IMPORT SUBSTITUTION IN LEONTIEF MODELS

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INTRODUCTION

In a typical Leontief model, there is a unique method of production for each product. Hence, if a given bill of final goods is specified, the output of each industry is uniquely determined, there being no possibilities of substitution. Professor Chenery, in a contribution to a study of the industrial structure of the Italian economy (11, Chapter II, Section E), has considered a model more general in that each commodity may be either produced domestically by a unique process or imported, which involves a drain on foreign exchange. Some, at least, of the domestic industries operate under capacity limitations. There are then alternative ways of producing a given bill of goods; choice among them is to be made on the basis of minimizing the cost of imports in foreign currency. One would expect that the choice of production and import program would depend upon the relative prices of imports. The procedure actually used by Chenery is, however, independent of these prices. The purpose of the present note is to demonstrate that his procedure is correct under a wide variety of circumstances, i.e., that in spite of the presence of a substitution possibility, the optimal choice is independent of relative prices.

1. BASIC ASSUMPTIONS OF THE MODEL

I will briefly sketch the characteristics of the economy being analyzed. I will not argue its realism, though it is, I think, a rough approximation to the condition of Italy and certain other European countries. The main purpose of this paper, however, is the analytic contribution, which, it is hoped, may be capable of further extension.

The outstanding relevant features of the present model are the presence of unemployed labor and the imperfect competition on the market for exports due to import quotas in foreign countries. It is further assumed that the domestic production system can be described by a Leontief model, that is, that each commodity can be produced in just one way, the inputs being proportional to the scale of the output. It is assumed that foreign relative prices are not grossly out of line, in a sense to be made more exact below, with what domestic prices would be under competitive conditions. Finally, it is assumed that at least some industries have short-run capacity limitations. The aim of planning is to minimize the drain on foreign exchange. These assumptions will now be stated in a more precise way.

First of all, there are \( n \) commodities, each producible by a unique process.
which has only one output. Let \( a_{ij} \) be the amount of commodity \( i \) used in producing one unit of commodity \( j \). As usual (see Leontief [2]), we measure the quantities in dollars of some fixed base year. Since we are considering only commodities that are produced, so that labor and capital are excluded, the price of a unit produced must exceed the cost of the produced commodities used in production.

(1) \[ a_{ij} \geq 0, \quad a_{ii} = 0, \quad \sum_{i=1}^{n} a_{ij} < 1. \]

For analytic convenience, the assumption that exports cannot be expanded indefinitely at fixed prices will be stated in an extreme form. It will be assumed that exports can be sold at world prices freely up to a maximum beyond which none can be sold. This is an approximation to a declining demand curve, and, in view of widespread quantitative restrictions on imports, may have a certain degree of realism. Some such assumption is certainly widely made by planning authorities. The world prices in question are the prices in foreign markets less transport cost (in foreign currency). These prices will be referred to as “export prices.” It will also be assumed that commodities can be imported in unlimited quantities at fixed prices, which are equal to foreign prices plus transport costs.\(^4\)

In symbols, let \( y^i \) be the export of the \( i \)th commodity, \( y^i \) the import of the \( i \)th commodity, \( \eta_i \) the maximum possible export of the \( i \)th commodity, \( \pi^i \) the export price, and \( \pi^i \) the import price of the \( i \)th commodity. Correspondingly, let \( y^e, y^n, \eta, \pi^e, \) and \( \pi^i \) be the vectors of exports, imports, maximum possible exports, export prices, and import prices, respectively. From the discussions in the previous paragraph, it is assumed that

(2) \[ y^e \leq \eta.\]

From the definitions of import and export prices,

(3) \[ \pi^i \geq \pi^e. \]

The net drain on foreign currency is given by,

(4) \[ (\pi^e)' y^i - (\pi^e)' y^e. \]

Let \( \xi \), be the final demand (consumption plus investment) for the \( i \)th commodity, \( \xi \) the vector of final demands, \( x_i \) the domestic output of the \( i \)th commodity, and \( x \) the vector of domestic outputs. The demand by industry \( j \) for commodity \( i \) is then \( a_{ij} x_j \), and the net domestic output of commodity \( i \), which is total domestic output less derived demand by other industries, is \( x_i - \sum_{j=1}^{n} a_{ij} x_j \). In vector notation, the vector of net domestic output is \((I - A)x\), where \( I \) is the identity matrix of order \( n \) and \( A \) is the matrix \((a_{ij})\). The

\(^3\) I am indebted to a referee for stressing the distinction between export and import prices.

