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CHAPTER XI

# REPRESENTATION OF A PREFERENCE ORDERING BY A NUMERICAL FUNCTION*

by

Gerard Debreu

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

## 1. INTRODUCTION

It has often been assumed in economics that if a set X (usually in the finite Euclidean space of commodity bundles) is completely ordered by the preferences of some agent, it is always possible to define on that set a real-valued order-preserving function (utility, satisfaction). This is easily seen to be false.[1]

The particular case where there exists on X (the set of prospects) a certain algebra of combining (corresponding to the combination of probabilities) has been rigorously and extensively studied by J. von Neumann and O. Morgenstern [7], J. Marschak [6], I. N. Herstein and J. Milnor [5].

But, rather paradoxically, the general case, which is more basic and simpler, has received little attention from economists. H. Wold's study [8] indeed seems to be the only rigorous one; its assumptions are however restrictive.

This note gives conditions under which a complete order

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I am grateful to staff members and guests of the Cowles Commission and very particularly to I. N. Herstein for their comments. I owe to P. R. Halmos reference [4]. My greatest debt is to L. J. Savage who suggested, in the course of a valuable discussion that Cantor's postulate $x < z_i < y$ (see Lemma II) might be weakened to $x \leqq z_i \leqq y$.

(denoted by $\leqq$) can be represented by a numerical function. The most common preference ordering in economics is that of bundles of n commodities, i.e., of points of an n-dimensional Euclidean space. We shall however treat the problem in a more general frame since this involves no additional mathematical cost.

The familiar case of a set in a finite Euclidean space is covered by the following proposition which is a very special application of theorem II below:

Let X be a completely ordered subset of a finite Euclidean space. If for every $x' \in X$ the sets $\{x \in X | x \leqq x'\}, \{x \in X | x' \leqq x\}$ are closed (in X), there exists on X a continuous, real, order-preserving function.

The assumption that the set $\{x \in X | x' \leqq x\}$ is closed (in X) is equivalent to the more intuitive assumption: let $(x^k)$ be any sequence of points in X having a limit $x^\circ \in X$, if for all k, $x^k$ is at least as good as $x'$, then $x^\circ$ is at least as good as $x'$.

## 2. TWO REPRESENTATION LEMMAS

A complete ordering on X is, to be precise, a binary relation, denoted $\leqq$, satisfying

1) Given any two elements x, y of X; $x \leqq y$ and/or $y \leqq x$
2) Given three elements of X such that $x \leqq y$, $y \leqq z$ then $x \leqq z$.

From this relation can be derived two new ones:

$x \sim y$ (x indifferent to y) if $x \leqq y$ and $y \leqq x$
$x < y$ (y better than x) if $x \leqq y$ and not $y \leqq x$.

The quotient set $X/\sim$, i.e., the set of indifference classes in X, will be denoted by A.[2] The trivial case where all elements of X are indifferent (i.e. where A has just one element) will always be excluded.

The interval $[x', y']$ is the set $\{x \in X | x' \leqq x \leqq y'\}$.
The interval $]x', y'[$ is the set $\{x \in X | x' < x < y'\}$.

A real-valued function $\phi(x)$ defined on X is said to be order-preserving if $x \leqq y$ is equivalent to $\phi(x) \leqq \phi(y)$.

A natural topology on X is a topology[3] for which the sets $\{x \in X | x \leqq x'\}$, $\{x \in X | x' \leqq x\}$ are closed for all $x' \in X$.

Lemma I. Let X be a completely ordered set whose quotient A is countable. There exists on X a real, order-preserving function, continuous[3] in any natural topology.

Rank the elements of A; it is clearly possible to construct by induction on the rank an order-preserving function $\psi$ taking A into some finite real interval. Let $\lambda = \operatorname{Inf}_{a \in A} \psi(a)$, $\mu = \operatorname{Sup}_{a \in A} \psi(a)$.

