Cowles Foundation Paper 101

Reprinted from THE REVIEW OF ECONOMIC STUDIES Vol. XXIII No.1.

The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis[1]

Neither the Pareto distribution nor the Cobb-Douglas production function are firmly established as empirical regularities, but they are sufficiently consistent with the facts and with the notions of economic theory to be frequently useful as approximative or illustrative devices, an application for which their convenient analytical properties make them particularly attractive. The relation between the two functions which will here be demonstrated may, therefore, be of some interest. Perhaps of more importance is the problem in which this relation occurs, namely the aggregation of the technological possibilities of individual producing units (e.g. firms or machines) into a production function for a group of units (e.g. an industry).

We consider a set of " production cells," which may sometimes be thought of as firms, but in other cases as individual machines or labourers. Each cell is capable of producing one or more outputs, $x_0, \ldots, x_k$, by means of two or more inputs, $x_{k+1}, \ldots, x_m$. Some of these inputs, say, $x_{k+1}, \ldots, x_n$ $(n < m)$, are available in variable amounts, whereas the remainder, $x_{n+1}, \ldots, x_m$, are fixed for the cell and period considered. The nature of these fixed inputs determines the nature of the cell. If, for instance, certain entrepreneurial resources are regarded as fixed the cell is thereby characterised as a firm.

For each cell there is a number of " production possibilities " describing what combinations of inputs and outputs can be obtained. One of these possibilities always consists in not producing at all, in which case the fixed factors will yield no outputs, for it is assumed that there can be no output without a positive input of at least one variable factor and at least one fixed factor.

If the production possibilities actually utilised by each cell are known the outputs and variable inputs can be summed over all cells so as to obtain total outputs and inputs $X_0, \ldots, X_n$ (there is no particular reason for summing the fixed inputs, which may be non-measurable). The main concern of this note will be with these totals. Clearly, not every vector X is attainable, for the possibilities of each cell, and hence of the whole set, are limited by the fixed inputs. Some of the vectors X that are possible and attainable are more efficient than others. We are, therefore, led to extend Koopmans' concept of an efficient point[2] to the production possibilities of a set of cells (an " industry "). For that purpose it is convenient to treat the inputs as negative outputs.

Definition. An input-output vector $X = (X_0, \ldots, X_n)$ for a set of production cells is efficient if, given, the distribution of fixed inputs $x_{n+1}, \ldots, x_m$ between cells, X is attainable and there is no attainable $X' = (X_0', \ldots, X_n')$ such that $X' \geqslant X$[3].

[1] This note is based on work done mostly at the Cowles Commission for Research in Economics and completed at Stanford University while under contract with the Office of Naval Research N6onr-25133. I am much indebted to my former colleagues, M. J. Beckmann, C. B. McGuire and L. G. Telser, for helpful discussions.

[2] Cf. T. C. Koopmans (editor), *Activity Analysis of Production and Allocation*, Cowles Commission Monograph No. 13 (New York, 1951), p. 60.

[3] $X' \geqslant X$ means that every component of X' is at least as large as the corresponding component of X.

Next we give a simple proof of a well-known result :

Theorem : If each cell utilises that production possibility which maximises its profit $\sum_{i=0}^{n} p_i x_i$, where $p_i > o$, then the total input-output vector X is efficient.

Proof: Evidently X is attainable. In order to get another vector X' some cells will have to change their outputs and variable inputs by amounts $\triangle x_i$, so that their profit will become $\Sigma p_i (x_i + \triangle x_i) \leqslant \Sigma p_i x_i$; it is not possible that $\Sigma p_i (x_i + \triangle x_i) > \Sigma p_i x_i$ for then profit would not have been at a maximum before the change. Total revenue for all cells will be $\Sigma p_i (X_i + \triangle X_i) = \Sigma p_i X_i' \leqslant \Sigma p_i X_i$. Hence it is not possible that $X' \geqslant X$; at least one of the X_i' must be less than the corresponding X_i, and X_i is, therefore, efficient.