\(^4\) All vectors are considered to be column vectors unless a prime is attached to the symbol, in which case the transpose is taken.

\(^5\) By this notation is meant, as usual, that \( y^i_i \leq \eta_i \) for all \( i \); see Koopmans [3], p. 45.
net domestic output vector must be such that, together with imports, it suffices to take care of final demands and exports.

\[(I - A)x + y^i - y^r \geq \xi.\]

For each industry, let \(\xi_i\) be the total capacity. For an industry in which there are no capacity limitations, i.e., where even in the short run output can be increased by a proportional increase in all inputs, \(\xi_i\) can be regarded as equal to infinity. Let \(\xi\) be the vector of capacities. Then,

\[0 \leq x \leq \xi.\]

Next, we will make a critically important assumption on import and export prices. It is assumed that, for each product, if the inputs (other than labor) are imported and the product then exported at the going import and export prices, there will be a net gain in foreign exchange. To see why this assumption may be reasonable, consider the excess of the export price over the import cost of the produced inputs for the \(j\)th product. Then we can write it as,

\[\pi_j^* - \sum_{i=1}^{n} a_{ij} \pi_i^*.\]

Let \(b_{ij}\) be the foreign input-output coefficient, i.e., the amount of commodity \(i\) used in producing one unit of commodity \(j\) in the foreign world. Let \(\pi\) be the vector of prices in the foreign country. Then, purely as an identity,

\[\pi_j^* - \sum_{i=1}^{n} a_{ij} \pi_i^* = (\pi_j - \sum_{i=1}^{n} b_{ij} \pi_i) - (\pi_j^* - \pi_j^*)\]

\[= - \sum_{i=1}^{n} a_{ij} (\pi_i^* - \pi_i) + \sum_{i=1}^{n} (b_{ij} - a_{ij}) \pi_i.\]

The "export surplus" obtainable by producing commodity \(j\) out of foreign components is thus expressed as the value added abroad less the transport cost of the final product less the transport cost of the inputs needed for the product plus a term which depends upon the difference between the foreign and domestic technologies. Now, since labor certainly enters into the costs of production abroad and there are normally profit elements also, the first term, which is value added per unit output abroad, is certainly positive and usually fairly considerable. If we can assume that transport costs are relatively small, the second and third terms will be small; while if the technology abroad does not differ too greatly from that domestically, the fourth term will be small because the \(b_{ij}\)'s will be close to the \(a_{ij}\)'s. The negligibility of transport costs and the identity of foreign and domestic production structures are, of course, standard assumptions of foreign trade theory. If this be accepted, the second, third, and fourth terms will be small, while the first term, the value added, must be definitely positive. Hence we can assume that the export surplus must be positive.

\[\pi_j^* \geq \sum_{i=1}^{n} a_{ij} \pi_i^*.\]

Finally, we assume that the only factor of production other than the produced commodities is labor and that there is unemployment, so that labor is a
free good from the national point of view. Under these conditions, any triplet of vectors \( x, y^i, y^e \) is feasible if it satisfies (2), (5) and (6).

Among the feasible programs, the optimal one is that which minimizes the drain upon foreign currency, given by (4).

**Theorem:** Under the above assumptions (1), (3), and (7) on the technology and the import and export prices, the optimal program has the following properties: (1) no commodity is both imported and exported, (2) there is no industry such that simultaneously the domestic industry has excess capacity and the exports are less than the maximum possible.\(^6\) There is precisely one feasible program satisfying these conditions, so that the optimal program is independent of import and export prices so long as they satisfy (3) and (7).

The theorem is demonstrated in the next section. To avoid misapprehension, it should be remarked that the theorem does not imply, even when applicable, that relative import and export prices are of no relevance for the planner. The theorem permits a considerable simplification of the planning process in that, within the limits of the theorem's validity, the minimum-cost import program for a given set of final demands can be determined in a fairly simple manner. But the cost itself depends upon import and export prices, so that the choice among alternative bills of final demands will still depend upon those prices.

2. **Mathematical Analysis**

The problem of minimizing the linear form \((\pi)^T y^e - (\pi)^T y^e\) subject to (2), (5), and (6) is of course a problem in linear programming. In the present case, some preliminary considerations lead to a relatively simple solution.

