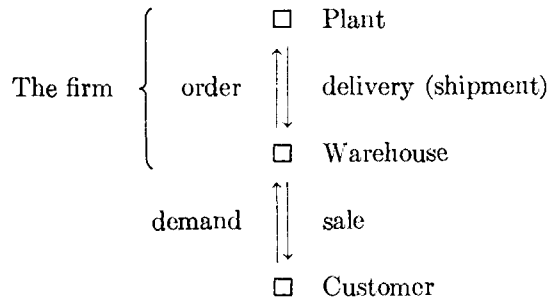


Fig. 1

demanded during any time interval of fixed length is subject to a Poisson distribution.

Ordering (by the warehouse manager) may take place at any time, i.e. not only at the beginning of discrete periods as assumed in the previous papers [1] [2] on inventory policy. For convenience of language we shall use the time unit "day." There is no implication, however, that transactions should occur only at the beginning of the day.

Demand is satisfied out of a warehouse *stock*. When stocks are exhausted a shortage may arise defined as the amount of unsatisfied demand. This we shall treat as a negative stock. It is assumed that demand is postponable indefinitely.

Delivery is now considered to be immediate.

Costs consist of three parts: the cost of ordering, the carrying cost, and the cost of deferred demand.

The *cost of ordering* is assumed to be linear: it consists of a fixed part which is independent of the order size, and of a variable part which is proportional to the amount ordered.

The *carrying cost* of inventories will be assumed proportional to the maximal inventory. If the firm stores only one product, it is this maximal inventory which determines the size of storage facilities and the amount of funds tied down. The *cost of deferred demand*, mainly losses of goodwill and concessions that must be made to waiting customers, is assumed here in the form of a fixed penalty per day of shortage; the assumption being that the loss of goodwill is associated with the frequency of being out of stock rather than the amount of each shortage. On the average, the amount of a shortage is proportional to its duration, so that our assumption is an admissible first approximation.

The aim of inventory policy as considered here is to minimize the expected value of discounted cost over the entire future at the beginning of the enterprise.

Notation

- t time (in days as units)
- $x(t)$ stock
- λ mean demand per day

- K fixed ordering cost
 k proportional ordering cost (price) plus the corresponding interest charge
 α daily interest rate
 c carrying cost per unit of the maximal amount stocked
 $l(x)$ expected loss, a function of the stock level only
 n order size
 s reordering point

We first derive an expression for the function $l(x)$. After a sale of one unit the expected loss is either $l(x - 1)$ or, if an order of n units is planned, $l(x - 1 + n) + K + kn$. Define a function $\delta(n) = \begin{matrix} 0 \\ 1 \end{matrix}$ according as $n \geq 0$. The ordering policy for a stock level $x - 1$ will be such as to minimize with respect to n the quantity $l(x + n - 1) + K\delta(n) + kn$.

To obtain an expression for $l(x)$ consider its variation during a small time interval dt . Either one item will be demanded or no demand will arise. The discounted loss dt units of time later is

$$e^{-\alpha dt} \text{Min}_n [l(x + n - 1) + K\delta(n) + kn]$$

with probability λdt or $e^{-\alpha dt}l(x)$ with probability $1 - \lambda dt$.

Since no penalty arises during dt , when $x \geq 0$, and the carrying cost is $c \cdot \max_t (x(t)) dt$ the expected loss function satisfies a recursive equation.

$$l(x) = e^{-\alpha dt} [(1 - \lambda dt)l(x) + \lambda dt \text{Min}_n (l(x + n - 1) + K\delta(n) + kn)] + c \max x$$

provided $x \geq 0$. For small dt $e^{-\alpha dt} \approx 1 - \alpha dt$. An easy calculation for $dt \rightarrow 0$ now gives

$$(2.1) \quad l(x) = \rho \text{Min}_n [l(x + n - 1) + K\delta(n) + kn] + c^* \max x$$

where¹

$$\rho = \frac{\lambda}{\alpha + \lambda} = \frac{1}{1 + \frac{\alpha}{\lambda}}$$

$$c^* = \frac{c}{\alpha}$$

If this equation has a finite non-negative solution then $\max x$ is finite. Let $\max x = s + n$ where

$$s = \min_t x(t)$$

Then (2.1) implies:

$$l(x) = \rho \text{Min}_n [l(s + n) + K\delta(n) + kn] + c^* \cdot (s + n) \quad \text{for } x = s + 1$$

$$l(x) = \rho l(x - 1) + c^* \cdot (s + n) \quad \text{for } s + 1 < x \leq s + n$$

¹ $1/(1 + (\alpha/\lambda))$ is the discount factor with reference to the "natural time unit" $1/\lambda$.