If $\alpha'$ satisfies $\lambda < \alpha' < \mu$ and $\alpha' \notin \psi(A)$, four cases may occur: the set $\{\alpha \in \psi(A) | \alpha < \alpha'\}$ (1) may, or (2) may not, have a largest element; and the set $\{\alpha \in \psi(A) | \alpha' < \alpha\}$ (1') may, or (2') may not, have a smallest element. We wish to eliminate the gaps of type (1-2'), (2-1') and (2-2'); this can easily be done by means of a non-decreasing step function $\Theta(\alpha)$, the height of each step being equal to the length of the corresponding gap. The new function $\phi^*(a) = \psi(a) - \Theta[\psi(a)]$ is still order-preserving and $\phi^*(A)$ has no gaps of the unwanted types. Denote by a(x) the indifference class a to which x belongs; we finally define $\phi(x) = \phi^*[a(x)]$. To show that $\phi$ is continuous in any natural topology on X consider a number $\alpha'$, $\lambda < \alpha' < \mu$ and the set $X_{\alpha'} = \{x \in X | \phi(x) \leqq \alpha'\}$.

1) If $\alpha' \in \phi(X)$, let $x' \in X$ be such that $\alpha' = \phi(x')$. $X_{\alpha'} = \{x \in X | x \leqq x'\}$ and is therefore closed.

2) If $\alpha' \notin \phi(X)$ and if the set $R_{\alpha'} = \{\alpha \in \phi(X) | \alpha < \alpha'\}$ has a largest element $\alpha''$, $X_{\alpha'} = X_{\alpha''}$ which is closed by 1).

3) If $\alpha' \notin \phi(X)$ and if the set $R_{\alpha'}$ has no largest element, then the set $R^{\alpha'} = \{\alpha \in \phi(X) | \alpha' < \alpha\}$ has no smallest element since $\phi(X)$ has no gap of type (2-1'). Thus $X_{\alpha'} = \bigcap_{\alpha \in R^{\alpha'}} X_\alpha$ and $X_{\alpha'}$ is closed as an intersection of closed sets.

Similarly one proves that for any number $\alpha'$ the set $X^{\alpha'} = \{x \in X | \alpha' \leqq \phi(x)\}$ is closed. It follows that the inverse image by $\phi$ of any closed set of the real line R is a closed set of X.

Lemma II. Let X be a completely ordered set, $Z = (z_0, z_1, \cdots)$ a countable subset of X. If for every pair x, y of elements of X such that $x < y$, there is an element $z_i$ of Z such that $x \leqq z_i \leqq y$, then there exists on X a real, order-preserving function, continuous in any natural topology.

The assumption made is a weakening of the postulate $(x < z_i < y)$ used by G. Cantor in [3].

Take first the quotient sets $X/\sim = A$ and $Z/\sim = C$. C is clearly countable and plays for A the role that Z played for X. If A has a smallest and/or a largest element, we can assume, without any loss of generality, that they are contained in C.

Define a new equivalence relation[2] among elements of A by: aFb if and only if between a and b there is a finite number of elements of A. The binary relation F is indeed reflexive, symmetric and transitive. Equivalence classes for F are denoted by $[a]_F$, $[b]_F, \cdots$

Every equivalence class is clearly countable. Moreover an equivalence class $[c]_F$ containing more than one element of A contains an element of C and thus the equivalence classes $[c]_F$ form a countable set. Summing up, C' the union over these classes $[c]_F$, is countable and so is $D = C \cup C'$.

Construct now on D the function $\phi^*$ as in the proof of Lemma I. $\phi^*$ is extended from D to A as follows. Let $a \epsilon A$ and $a \notin D$; the set $D_a = \{d \epsilon D | d < a\}$ has no largest element. To see this consider any $d' \epsilon D_a$. Since $a \notin D$, $a \notin C'$ and there is an infinity of elements of A between d' and a, there is therefore an infinity of elements of C, i.e. of D, between d' and a. Similarly the set $D^a = \{d \epsilon D | a < d\}$ has no smallest element. As a consequence the values $\sup_{d \epsilon D_a} \phi^*(d)$ and $\inf_{d \epsilon D^a} \phi^*(d)$ are not taken on. Moreover these two values are equal since $\phi^*(D)$ has no gap of the (2-2') type; they define $\phi^*(a)$. The function $\phi^*(a)$, and therefore the function $\phi(x) = \phi^*[a(x)]$, are clearly order-preserving and, since $\phi(X) = \phi^*(A)$ has no gaps of types (1-2') or (2-1'), $\phi(x)$ is continuous in any natural topology on X (the proof is the same as for Lemma I).