2.1. First of all, we can assume without loss that in an optimal program, there is no commodity which is both exported and imported. For suppose there were, and we reduced both exports and imports of that commodity slightly by the same amount. The upper bound (2) on exports would certainly be maintained. In the condition (5) that the given final demands be at least met, only the difference between exports and imports enters, so (5) is still satisfied. In (6), the exports and imports do not enter explicitly. Hence, the new program is still feasible. Further, from (3), which states that import prices are at least as high as export prices, adverse trade balance cannot be increased and will in fact be reduced if the import price is actually greater than the export price, as we would expect if transport costs are significant. Hence, we can assume without loss of generality that,

\[(8) \quad \text{for any } j, \quad \text{either } y^i_j = 0 \text{ or } y^e_j = 0.\]

In view of (8) we can simplify the formulation of the problem by introducing the vector,

\[(9) \quad y = \eta + y^i - y^e.\]

\(^6\) Under certain unimportant circumstances, there may be other optimal programs equally as good (in terms of minimizing the adverse trade balance) as the one satisfying (1) and (2), but there is never any loss in restricting oneself to such programs. For the circumstances, see below in 2.1 and 2.3.
The vector $y$ may be interpreted as the excess of maximum possible exports over actual exports, imports being taken as negative exports. Given the vector $y$, we can easily find the vectors $y^i$ and $y^v$ from which it was derived; since $y - \eta = y^i - y^v$, where both latter vectors are nonnegative, then it follows from (8) that $y^i$ can be obtained by replacing all negative components of $y - \eta$ by 0, while $y^v$ can be obtained from $y - \eta$ by replacing all positive components by 0 and changing the sign of all negative components. If $y \geq 0$, then each negative component of $y - \eta$ will be greater than or equal to the corresponding component of $-\eta$, so that $y^i \leq \eta$, that is, (2) is satisfied. Conversely, if (2) is satisfied, it follows from (9) that $y \geq 0$. Hence, (2) is equivalent to the condition that,

$$(10) \quad y \geq 0.$$

Condition (5) can be written,

$$(11) \quad (I - A)x + y \geq \zeta + \eta.$$  

A feasible program can then be characterized by a pair of vectors $x$, $y$, satisfying (6), (10), and (11). For each such program, the corresponding import and export vectors $y^i$ and $y^v$ can be derived, as noted above, and the loss of foreign exchange evaluated from (4).

2.2. We will make use of a fundamental property of Leontief matrices. If we consider a matrix $A$ having the properties stated in (1), i.e., a nonnegative matrix whose column sums are all less than one, then it is well known that the matrix $I - A$ is non-singular and the elements of $(I - A)^{-1}$ are all nonnegative (see for example, Solow [4], p. 37). Any principal minor $\tilde{A}$ of $A$ (formed by deleting rows and corresponding columns) also enjoys the properties (1), so that $(I - \tilde{A})^{-1}$ is also defined and consists of nonnegative elements.

2.3. Suppose the vectors $x$, $y$ constitute an optimal program, so that (6), (10), and (11) are satisfied. In particular, (11) may be satisfied with the inequality holding for at least one component, that is, production plus imports exceeds required final demand plus exports for at least one commodity. A new program will be constructed which will also be feasible, will be at least as good from the viewpoint of minimizing loss of foreign exchange, and will involve equality in all components of (11), i.e., no excess production. Assume then that (11) holds with an inequality in at least one component.

$$(12) \quad (I - A)x + y \geq \zeta + \eta.$$  

(12) can also be written,

$$(13) \quad (I - A)x + y = \zeta + \eta + u, \quad \text{where } u \geq 0.$$  

If a program is optimal, there cannot be another feasible program $\tilde{x}$, $\tilde{y}$ for which $y \geq \tilde{y}$, for a decrease in one component of $y$ implies either an increase in exports or a decrease in imports, either of which reduces the drain on foreign ex-