In particular

$$\begin{aligned} l(s+n) &= \rho^{n-1}l(s+1) + c^*(s+n) \sum_0^{n-2} \rho^i \\ &= \rho^{n-1}[\rho(l(s+n) + K + kn) + c^*(s+n)] + c^*(s+n) \sum_0^{n-2} \rho^i \\ &= \frac{\rho^n}{1-\rho^n} (K + kn) + \frac{c}{\alpha} (s+n) \end{aligned}$$

Now n is determined by the minimization with respect to m , given s , of

$$l(s+m) + K + km \equiv \frac{1}{1-\rho^m} (K + km) + c^*(s+m)$$

That is to say n is the minimizer of

$$(2.2) \quad \frac{K + km}{1 - \rho^m} + c^*m$$

and therefore n is independent of s .

While for given s the solution n of (2.1) is thus uniquely determined, s may still be chosen. The optimal s , which makes all $l(x)$ as small as possible, is clearly $s = 0$.

In order to find the approximate minimizer of (2.2) we may treat n as a continuous variable. n is the nearest integer to the solution of

$$(2.3) \quad \frac{\left(\frac{K}{k} + n\right) |\log \rho|}{\frac{1}{\rho^n} + \rho^n - 2} - \frac{1}{1 - \rho^n} = \frac{c}{k\alpha}$$

For $\alpha \ll \lambda$ the minimand (2.2) may be replaced by

$$\frac{K + km}{m(1 - \rho)} + \frac{c}{\alpha} m$$

yielding

$$n \approx \sqrt{\frac{K\lambda}{\rho c}}$$

If λ is regarded as a fixed rate of sales (2.4) is the well known "economic lot size" formula which has been proposed for the case of certainty of demand (cf. Whitin [5, esp. p. 38]). It results here as an approximate formula for the case of uncertainty without lag in delivery. If, for example,²

$$\begin{aligned} \lambda &= 1 \text{ unit per day} \\ \alpha &= 0.0002 \text{ per day} \\ K &= \$1.80 \text{ per order} \\ k &= \$0.30 \text{ per unit} \\ c &= \$0.002 \text{ per day} \end{aligned}$$

² This set of parameters is taken from an actual case discussed later in Section 5.

then the optimum order size is 78 units, and the average time span between orders is 78 days.

3. Case of Lagged Delivery

3.1. We now formulate the main problem of this paper. The assumptions concerning demand and costs will be retained. *Delivery* (from plant) shall follow upon orders (by the warehouse manager) by a lag of T days, where T is not necessarily an integer.

Stock (positive or negative) plus orders outstanding will be called *net stock*. Net stock equals the stock T days hence when no additional demand arises during the intervening T days. Now present stocks equal net stocks T days ago minus sales, or zero, whichever is larger. It follows that the maximal level of stock is given by the maximal level of net stock. *Carrying cost* shall therefore be assumed to be proportional to the maximal net stock. All *ordering costs* are charged to the firm at the time at which the shipment is received, i.e. at which it materializes into a stock. Although costs may actually arise before delivery, they can always be treated (for reasons of mathematical convenience) as if they were due upon delivery, with proper interest added.

3.2. The main mathematical step is to find an expression for the discounted expected cost. Since all actions of inventory policy taken at a time t or after, affect the cost only after the lag period, i.e. at time $t + T$ or later, it is appropriate to maximize the expected discounted cost after T days, i.e. as of time $t + T$. What are the variables known at time t that the expected discounted cost at time $t + T$ depends on?

