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*VALUATION EQUILIBRIUM AND PARETO OPTIMUM**

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For an economic system with given technological and resource limitations, individual needs and tastes, a valuation equilibrium with respect to a set of prices is a state where no consumer can make himself better off without spending more, and no producer can make a larger profit; a Pareto optimum is a state where no consumer can be made better off without making another consumer worse off. Theorem 1 gives conditions under which a valuation equilibrium is a Pareto optimum. Theorem 2, in conjunction with the Remark, gives conditions under which a Pareto optimum is a valuation equilibrium. The contents of both theorems (in particular that of the first one) are old beliefs in economics. Arrow[1] and Debreu[2] have recently treated this question with techniques permitting proofs. A synthesis of their papers is made here. Their assumptions are weakened in several respects; in particular, their results are extended from finite dimensional to general linear spaces. This extension yields as a possible immediate application a solution of the problem of infinite time horizon (see sec. 6). Its main interest, however, may be that by forcing one to a greater generality it brings out with greater clarity and simplicity the basic concepts of the analysis and its logical structure. Not a single simplification of the proofs would indeed be brought about by restriction to the finite dimensional case.

As far as possible the mathematical structure of the theory has been dissociated from the economic interpretation, to be found in brackets.

1. *The Economic System.*—Let L be a linear space (on the reals R).[3] The economic system can be described as follows:

The ith consumer ($i = 1, \ldots, m$) chooses a point x_i [his consumption] in a given subset X_i [his consumption-set] of L. [x_i completely describes the quantities of commodities he actually consumes, to be thought of as positive, and the quantities of the various types of labor he produces, to be thought of as negative. X_i is determined by constraints of the following types: quantities of commodities consumed (labor produced) must be nonnegative (nonpositive), and, moreover, they must enable the individual to survive.] There is on X_i a complete ordering, denoted by $\underset{i}{\leqq}$ [corresponding to the preferences of that consumer].[4] x_i^0 is a saturation point of X_i, if, for all $x_i \in X_i$, one has $x_i \underset{i}{\leqq} x_i^0$.

The jth producer ($j = 1, \ldots, n$) chooses a point y_j [his production] in a given subset Y_j [his production-set] of L. [y_j is a complete description of all his outputs, to be thought of as positive, and his inputs, to be thought of as negative. Y_j is determined by technological limitations.]

Denote $x = \sum_i x_i$, $y = \sum_j y_j$; they are constrained to satisfy the equality $x - y = \zeta$, where ζ is a given point of L. [ζ corresponds to the exogenous resources available (including all capital existing at the initial date). $x - y$ is the *net* consumption of all consumers and all producers together. It must clearly equal ζ.][5]

A $(m + n)$-tuple $[(x_i), (y_j)]$, one x_i for each i, one y_j for each j, is called a state of the economy. [It is a complete description of the activity of every consumer and every producer.] A state $[(x_i), (y_j)]$ is called attainable if $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j, $x - y = \zeta$.

2. *Valuation Equilibrium.*—$v(z)$ will denote a (real-valued) linear form on L.[6] [It gives the value of the commodity-point z. When L is suitably specialized, this value can be represented by the inner product $p \cdot z$, where p is the price system.] A state $[(x_i^0), (y_j^0)]$ is a valuation equilibrium with respect to $v(z)$ if:

(2.1) $[(x_i^0), (y_j^0)]$ is attainable.

(2.2) For every i « $x_i \in X_i$, $v(x_i) \leqq v(x_i^0)$ » implies « $x_i \underset{i}{\leqq} x_i^0$ ». [Best satisfaction of preferences subject to a budget constraint.]

(2.3) For every j « $y_j \in Y_j$ » implies « $v(y_j) \leqq v(y_j^0)$ ». [Maximization of profit subject to technological constraints.]

3. *Pareto Optimum.*—The set $X_1 \times \ldots \times X_m$ of m-tuples (x_i), one x_i for each i, is (partially) ordered as follows: $(x_i') \geqq (x_i)$ if and only if $x_i' \underset{i}{\geqq} x_i$ for all i.

A state $[(x_i^0), (y_j^0)]$ is a Pareto optimum if:

(3.1) $[(x_i^0), (y_j^0)]$ is attainable.

(3.2) There is no attainable state $[(x_i), (y_j)]$ for which $(x_i) > (x_i^0)$. [It is impossible to make one consumer better off without making another one worse off.]