$^7$ By the symbol $u \geq v$, where $u$ and $v$ are vectors, is meant $u_i \geq v_i$ for all $i$, $u_i > v_i$ for at least one $i$. Similarly, by $u > v$ is meant $u_i > v_i$ for all $i$. See Koopmans, [3], p. 45.
change. Let \( v \) be defined by letting \( v_i = \min(y_i, u_i) \); subtract \( v \) from both sides of (13), so that,

\[
(I - A)x + (y - v) = \xi + \eta + (u - v) \geq \xi + \eta,
\]

since \( u \geq v \). As also \( y - v \geq 0 \), the program \( x, y - v \) is feasible. If \( v_i > 0 \) for any component \( i \), then \( y - v \leq y \), which is impossible if, as assumed, the program \( x, y \) is optimal. Hence, \( v = 0 \); that is,

\[
(14) \quad \text{if } y_i > 0, \quad u_i = 0.
\]

Partition the vectors \( x, y \) and \( u \) so that all the positive components of \( y \) are in \( y^{(2)} \) and the zero components in \( y^{(1)} \). With the aid of (14), then,

\[
(15) \quad y^{(1)} = 0, \quad y^{(2)} > 0, \quad u^{(1)} \geq 0, \quad u^{(2)} = 0.
\]

Partition the matrix \( A \) correspondingly, both as to rows and as to columns. Then (13) can be written,

\[
(16) \quad (I - A_{11})x^{(1)} - A_{12}x^{(2)} = \xi^{(1)} + \eta^{(1)} + u^{(1)},
- A_{21}x^{(1)} + (I - A_{22})x^{(2)} + y^{(2)} = \xi^{(2)} + \eta^{(2)}.
\]

Define vector functions \( x(t), y(t) \) of the real variable \( t \) over the interval from 0 to 1, as follows:

\[
x^{(1)}(t) = (I - A_{11})^{-1} [A_{12}x^{(2)} + \xi^{(1)} + \eta^{(1)} + (1 - \xi)u^{(1)}],
\]

\[
x^{(2)}(t) = x^{(2)},
\]

\[
y^{(1)}(t) = 0,
\]

\[
y^{(2)}(t) = A_{21}x^{(1)}(t) - (I - A_{22})x^{(2)} + \xi^{(2)} + \eta^{(2)}.
\]

By solving for \( x^{(1)} \) in the first equation of (16) and for \( y^{(2)} \) in the second, it is easily verified that,

\[
(18) \quad x(0) = x, \quad y(0) = y.
\]

The expression in brackets in the first equation of (17) is composed of nonnegative elements; since \( (I - A_{11})^{-1} \) has only nonnegative elements, \( x^{(1)}(t) \geq 0 \). Further, since \( u^{(1)} \geq 0 \), the expression in brackets is decreasing in at least one component as \( t \) increases; from the nonnegativity of \( (I - A_{11})^{-1} \), each component of \( x^{(1)}(t) \) must be nonincreasing as \( t \) increases. Since \( x, y \) was feasible, \( x \leq \xi \), by (6).

\[
(19) \quad 0 \leq x(t) \leq \xi \quad \text{for} \quad 0 \leq t \leq 1.
\]

If we multiply through the first equation in (17) by \( I - A_{11} \) and perform some obvious transpositions, it is easily seen that,

\[
(20) \quad (I - A)x(t) + y(t) = \xi + \eta + (1 - t)u \geq \xi + \eta \quad (0 \leq t \leq 1).
\]
The matrix $A_2$ has only nonnegative components; since all components of $x^{(i)}(t)$ are nonincreasing, all components of $y^{(i)}$ are also nonincreasing, by the last line of (17). From (17), (18), and (15), for $t$ sufficiently small but positive, $y(t) \geq 0$; in conjunction with (19) and (20), then, the program $x(t)$, $y(t)$ is feasible. If $y(t)$ had one component which is strictly decreasing in $t$, then $y \geq y(t)$, which is impossible since it was assumed that $x$, $y$ is optimal. Hence each component of $y(t)$ is neither increasing nor decreasing, so that $y(1) = y \geq 0$. If $t$ is set equal to 1 in (20),

$$\begin{align*}
(I - A)x(1) + y &= \xi + \eta.
\end{align*}$$

With the aid of (19), it is clear that the program $x(1)$, $y$ is feasible. Since the net drain on foreign exchange depends only on $y$, it appears that starting with an optimal program we can find another as good by imposing the restriction that equality hold in every component in (11). That is, in searching for an optimal program, we may replace (11) by the stronger conditions,

$$\begin{align*}
(I - A)x(1) + y &= \xi + \eta.
\end{align*}$$

It may be interesting to inquire to what extent the condition (22) is actually essential for an optimum. That is, how can it happen that there is more than one optimal program, not all satisfying (22)? To answer this question, subtract (21) from (13). Then,