Because of the nature of the Poisson process future demand is at any time independent of the past. Future cost at time $t + T$ therefore depends on the availability of stocks at and after time $t + T$ only. Since demand was assumed postponable, the age composition of the orders outstanding at time t does not affect the availability of stocks at time $t + T$.³ Therefore there is no reason why the policy at time t and afterwards, regarding stock availability at time $t + T$ and afterwards, should depend on the dating of the orders outstanding. They may all be lumped together into a "net stock" as far as their effect on stocks and the policies effecting stocks after T days are concerned. We conclude that expected discounted loss computed at time t for the time $t + T$ and thereafter should be a function of net stock at time t only.

3.3. Suppose that net stocks are at their maximal level under an optimal inventory policy. As this level is diminished successively by one unit at a time a level of net stock will be reached ultimately, where ordering cost is less than the advantage of increasing net stock.⁴ Net stock will then be returned to its first, maximal, level, thus completing a cycle. The ordering policy is reduced to two

³ Namely an order τ days old at time t will mature into a physical stock at time $t + T - \tau$, so that at time $t + T$ all orders outstanding at time t , and only these, will have grown into stock.

⁴ The case that it does not pay to keep an inventory but rather to let it run out, will be excluded as uninteresting.

constants, the upper and lower levels of this cycle. Let the lower level be called "re-ordering point" and the difference of upper and lower level "order size." The optimal value of re-ordering point and order size may be found by minimizing the initial expected discounted total cost with respect to these two parameters. In the following section this program of calculations is carried out. $x(t)$ shall now denote the net stock and $l(x)$ the expected discounted loss starting T days hence conditional on a current level of net stock x .

Additional notation

T the lag

A the penalty per day of shortage independent of the amount of shortage

$A^* = \frac{A}{\lambda}$ $c^* = \frac{c}{\lambda}$ penalty and carrying cost per natural time unit

$p(y) = \frac{(\lambda T)^y}{y!} e^{-\lambda T}$ the probability that y units will be demanded during T days

Ordering costs K and k and carrying cost c will be interpreted as containing a discount factor for T days.

3.4. With probability $p(y) = (\lambda T)^y / y! e^{-\lambda T}$ an amount y will be demanded during a period of length T . The probability of a shortage during the time interval from T to $T + dt$ is therefore $P(x) = \sum_{x+1}^{\infty} p(y)$ and the expected discounted penalty $e^{-\alpha t} AP(x)$. Using the reasoning of 1.4 the expected loss at time t may be expressed in terms of the expected loss at time $t + dt$ as follows

$$\begin{aligned} l(x) = e^{-\alpha dt} (1 - \lambda dt) \text{Min}_n [l(x + n - 1) + K\delta(n) + kn] dt \\ + e^{-\alpha dt} \lambda dt l(x) + e^{-\alpha t} AP(x) dt \\ + c \text{Max } x dt \end{aligned}$$

As $dt \rightarrow 0$ we obtain the recursive relation

$$(3.1) \quad l(x) = \text{Min}_n [l(x + n - 1) + K\delta(n) + kn] + A^* p P(x) + c^* \text{Max } x$$

Except in those uninteresting cases in which an infinite inventory or no inventory at all would be optimal, there exists one largest net stock level at which it pays to place a positive order. This means that there exists a maximal x , say $x = s$, such that the minimizing n in (3.1) is positive.

$$(3.2) \quad l(s + 1) = \rho [l(S) + K + k(S - s)] + A^* p P(S) + c^* \cdot S$$

where

$$(3.3) \quad l(S) + K + k(S - s) = \text{Min}_n l(s + n) + K\delta(n) + kn$$

For $s + 1 < x \leq S$ (3.1) assumes the form

$$(3.4) \quad l(x) = \rho l(x - 1) + A^* p P(x) + c^* S$$

Solving (3.4) we obtain

$$l(S) = \rho^{n-1}l(s+1) + A^* \sum_{j=1}^{n-1} P(s+j)\rho^{n-1-j} + c^*S \frac{1-\rho^{n-1}}{1-\rho}$$

Substituting $l(s+1)$ from (3.2) yields after some light calculation

$$(3.5) \quad l(S) = \frac{\rho^n}{1-\rho^n} (K + kn) + \frac{A^*}{1-\rho^n} \sum_{j=0}^{n-1} P(S-j)\rho^{j+1} + \frac{c^*S}{1-\rho}$$