It will be noticed that this theorem assumes nothing more than that profit for each cell reaches its maximum. An immediate corollary is that among all efficient vectors the one attained gives maximum total profit. We now introduce the following :

Definition. A production function is a function $F (X_0, \ldots, X_n)$ which equals one if and only if the input-output vector $(X_0, \ldots, X_n)$ is efficient.

It can then be said that for the entire set of production cells the following maximum problem is solved by the price mechanism :

$$\sum_{i=0}^{n} p_i x_i \text{ max. if } F (X_0, \ldots, X_n) = 1 \tag{1}$$

To make use of (1) we have to put some restrictions on the production possibilities. It will be assumed that the cells are small and numerous, that production possibilities tend to be different between them, that there are no indivisibilities apart from those due to the fixed factors, and that cases in which two or more production possibilities are equally profitable to a cell are so infrequent that they can be neglected. The more nearly those assumptions are fulfilled, the closer the aggregate efficient points will be together. In the limit they will form a continuous surface in the commodity space, and F, which describes that surface, will thus be continuous.

The relation between the aggregate production function F and the production possibilities of the individual cells can be analysed along these lines with considerable generality. The results that have so far been obtained for the more general cases are too formal to be of much interest, however. We shall, therefore, confine our further discussion to the special case of fixed proportions between outputs and variable inputs ; fixed, that is to say, for each cell, but with different proportions for different cells.[1] To simplify the exposition it will, moreover, be assumed that there is only one output x_0 and two variable inputs x_1 and x_2. The analysis for a greater number of commodities presents no new difficulties if proportions are fixed. The number of fixed inputs is irrelevant, provided there is at least one.

Suppose then that for a particular cell, say, the jth, a_{1j} units of x_1 and a_{2j} units of x_2 are needed to produce one unit of x_0, and that the fixed inputs for that cell limit its possible output to φ_j units of x_0. Let the prices of x_0, x_1 and x_2 be p_0, p_1 and p_2. Profit will then be positive if $p_0 - a_{1j}p_1 - a_{2j}p_2 > 0$, and if this inequality is fulfilled it will be

[1] Fixed proportions are also postulated in input-output analysis, but there they apply to a whole industry rather than to individual firms. This implies that either the proportions are the same for all firms, or that, if they differ between firms, the share of each firm in the industry's output is constant. Neither of these alternative implications is very plausible in practice, though it should, of course, be pointed out that input-output analysis does not pretend to be more than an approximative method. For a recent survey of input-output analysis, including a discussion of the necessity of the fixed-proportions postulate, see Robert Dorfman, " The Nature and Significance of Input-Output," *Review of Economics and Statistics*, 36 (1954), pp. 121-34.

profitable to produce to capacity, so that the variable inputs will be $a_{1j}\ \varphi_j$ and $a_{2j}\ \varphi_j$.[1] If $p_0 - a_{1j}p_1 - a_{2j}p_2 < 0$ it will evidently be most profitable to produce nothing. What happens if $p_0 - a_{1j}p_1 - a_{2j}p_2 = 0$ is a matter of arbitrary convention about which we need not worry, since it has already been assumed that such cases are infrequent enough to be negligible.