## 3. TWO REPRESENTATION THEOREMS

Before stating Theorem I we recall two definitions. A (topological) space X is separable if it contains a countable subset whose closure is X. A (topological) space X is connected if there is no partition of X into two disjoint, non-empty, closed sets.

Theorem I. Let X be a completely ordered, separable, and connected space. If for every $x' \epsilon X$ the sets $\{x \epsilon X | x \leqq x'\}$ and $\{x \epsilon X | x' \leqq x\}$ are closed, there exists on X a continuous, real, order-preserving function.[4]

This theorem can easily be derived from the results of S. Eilenberg [4]. It will be proved here as an immediate consequence of Lemma II. A much more direct proof could assuredly be given: the motivation for the two lemmas is Theorem II.

Call $Z$ the countable set dense in $X$ and consider a pair $x', y'$ of elements of $X$ such that $x' < y'$. The sets $\{x \in X | x \leqq x'\}$ and $\{x \in X | y' \leqq x\}$ are disjoint, non-empty and closed, they cannot exhaust $X$ which is connected, therefore the open interval $]x', y'[$ is not empty, and it must contain an element $z_i \in Z$. The theorem is proved since the topology on $X$ is a natural topology.

The assumption of connectedness is however very strong. We give a second theorem where it is removed at the cost of a slightly stronger separability assumption.

A topological space $X$ is perfectly separable if there is a countable class (S) of open sets such that every open set in $X$ is the union of sets of the class (S).

We remark that a separable metric space is perfectly separable, that a subspace of a perfectly separable space is perfectly separable.

Theorem II. Let $X$ be a completely ordered, perfectly separable space. If for every $x' \in X$ the sets $\{x \in X | x \leqq x'\}$ and $\{x \in X | x' \leqq x\}$ are closed, there exists on $X$ a continuous, real, order-preserving function.

Choose an element in each non-empty set S; they form a countable set $Z''$.

Consider then the pairs $a', b'$ of elements of $A$ such that $a' < b'$ and the interval $]a', b'[$ is empty. The set of those pairs is countable. To see this, associate with each such pair a set $S_{b'}$ as follows: take two elements $x', y'$ in the indifference classes $a', b'$ respectively. The set $\{x \in X | x < y'\}$ is open and therefore there exists a set $S_{b'}$ in the class (S) such that $x' \in S_{b'} \subset \{x \in X | x < y'\}$. If $a'', b''$ is another pair with the same properties, $S_{b''}$ is different from $S_{b'}$ for one has $a' < b' \leqq a'' < b''$, in which case $x'' \in S_{b''}$ and $x'' \notin S_{b'}$ or $a'' < b'' \leqq a' < b'$, in which case $x' \in S_{b'}$ and $x' \notin S_{b''}$. The pairs $a', b'$ are thus in one-to-one correspondence with a subclass of the countable class (S). Choose then an element $x'$ in each class $a'$ and an element $y'$ in each class $b'$. All those $x'$ and $y'$ form a countable set $Z'$.

Consider finally the countable set $Z = Z' \cup Z''$; it has all the properties required by Lemma II. Let $x, y$ be a pair of elements of $X$ such that $x < y$. If the open set $]x, y[$ is not empty, it contains a non-empty set S and therefore an element of $Z''$. If the set ]x,y[ is empty, $x \sim x' \in Z'$ and $y \sim y' \in Z'$. So that in any case [x,y] contains an element of Z.

## FOOTNOTES

1. Consider the lexicographic ordering of the plane: a point of coordinates (a',b') is better than the point (a,b) if "a' > a" or if "a' = a and b' > b". Suppose that there exists a real order-preserving function $\alpha$(a,b). Take two fixed numbers $b_1 < b_2$ and with a number a associate the two numbers $\alpha_1(a) = \alpha(a,b_1)$ and $\alpha_2(a,b_2)$. To two different numbers a,a' correspond two disjoint intervals $[\alpha_1(a), \alpha_2(a)]$ and $[\alpha_1(a'), \alpha_2(a')]$. One obtains therefore a one-to-one correspondence between the set of real numbers (non-countable) and a set of non-degenerate disjoint intervals (countable).

2. For definitions relating to an equivalence relation see [1].

3. For definitions of a topology and of a continuous function see [2, § 1] and [2, § 4] respectively.

4. The closedness assumptions have already been used in a similar context by I. N. Herstein in an earlier unpublished version of [5].

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