Equation (3.5) expresses the conditional expected loss given a present net stock S , as a function of two choice parameters: the maximum stock S and the order size n . The optimal inventory policy will be completely specified by requiring that the expression (3.5)—plus a term allowing for the initial investment in the inventory—is a minimum with respect to⁵ S and n . If b denotes the initial purchasing price per unit of inventory and we consider the enterprise to be at its beginning the problem is to determine

$$\text{Min } \frac{\rho^n}{1-\rho^n} (K + kn) + \frac{A^*}{1-\rho^n} \sum_{j=0}^{n-1} P(S-j)\rho^{j+1} + \left(\frac{c}{\alpha\rho} + b\right)S$$

The two conditions of minimality are that the first differences of this expression with respect to S and n be approximately zero.

$$(3.6) \quad \frac{A^*}{1-\rho^n} \sum_{j=0}^{n-1} \rho^j p(S-j+1) - \left(\frac{c}{\alpha\rho} + b\right) \simeq 0$$

$$(3.7) \quad A^* \sum_{j=0}^{n-1} \rho^{j+1} [P(S-n) - P(S-j)] - K - k \left(n - \sum_{j=0}^{n-1} \rho^{j+1}\right) \simeq 0$$

If $\alpha \ll \lambda$ so that $\rho \simeq 1$ the coefficient of k nearly vanishes. In this case (3.7) simplifies to

$$(3.8) \quad A^* \sum_{j=0}^{n-1} \rho^{j+1} [P(S-n) - P(S-j)] - K \simeq 0$$

4. Discussion

Economically speaking, k nearly drops out of (3.7) because one must pay the proportionate part of ordering cost regardless of how often one orders. The fact that future outlays are discounted by the factor α leads to only a negligible difference in the present value of this part of ordering cost from, say, ordering 30 units now rather than 15 units now and 15 units two weeks hence. The major difference arises from the fixed cost in ordering.

While the present form of equation (3.8) is convenient for computation, a transformation of the sum permits a more plausible interpretation

$$(4.1) \quad A^* \sum_{j=0}^{n-1} p(S-j) \sum_{i=1}^{j+1} \rho^i \simeq K$$

Suppose that net stock is S at the present moment. $p(S-j)$ is the probability that on the average the actual stock T days later will be j . On the average this

⁵ With no change in final result we could also have chosen s and n , or S and s , as the parameters of minimization.

stock will last $(j + 1)/\lambda$ days. $A^* \sum_{i=1}^{j+1} \rho^i$ is the amount of penalty saved during these days. The left hand term in (4.1) is therefore the average saving in penalty achieved T days from now by having raised net stock to S . The right hand side denotes the cost of doing so, K . Equation (4.1) is therefore of the type "marginal revenue equals marginal cost."

In equation (3.6) $A^* p(S - j + 1)\rho^j$ is the discounted penalty times the probability that a shortage arises $(T + j)/\lambda$ days from now, given that present net stock is S . The first term of (3.6) is therefore the discounted marginal penalty, and $c/(\alpha + b)$ is of course the marginal cost of a maximal stock of S units.

5. An Example

5.1. For purposes of computation and illustration it is convenient to rearrange (3.6) and (3.7) slightly to obtain:

$$(5.1) \quad \phi_1(S, n) = \sum_{j=0}^{n-1} \rho^{j+1} p(S - j + 1) \simeq \frac{(c/\alpha) + b}{A^*} (1 - \rho^n) = \varphi_1(n)$$

$$(5.2) \quad \phi_2(S, n) = P(S - n) \sum_{j=0}^{n-1} \rho^{j+1} - \sum_{j=0}^{n-1} \rho^{j+1} P(S - j) \simeq \frac{K}{A^*} + \frac{k}{A^*} \left(n - \sum_{j=0}^{n-1} \rho^{j+1} \right) = \varphi_2(n)$$

For given values of the parameters the solution is easily obtained by numerical calculation. Values of $P(u)$ and $p(u)$ have been tabulated by Molina [4]; using these ϕ_1 and ϕ_2 can be found by cumulative multiplication, given values of S and n . Hence, for any given S one can find the values of n which satisfy equations (5.1) and (5.2) respectively. By choosing different values of S , repeated calculations yield values of S and n which satisfy both these equations.