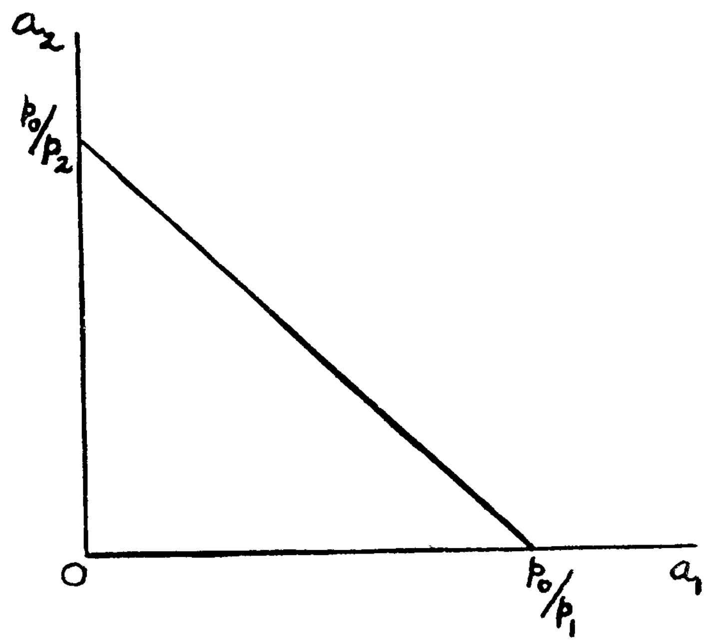

FIG. I

Fig. 1 shows for which values of a_{1j} and a_{2j} it is profitable, at the given prices p_0, p_1 and p_2, to produce up to capacity. At all points below the straight line, which has the equation $p_0 - a_{1j}p_1 - a_{2j}p_2 = 0$, profit is positive, and at all points above profit is negative.

Consider now all cells for which a_{1j} lies between a_1 and $a_1 + da_1$, and for which a_2 lies between a_2 and $a_2 + da_2$. If the production possibilities are continuously dispersed as assumed above, and if da_1 and da_2 are small, the total capacity of all those cells may be represented by :

$$\varphi\ (a_1,\ a_2)\ da_1 da_2 \qquad (2)$$

The function φ will be called the *input-output distribution* for the set of cells (or " industry ") concerned. By integrating over all the values of a_1 and a_2 for which production is profitable, that is over the triangle below the straight line in Fig. 1, we obtain total output X_0 as :

$$X_0 = \int_0^{p_0/p_1} \int_0^{\frac{p_0 + a_1 p_1}{p_2}} \varphi\ (a_1,\ a_2)\ da_2 da_1 \qquad (3)$$

[1] The previous convention to treat inputs as negative outputs will now be abandoned, as from here on it would complicate the notation. All inputs and outputs are now to be regarded as non-negative.

and similarly for the variable inputs :

$$X_1 = \int_0^{p_0/p_1} \int_0^{\frac{p_0 + a_1 p_1}{p_2}} a_1 \varphi\ (a_1,\ a_2)\ da_2 da_1 \qquad (4)$$

$$X_2 = \int_0^{p_0/p_1} \int_0^{\frac{p_0 \div a_1 p_1}{p_2}} a_2 \varphi\ (a_1,\ a_2)\ da_2 da_1 \qquad (5)$$

These three expressions depend on three parameters, p_0, p_1 and p_2, but since a proportional change in all prices does not affect the profitability of any production possibility no generality is lost by putting one of the prices, say, p_0, equal to one. We are then left with three expressions in two parameters ; hence there must be a functional relation between X_0, X_1 and X_2. This is clearly the aggregate production function $F\,(X_0,\ X_1,\ X_2)$ defined above.

There appears to be no generally applicable method of eliminating p_1 and p_2 from (3), (4) and (5) and thus deriving F directly from φ. In at least one special case, however, this derivation is possible, viz. when the input-output distribution is of the (generalised) Pareto type :

$$\varphi\ (a_1,\ a_2) = A a_1^{\alpha_1 - 1}\ a_2^{\alpha_2 - 1} \qquad (\alpha_1 \geqslant 1,\ \alpha_2 \geqslant 1) \qquad (6)$$

where A, α_1 and α_2 are constants. The aggregate production function then has the Cobb-Douglas form :

$$X_0 = C X_1^{\gamma_1} X_2^{\gamma_2} \qquad (7)$$

where C, γ_1 and γ_2 are expressions involving A, α_1 and α_2 as specified later. Both (6) and (7) are linear in the logarithms of the variables.