$$\begin{align*}
(I - A)[x - x(1)] &= u.
\end{align*}$$

Multiply both sides by $(I - A)^{-1}$. Since $(I - A)^{-1}$ has only nonnegative elements and $u \geq 0$ (that is, $u$ has at least one positive element),

$$\begin{align*}
x - x(1) &= (I - A)^{-1}u \geq 0,
\end{align*}$$

so that,

$$\begin{align*}
x(1) \leq x \leq \xi.
\end{align*}$$

One special case is that in which the strict inequality holds, i.e., $x(1) < \xi$. It will appear in the next section that under these circumstances the program $x(1)$, $y$ can be optimal only if $y = 0$. That is, if it is possible to satisfy all final demands and export the maximum possible of every commodity without any imports and without reaching capacity in any industry, then there exist alternative optimal programs with excess production in some industries. More complicated cases where optimal programs with excess production also exist with some industries at capacity and even with imports in some industries; a detailed analysis of these cases does not seem worthwhile.

2.4. As a result of the last section, the problem is reduced to selecting a pair of vectors $x$, $y$ satisfying (6), (10), and (22) so as to minimize (4). We shall now make use of (7) to establish the basic property of an optimal program, property (2) in the statement of the Theorem.
Suppose some industry is not operating at full capacity and at the same time is exporting less than the maximum possible. Then by increasing the output of the industry slightly, exporting the increase, and importing all the inputs needed for the increase, there will be, according to (7), an improvement in the foreign exchange position. For the right-hand side gives the cost in foreign exchange of a unit increase in production, while the left-hand side gives the additional receipts of foreign exchange.

In an optimal program, for each commodity \( j \),

\[
\text{either } x_j = \xi_j \quad \text{or} \quad y_j = 0.
\]

Simple as this step is, it is the critical one. For it will be shown that there can be only one program satisfying (6), (10), (22), and (23) simultaneously.

2.5. Suppose there were two different programs which satisfied (6), (10), (22), and (23) simultaneously; denote them by \( x, y \), and \( \bar{x}, \bar{y} \), respectively. If \( x = \bar{x} \), then \( y = \bar{y} \), by (22); hence, we must have \( x \neq \bar{x} \). Without loss of generality, we may suppose \( x_i < \bar{x}_i \) for at least one \( i \). Partition the vectors \( x, \bar{x}, y, \bar{y} \), so that

\[
x^{(1)} < \bar{x}^{(1)}, \quad x^{(2)} \geq \bar{x}^{(2)}.
\]

Since \( \bar{x}^{(1)} \leq \xi^{(1)}, x^{(1)} < \xi^{(1)} \); by (23),

\[
y^{(1)} = 0.
\]

Partition the matrix \( A \) correspondingly. In (22), applied to the two programs in question, consider only those equations corresponding to elements of \( y^{(1)} \).

\[
(I - A_{11})x^{(1)} - A_{12}x^{(2)} + y^{(1)} = \xi^{(1)} + \eta^{(1)},
\]

\[
(I - A_{11})\bar{x}^{(1)} - A_{12}\bar{x}^{(2)} + \bar{y}^{(1)} = \xi^{(1)} + \bar{\eta}^{(1)}.
\]

Solve for \( x^{(1)} \) in the first equation and for \( \bar{x}^{(1)} \) in the second, and take account of (25).

\[
x^{(1)} = (I - A_{11})^{-1} [A_{12}x^{(2)} + \xi^{(1)} + \eta^{(1)}],
\]

\[
\bar{x}^{(1)} = (I - A_{11})^{-1} [A_{12}\bar{x}^{(2)} + \xi^{(1)} + \bar{\eta}^{(1)} - \bar{y}^{(1)}].
\]

Since \( A_{12} \) contains only nonnegative elements, it follows from the second half of (24) that \( A_{12}x^{(2)} \geq A_{12}\bar{x}^{(2)} \); since also \( \bar{y}^{(1)} \geq 0 \), by (10), the bracketed expression in the first equation is at least as large, in each component, as that in the second. Since \( (I - A_{11})^{-1} \) has only nonnegative elements, it follows that \( x^{(1)} \geq \bar{x}^{(1)} \), which contradicts the first half of (24).