To illustrate we shall present a numerical example based upon the stocking of repair parts by a manufacturing firm in Chicago. The following approximate values of the parameters are estimates from the firm's experience:

λ	= 1 unit per day
T	= 90 days
α	= 0.0002 (corresponding to an annual rate of approximately 6 per cent)
K	= \$1.80
k	= \$0.30
$\frac{c}{\alpha\rho} + b$	= \$10.00
A	= \$1.80 per day

As a first step we take $S = 130$. Figures 1A and 1B illustrate graphically the solution for n for this value of S . ϕ_1 and ϕ_2 are convex⁶ functions of n in the relevant range. The intersections of these curves with φ_1 and φ_2 , which are nearly linear in n , give the values of n ($= 21$ and 24 , respectively) which satisfy these equations for the given value of S . Repeating the calculations for different values

⁶That is, they increase at an increasing rate.

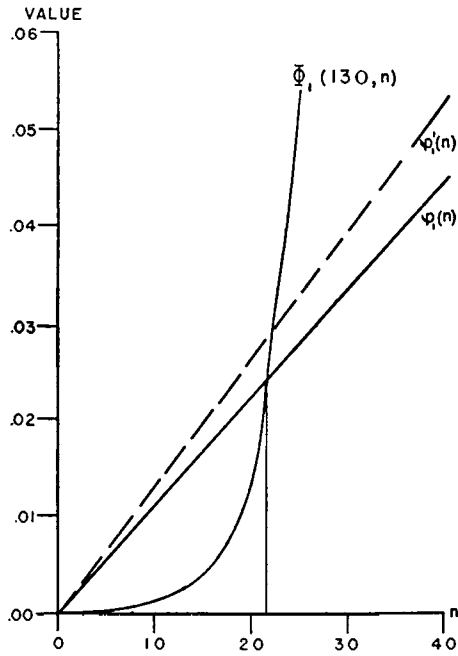


FIG. 1A

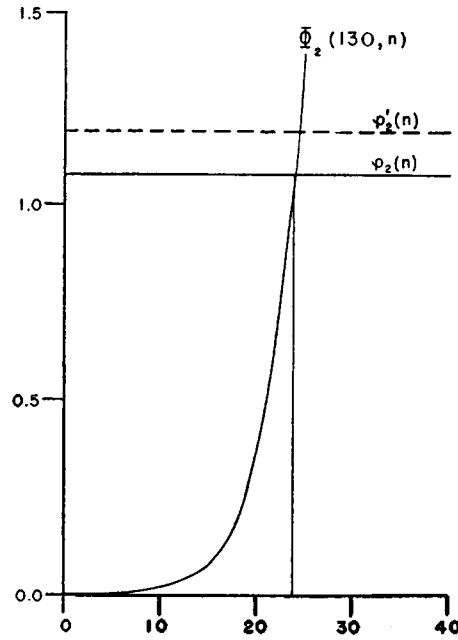


FIG. 1B

of S we generate two curves similar to those shown in figure 2. The steeper of these is the locus of values of S and n which satisfy (5.1), the flatter the locus of those satisfying (5.2). Their intersection gives the optimal values of these variables, in this example $S = 150$ and $n = 33$.

It is interesting to note that in this case the average number of orders placed per ninety day period is almost three. Thus roughly two unfilled orders are outstanding when a new order is placed. Also the optimal ordering point $s = S - n = 107$ and since mean demand during the replenishment period is 90 units the "safety margin" is 17 units. Referring to Molina's tables one finds that the probability of a shortage occurring at the end of the replenishment period, that is the probability that the demand will exceed 107 units during this period, is approximately 3.5%. Even with a relatively low penalty, loss is minimized by adopting an ordering policy which will lead to backlogs only once in thirty times on the average. Compared with the unlagged case above (p. 7) the introduction of a lag of 90 days and a shortage penalty reduces order size by about a half (from 78 to 33) and almost doubles the maximum stock level (from 78 to 140).