4. *A Valuation Equilibrium Is a Pareto Optimum.*—The following assumptions will be made:

I. *For every i, X_i is convex.*

II. *For every i, « $x_i' \in X_i$, $x_i'' \in X_i$, $x_i' \underset{i}{<} x_i''$ » implies » $x_i' \underset{i}{<} (1 - t)\, x_i' + t x_i''$ for all t, $0 < t < 1$ ».*

These two axioms on the convexity of the consumption-sets and the convexity of preferences have been used by Arrow and Debreu[7] in a different context.

THEOREM 1. *Under assumptions I and II, every valuation equilibrium $[(x_i^0), (y_j^0)]$, where no x_i^0 is a saturation point, is a Pareto optimum.*

Proof: (4.1) « *$x_i \in X_i$ and $x_i \underset{i}{>} x_i^0$* » *implies* « $v(x_i) > v(x_i^0)$.»

This is a trivial consequence of definition (2.2).

(4.2). « *$x_i \in X_i$ and $x_i \underset{i}{\sim} x_i^0$* » *implies* « $v(x_i) \geqq v(x_i^0)$.»

Since x_i^0 is not a saturation point, there is $x_i' \in X_i$, such that $x_i' \underset{i}{>} x_i^0$, hence $x_i' \underset{i}{>} x_i$. Consider $x_i(t) = (1 - t)\, x_i + t x_i'$. By assumption II, for all t, $0 < t < 1$, $x_i(t) \underset{i}{>} x_i$, hence $x_i(t) \underset{i}{>} x_i^0$, so (by [4.1]) $v(x_i^0) < v(x_i(t)) = (1 - t)\, v(x_i) + t v(x_i')$. Let t tend to zero; in the limit $v(x_i^0) \leqq v(x_i)$.

To complete the proof we consider a state $[(x_i), (y_j)]$, where $x_i \epsilon X_i$ for all i, $y_j \epsilon Y_j$ for all j, and show that if $(x_i) > (x_i^0)$, the state is not attainable, i.e., $x - y \neq \zeta$.

$(x_i) > (x_i^0)$ means that for all i, $x_i \underset{i}{\geqq} x_i^0$, and for some i', $\overset{0}{x}_{i'} \underset{i'}{>} \overset{0}{x}_{i'}$; so by (4.1) and (4.2) $\sum\limits_i v(x_i) > \sum\limits_i v(x_i^0)$, i.e., $v(x) > v(x^0)$. On the other hand, (2.3) implies $v(y) \leqq v(y^0)$, so $v(x) - v(y) > v(x^0) - v(y^0)$. Since $x^0 - y^0 = \zeta$, $v(x - y) > v(\zeta)$, which rules out $x - y = \zeta$.

5. *A Pareto Optimum Is a Valuation Equilibrium.*—In this section L is a topological linear space.[8] Let x_i', x_i'' be points of X_i; we define $I(x_i', x_i'') = \{t | [(1 - t)x_i' + tx_i''] \epsilon X_i\}$. When X_i is convex, $I(x_i', x_i'')$ is a real interval with possibly one or two end-points excluded. In addition to assumptions I and II, three further assumptions are needed here.

III. *For every i, x_i, x_i', x_i'' in X_i the sets $\{t \epsilon I(x_i', x_i'') | (1 - t)x_i' + tx_i'' \underset{i}{\geqq} x_i\}$ and $\{t \epsilon I(x_i', x_i'') | (1 - t)x_i' + tx_i'' \underset{i}{\leqq} x_i\}$ are closed in $I(x_i', x_i'')$.*

This weak axiom of continuity for preferences has been introduced by Herstein and Milnor[9] in another context. We define $Y = \sum\limits_j Y_j$ (the set of all $y = \sum\limits_j y_j$, where $y_j \epsilon Y_j$ for all j).

IV. *Y is convex.* [The assumption that the aggregate production-set is convex is strictly weaker than the assumption that the individual production-sets Y_j are all convex.]

V. *L is finite dimensional and/or Y has an interior point.* [The assumption that Y has an interior point will be shown in section 6 to be implied by free disposal of commodities.]

THEOREM 2. *Under assumptions I–V, with every Pareto optimum $[(x_i^0), (y_j^0)]$, where some x_i^0 is not a saturation point, is associated a (nontrivial) continuous linear form $v(z)$ on L such that:*

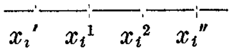
(5.1) *For every i « $x_i \epsilon X_i$, $x_i \underset{i}{\geqq} x_i^0$ » implies « $v(x_i) \geqq v(x_i^0)$.»*

(5.2) *For every j « $y_j \epsilon Y_j$ » implies « $v(y_j) \leqq v(y_j^0)$.»*

Proof: From assumptions I, II, and III follows:

a) « *x_i', x_i'' in X_i, $x_i' \underset{i}{\leqq} x_i''$* » *implies* « *for all t, $0 \leqq t \leqq 1$, $x_i' \underset{i}{\leqq} [(1 - t)x_i' + tx_i''] \epsilon X_i$.*»