That is, it has been shown that there can be only one program which satisfies all the conditions (6), (10), (22), and (23). But on the one hand, under the present assumptions, any optimal program must satisfy these conditions, and on the other hand, the conditions do not involve the import and export prices,
so long as they satisfy the conditions (3) and (7). Hence, the Theorem has been demonstrated.\footnote{This proof, which is considerably simplified from my original argument, is due to Mr. John Pei.}

2.6. A simple diagram may illustrate the reasoning of the last section, in the special case of two commodities where export and import prices are equal, that is, \( \pi^* = \pi^i \). The aim of minimizing the drain on foreign exchange, as given by (4), is then simplified to that of minimizing \( \pi^*(y^e - y^i) \), where \( \pi \) stands for the common value of export and import price vectors. If, as before, we let \( y \) represent the extent to which exports fall short of the maximum possible, where imports are counted as negative exports, then \( y - \eta \) represents net imports, or \( y^i - y^e \). The net drain on foreign exchange is thus given by,

\[
\pi_1(y_1 - \eta_1) + \pi_2(y_2 - \eta_2).
\]

The restriction (7) on prices can be written,

\[
\pi_1 - a_{21}\pi_2 > 0, \quad \pi_2 - a_{12}\pi_1 > 0.
\]

These conditions say that if labor costs are disregarded, each commodity can be produced at a profit in terms of foreign prices. The condition (22), that net domestic output plus net imports equals given final demands, can be written,

\[
x_1 - a_{12}x_2 + y_1 - \eta_1 = \xi_1, \quad x_2 - a_{21}x_1 + y_2 - \eta_2 = \xi_2,
\]

which can be solved for \( y_1 \) and \( y_2 \),

\[
-y_1 = x_1 - a_{12}x_2 - (\xi_1 + \eta_1), \quad -y_2 = x_2 - a_{21}x_1 - (\xi_2 + \eta_2).
\]

From (28) and (26), the optimal program can be characterized as maximizing,

\[
\pi_1(x_1 - a_{12}x_2 - \xi_1) + \pi_2(x_2 - a_{21}x_1 - \xi_2)
\]

\[
= (\pi_1 - a_{21}\pi_2)x_1 + (\pi_2 - a_{12}\pi_1)x_2 - (\pi_1^*\xi_1 + \pi_2^*\xi_2).
\]

Since the last term is a constant uninfluenced by the choice of program, the optimal program is that feasible program which maximizes,

\[
(\pi_1 - a_{21}\pi_2)x_1 + (\pi_2 - a_{12}\pi_1)x_2.
\]

A basic restriction on the choice of a program is that exports shall not exceed a certain quota \( \eta \), which is expressed by the conditions \( y_1 \geq 0, y_2 \geq 0 \). In view of (28), these may be expressed as the following restrictions on \( x_1 \) and \( x_2 \):

\[
x_1 - a_{12}x_2 \leq \xi_1 + \eta_1, \quad x_2 - a_{21}x_1 \leq \xi_2 + \eta_2.
\]

The choice of \( x_1 \) and \( x_2 \) is further restricted by the capacity restrictions (6),

\[
0 \leq x_1 \leq \xi_1, \quad 0 \leq x_2 \leq \xi_2.
\]
The problem is to choose $x_1$ and $x_2$ satisfying (30) and (31) so as to maximize (29). The solutions of the equations,

$$x_1 - a_{12}x_2 = \xi_1 + \eta_1, \quad x_2 - a_{21}x_1 = \xi_2 + \eta_2,$$

must lie in the positive quadrant since $0 < a_{12}, a_{21} < 1$. Hence, the feasible set is the horizontally shaded area in Figure 1. It is immediately apparent that the point 0 dominates any other feasible point, in the sense that each coordinate of 0 is at least as large as any other feasible point and at least one coordinate is actually large. On the other hand, if (27) and (29) are compared, the objective is that of maximizing a positively-weighted sum of $x_1$ and $x_2$ among all feasible points. Hence, point 0 must be better than any other feasible point for any set of prices, so long as they satisfy (27), which is the main content of the theorem. It is further to be noted that point 0 uses the full capacity of industry 2 but is below capacity in industry 1. It is on the line $x_1 - a_{12}x_2 = \xi_1 + \eta_1$, which, from (28), implies that $y_1 = 0$, i.e., that exports are at the maximum possible. That is, the commodity which is being produced domestically below capacity is being exported to the maximum possible extent.