The derivation of (7) from (6) is straightforward but somewhat laborious, and we shall, therefore, only indicate the principal steps of the proof, leaving the details to the interested reader. Take, for instance, X_1 ; since $p_0 = 1$ we get :

$$X_1 = A \int_0^{\frac{1}{p_1}} \int_0^{\frac{1 - a_1 p_1}{p_2}} a_1^{\alpha_1}\ a_2^{\alpha_2 - 1}\ da_2 da_1.$$

$$= \frac{A}{a_2} \int_0^{\frac{1}{p_1}} a_1^{\alpha_1} \left(\frac{1 - a_1 p_1}{p^2}\right)^{\alpha_2} da_1$$

Put $a_1 p_1 = t$, then :

$$X_1 = \frac{A}{\alpha_2 p_1{}^{\alpha_1 + 1}\ p_2{}^{\alpha_2}} \int_0^1 t^{\alpha_1}\ (1 - t)^{\alpha_2} dt$$

$$= \frac{A}{\alpha_2 p_1^{\alpha_1 + 1}\ p_2^{\alpha_2}}\ B\ (\alpha_1 + 1,\ \alpha_2 + 1)$$

where $B\ (\alpha_1 + 1,\ \alpha_2 + 1)$ is a constant which does *not* depend on p_1, p_2. Similarly :

$$X_0 = \frac{(\alpha_1 + \alpha_2 + 1) A}{\alpha_1 \alpha_2 p_1^{\alpha_1}\ p_2^{\alpha_2}}\ B\,(\alpha_1 + 1,\ \alpha_2 + 1)$$

$$X_1 = \frac{A}{\alpha_1 p_1^{\alpha_1}\ p_2^{\alpha_2} + 1}\ B\ (\alpha_1 + 1,\ \alpha_2 + 1).$$

Hence, for all values of p_1 and p_2 :

$$X_0 = (\alpha_1 + \alpha_2 + 1)\left\{\frac{AB(\alpha_1 + 1, \alpha_2 + 1)}{{\alpha_1}^{\alpha_1+1}\,{\alpha_2}^{\alpha_2+1}}\right\}^{\frac{1}{\alpha_1+\alpha_2+1}} X_1^{\frac{\alpha_1}{\alpha_1+\alpha_2+1}} X_2^{\frac{\alpha_2}{\alpha_1+\alpha_2+1}} \qquad (8)$$

which corresponds to (7) if the first two factors together are called C and the exponents are called γ_1 and γ_2 respectively.

The exponents of X_1 and X_2 in (8) add up to $(a_1 + a_2) / (a_1 + a_2 + 1)$ which is less than one. (7) or (8) are, therefore, not homogeneous of degree one as production functions are usually represented to be ; the fixed inputs are, of course, responsible for the diminishing returns of the variable inputs. Formally (8) could be made homogeneous of degree one by introducing an additional variable x_3 for the fixed inputs and writing :

$$X_0 = CX_1^{\frac{\alpha_1}{\alpha_1+\alpha_2+1}} X_2^{\frac{\alpha_2}{\alpha_1+\alpha_2+1}} X_3^{\frac{1}{\alpha_1+\alpha_2+1}} \qquad (9)$$

but this would be misleading, since the distribution of the fixed inputs between cells is an essential element of the problem ; aggregating the fixed inputs is therefore not legitimate. It is true, however, that the fixed inputs receive a constant share of the value of output, for by a well-known property of the Cobb-Douglas function (easily derivable from the formulae preceding (8)) the value of the variable inputs is a constant fraction of the value of output. Since the remainder accrues to the fixed inputs it must also be a constant fraction.

To test whether the approach here outlined has any interest for empirical research three kinds of investigation are necessary. In the first place data would have to be gathered on the distribution of input-output ratios between firms (or single items of equipments such as cotton spindles) ; such information would be very valuable in any case, for recent empirical and theoretical research displays an undue tendency to neglect the variability of production possibilities between firms. In the second place, new attempts should be made to estimate production functions for industries ; previous attempts have been severely criticised because of their defective statistical methods.[1] Finally, the above theory should be extended, particularly to take more account of the fact that virtually all industries produce many different products, a complication which cannot be satisfactorily dealt with by the present approach.

Stanford, California. H. S. HOUTHAKKER.

[1] Cf. H. Mendershausen. "On the Significance of Professor Douglas' Production Function." *Econometrica*, 6 (1938), pp. 143-53 ; J. Marschak and W. H. Andrews, Jr., " Randon Simultaneous Equations and the Theory of Production," ibid., 12 (1944), pp. 143-205.