By assumption III, the set $\{t \epsilon I (x_i', x_i'') | (1 - t) x_i' + tx_i'' \underset{i}{<} x_i'\}$ is open in $I(x_i', x_i'')$. Its intersection with the interval]0, 1[(end-points excluded) is open. We wish to show that this intersection is empty. If it were not, it would contain two numbers $t_1 < t_2$. Take the corresponding points x_i^1, x_i^2. Then $x_i^1 \underset{i}{<} x_i' \underset{i}{\leqq} x_i''$. By assumption II, $x_i^1 \underset{i}{<} x_i''$ gives $x_i^1 \underset{i}{<} x_i^2$.



Similarly, $x_i^2 \underset{i}{<} x_i'$ gives $x_1^2 \underset{i}{<} x_i^1$, a contradiction.

As an immediate consequence of (a), for all i, the sets $X_{i(\overset{0}{x_i})} = \{x_i \epsilon X_i | x_i \underset{i}{\geqq} \overset{0}{x_i}\}$ and $\mathring{X}_{i(\overset{0}{x_i})} = \{x_i \epsilon X_i | x_i > x_i^0\}$ are convex.

Let i' be a value of i for which $x_{i'}^0$ is not a saturation point, and consider the set

$$Z = \mathring{X}_{i'(x_{i'}^0)} + \sum_{i \neq i'} X_{i(x_i^0)} - \sum_j Y_j.$$

$\zeta \notin Z$, this is the definition of a Pareto optimum $[(x_i^0), (y_j^0)]$. Z is convex as it is the sum of convex sets. If $Y = \sum_j Y_j$ has an interior point, Z also has one. The Hahn-Banach theorem[10] can therefore be applied to Z and ζ. There is a (non-trivial) continuous linear form $v(z)$ on L such that $v(z) \geqq v(\zeta)$ for all $z \in Z$, i.e., since

$$\zeta = \sum_i x_i^0 - \sum_j y_j^0, \qquad v\left[\sum_i (x_i - x_i^0) - \sum_j (y_j - y_j^0)\right] \geqq 0$$

for all $x_{i'} \in \mathring{X}_{i'(x_{i'}^0)}$, $x_i \in X_{i(x_i^0)}$ (for $i \neq i'$), $y_j \in Y_j$ (for all j).

In this statement $\mathring{X}_{i'(x_{i'}^0)}$ can be replaced by $X_{i'(x_{i'}^0)}$, for every $x_{i'} \in X_{i'}$, $x_{i'} \underset{i'}{\sim} x_{i'}^0$, can be exhibited, as in the proof of (4.2), as a limit of points belonging to $\mathring{X}_{i'(x_{i'}^0)}$. Therefore,

b) $\sum_i v(x_i - x_i^0) + \sum_j v(y_j^0 - y_j) \geqq 0$ for all $x_i \in X_i(x_i^0)$, $y_j \in Y_j$.

By making all but one of the x_i, y_j equal to the corresponding x_i^0, y_j^0, one proves that for the remaining term in (*b*) $v(x_i - x_i^0) \geqq 0$ for all $x_i \in X_i(x_i^0)$ (or $v(y_j^0 - y_j) \geqq 0$ for all $y_j \in Y_j$) which is precisely the statement of Theorem 2.

(5.2) is identical to (2.3), but (5.1) does not necessarily imply (2.2), and Theorem 2 does not quite correspond to the title of this section. The following Remark, due to Arrow[11] in its essence, tries to fill this gap:

REMARK. *Under assumptions I and III, if there is, for every i, an* $x_i' \in X_i$ *such that* $v(x_i') < v(x_i^0)$, *then* (5.1) *implies* (2.2).

Consider an $x_i \in X_i$, $v(x_i) \leqq v(x_i^0)$. Let $x_i(t) = (1 - t)\, x_i + t x_i'$. For all t, $0 < t < 1$, $v(x_i(t)) < v(x_i^0)$ and thus, by (5.1), $x_i(t) \underset{i}{<} x_i^0$. The set $\{t \in I\,(x_i, x_i') \mid (1 - t)\, x_i + t x_i' \underset{i}{\leqq} x_i^0\}$ contains the interval $]0, 1[$; since it is closed in $I(x_i, x_i')$ (by assumption III), it contains 0, i.e., $x_i \underset{i}{\leqq} x_i^0$.

[The condition that there is $x_i' \in X_i$ such that $v(x_i') < v(x_i^0)$ means that the consumer does not have such a low $v(x_i^0)$ that with any lower value he could not survive.]