3. Computational Methods

This section will justify the computational methods used by Chenery. Mathematically, they can be described as follows.

For two vectors $u$ and $v$, let $\min (u, v)$ be the vector whose $i$th component is $\min (u_i, v_i)$. Define sequences $w^n$, $x^n$, $y^n$ by the following recursive formulas:

$$w^0 = x^0 = \min (\xi, \xi + \eta), \quad y^0 = \xi + \eta - x^0,$$
\( w^{n+1} = \min (Aw^n, \xi - x^n), \)

\( x^{n+1} = x^n + w^{n+1}, \)

\( y^{n+1} = y^n + Aw^n - w^{n+1}. \)

In words, at the initial stage, we set the output of each industry equal to its capacity or the final demand for its product, including maximum possible exports, whichever is smaller. At each subsequent stage, the output of each industry is expanded to meet the derived demand of the previous round until capacity limits are reached; any remaining derived demand is met from imports. It will be shown that the sequences \( x^n, y^n \) converge to the optimal program.

From (33), \( w^{n+1} \leq Aw^n \); hence, from (35), \( y^{n+1} \geq y^n \). From (32), \( y^0 \geq 0 \). Therefore,

\( y^n \) is monotone increasing, \( y^n \geq 0 \) for all \( n \).

From (33), \( w^{n+1} \leq \xi - x^n \); then, \( x^{n+1} = x^n + w^{n+1} \leq x^n + \xi - x^n = \xi \). Hence,

\( x^n \leq \xi \) for all \( n \).

Also, \( w^0 \geq 0 \). Suppose \( w^n \geq 0 \); then \( Aw^n \geq 0 \). With the aid of (37) and (33), \( w^{n+1} \geq 0 \). Therefore, \( w^n \geq 0 \) for all \( n \); from (34), \( x^n \) is a monotone increasing sequence. But (37) implies that \( x^n \) is bounded. Hence,

\( \lim_{n \to \infty} x^n = x^* \) exists; \( 0 \leq x^* \leq \xi \); \( x^n \) is monotone increasing.

\[ y^n = \sum_{i=0}^{n-1} (y^{i+1} - y^i) + y^0 = y^0 + A \sum_{i=0}^{n-1} w^i - \sum_{i=1}^{n} w^i \]

\[ = y^0 + A \sum_{i=1}^{n} (x^i - x^{i-1}) + Aw^0 - \sum_{i=1}^{n} (x^i - x^{i-1}) \]

\[ = y^0 + Ax^{n-1} - Ax^0 + Aw^0 - x^n + x^0. \]

Let \( n \) approach infinity. Let \( y^* = \lim_{n \to \infty} y^n \).

\[ y^* = y^0 + Ax^* - Ax^0 + Aw^0 - x^* + x^0 = y^0 - (I - A)x^* + x^0. \]

With the aid of (32),

\( (I - A)x^* + y^* = \xi + \eta, \)

From (36),

\( y^* \geq 0. \)

From (38–40), the program to which the sequences \( x^n, y^n \) converge satisfies (6), (10), and (22). We seek to show that this program is optimal. It suffices to show that it satisfies (23).

Suppose for some \( j \) that \( x_j^* < \xi_j \). From (38), \( x_j^* < \xi_j \) for all \( n \). If \( n \) is replaced by \( n + 1 \), it follows from (34) that

\[ w_j^{n+1} < \xi_j - x_j^n, \]
and therefore, from (33), that \( w_{j}^{n+1} = \sum_{i} a_{ij} w_{i}^{n} \), for all \( n \). From (35), then, \( y_{j}^{n+1} = y_{j}^{n} \) for all \( n \), or \( y_{j}^{n} = y_{j}^{0} \) for all \( n \), and therefore \( y_{j}^{*} = y_{j}^{0} \). But since \( x_{i}^{0} < \xi_{i} \), \( x_{i}^{0} = \xi_{i} + \eta_{i} \), and \( y_{j}^{0} = 0 \). It has been shown that for all industries \( j \) for which \( x_{i}^{*} < \xi_{i} \), \( y_{j}^{*} = 0 \), which is precisely (23). The limiting program \( x^{*}, y^{*} \) obtained from the iterative procedure (32–35) is then the optimal program so long as import prices satisfy condition (7).

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REFERENCES