5.2. The effect of changes in the cost parameters may be seen from figures 1 and 2. First, let carrying cost, c , or the proportional order cost, k , increase. This increases ϕ_1 for any n but leaves ϕ_1 , ϕ_2 and ϕ_2 unchanged. That is, ϕ_1 shifts upward in 1A to, say, ϕ_1' . This means that the curve (5.1) in 2 shifts upward to, say, (5.1)' or that the optimal values of S and n decrease. Similarly, let K increase. This causes ϕ_2 to shift upward to, say, ϕ_2' and (5.2) to (5.2)' and the opti-

mal values of S and n increase. Both results are in accord with economic intuition; an increase in c , the carrying cost coefficient, leads to a decrease in average inventory while an increase in the fixed cost of ordering reduces average number of orders per unit time. If A , the shortage penalty, decreases then both (5.1) and (5.2) shift upward, say, to (5.1)' and (5.2)' in figure 2. Though not obvious from this diagram one would expect a decrease in A to decrease s . Finally we note that convexity of ϕ_1 and ϕ_2 implies that after a point an increase in either of the cost parameters brings about little change in the optimal values for S and n .

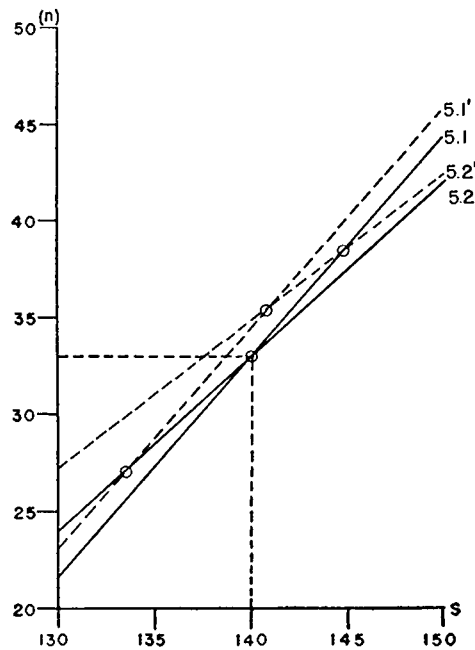


FIG. 2

6. Conclusion: Reordering Points Reconsidered

As a rule of thumb it is sometimes suggested that the reordering point s should equal the mean demand during the time that must elapse before delivery. Whenever the shortage penalty A is considerably larger than the carrying cost c the reordering point should actually be larger than the mean demand. The amount by which it should be larger in turn will be a function of the order size. The fact that optimal order size and reordering point are jointly dependent variables, has been brought out by Whitin [5, pp. 56-62].

Here we want to emphasize another fact about reordering points. The tacit assumption in the arguments of Whitin and the earlier writers criticized by him, is that not more than one order at a time is ever outstanding. If delivery lags are considerable this may force up the order size to the level of the mean demand in the lag period which may be considerable. The preceding analysis, which is

free from this assumption, shows however that the ordering point should not be given in terms of stock on hand alone, but in terms of the *sum of stocks and all orders* outstanding. (If the assumptions about postponable demand and a constant probability of one demand at a time are not satisfied, no ordering point exists, but instead there is a more complex "ordering function" which must involve the dates of previous unfilled orders.) When delivery lags are long and the value (a part of the proportional ordering cost) of the commodity is high, our equations show that it becomes optimal to stagger orders: the difference between the maximal net stock S and reordering point s may be less than the mean demand λT during the lag period T . This has been the case in the example.

References

- [1] K. ARROW, T. HARRIS AND J. MARSCHAK, "Optimal Inventory Policy," *Econometrica*, 19, (1951), pp. 250-272.
- [2] A. DVORETZKY, J. KIEFER, AND J. WOLFOWITZ, "The Inventory Problem I: Case of Known Distributions of Demand," *Econometrica*, 20 (1952), pp. 187-222.
- [3] WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, Vol. I, New York, John Wiley and Sons, Inc., 1952.
- [4] E. C. MOLINA, *Poisson's Exponential Binomial Limit*, New York, Van Nostrand Co., Inc., 1952.
- [5] THOMPSON M. WHITIN, *The Theory of Inventory Management*, Princeton, Princeton University Press, 1953.