6. *The Free Disposal Assumption.*—An example will show the economic justification of assumption V when L is not finite dimensional. Suppose that there is an infinite sequence of commodities [because, for example, economic activity takes place at an infinite sequence of dates, a case studied by Malinvaud[12] with different techniques]. The space L will be the set of infinite sequences of real numbers (z_h) such that $\text{Sup}\,|z_h| < +\infty$. L is normed by $\|z\| = \text{Sup}\,|z_h|$.

The assumption of free disposal for the technology means that if $y \in Y$ and $y_h' \leqq y_h$ for all h, then $y' \in Y$ [if an input-output combination is possible, so is one where some outputs are smaller or some inputs larger; it is implied that a surplus can be freely disposed of]. With this assumption, if Y is not empty, it clearly has an interior point: select a number $\rho > 0$ and a point $y \in Y$; consider y' defined by $y_h' = y_h - \rho$ for all h. The sphere of center y', radius ρ, is contained in Y.

Other examples of linear spaces in economics are provided by the case where there is a finite number l of commodities, and time and/or location is a continuous variable. The activity of an economic agent is then described by the l rates of flow of the commodities as functions of time and/or location. The space L is the set of l-tuples of functions of the continuous variable.

In any case, if L is properly chosen, the existence of an interior point for Y will follow from the free disposal assumption. Then application of Theorem 2 will give a continuous linear form $v(z)$.

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[1] K. J. Arrow, "An Extension of the Basic Theorems of Classical Welfare Economics," *Proceedings of the Second Berkeley Symposium* (Berkeley: University of California Press, 1951), pp. 507–532.

[2] G. Debreu, "The Coefficient of Resource Utilization," *Econometrica*, 19, 273–292, 1951.

[3] A real linear space L is a set where the addition of two elements $(x + y)$ and the multiplication of a real number by an element (tx) are defined and satisfy the eight axioms:

1. For all x, y, z in L, $(x + y) + z = x + (y + z)$.
2. There is an element $0 \in L$ such that for every $x \in L$, $x + 0 = x$.
3. For every $x \in L$, there is an $x' \in L$ such that $x + x' = 0$.
4. For all x, y in L, $x + y = y + x$. For all x, y in L, t, t' in R,
5. $t(x + y) = tx + ty$,
6. $(t + t')x = tx + t'x$,
7. $t(t'x) = (tt')x$,
8. $1x = x$.

[4] An order is a reflexive and transitive binary relation (generally denoted by $\leqq$). $x \sim x'$ means $x \leqq x'$ and $x' \leqq x$, while $x < x'$ means $x \leqq x'$ and not $x' \leqq x$. The order is complete (as opposed to partial) if for any x, x' one has $x \leqq x'$ and/or $x' \leqq x$.

One may object to completeness of the preference ordering as well as to its transitivity. The reader must therefore note that, with slight modifications of the definitions and the assumptions, Theorems 1 and 2 can easily be proved for *arbitrary* binary relations on the X_i.

[5] Usually the net consumption is only constrained to be at most equal to the available resources. But this implies that any surplus can be freely disposed of. Such an assumption on the technology should be made explicit (see sec. 6) while requiring at the same time $x - y = \zeta$.

[6] For all x, y, $v(x + y) = v(x) + v(y)$. For all t, x, $v(tx) = tv(x)$. $v(z)$ is said to be trivial if it vanishes everywhere.

[7] K. J. Arrow, and G. Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 1954.

[8] A topological linear space is a linear space with a topology such that the functions $(x, y) \to x + y$ from $L \times L$ to L and $(t, x) \to tx$ from $R \times L$ to L are continuous. For definition of a topology, of the topology on a product, of a continuous function see N. Bourbaki, *Eléments de mathematique* (Paris: Hermann, et Cie, 1940), Part I, Book 3, chap. i. For the representation of continuous linear forms on L see S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932), in particular, chap. iv, sec. 4.

[9] I. N. Herstein and J. Milnor, "An Axiomatic Approach to Measurable Utility," *Econometrica*, 21, 291–297, 1953.

[10] In a real topological linear space, if Z is a convex set with interior points, ζ a point which does not belong to Z, there is a closed hyperplane through ζ, bounding for Z. (See for example, N. Bourbaki, *Eléments de mathématique* [Paris: Hermann et Cie, 1953], Part I, Book 5, chap. ii, in particular, sec. 3.)

[11] Arrow, *op. cit*, Lemma 5.

[12] E. Malinvaud, "Capital Accumulation and Efficient Allocation of Resources," *Econometrica*, 21, 233–268, 1953.