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NUMERICAL REPRESENTATIONS OF TECHNOLOGICAL CHANGE *

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SUMMARY: § 1. Description of the economic system. – § 2. Interpretations of the model. – § 3. Definition of Technological Change. – § 4. Numerical representations – § 5 Invariance requirement – § 6 Intrinsic Price Systems. – § 7. A Numerical Representation of Technological Change. – § 8. Concrete Evaluation. – § 9 A Calculus Study.

§ 1. DESCRIPTION OF THE ECONOMIC SYSTEM

The activity of the economic system can be viewed as the transformation by n production units, and the consumption by m consumption units, of l commodities.

x_{hi} will denote the quantity of the h-th commodity consumed by the i-th consumption unit; x_{hi} is positive for a commodity actually consumed, it is negative for a commodity produced (for example a certain type of labor). The activity of the i-th consumption unit can thus be completely characterized by the l numbers x_{hi} ($h = 1, ..., l$) or better still by the point x_i of the l-dimensional commodity space

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I am greatly indebted, for their critical comments, to participants in the Princeton Conference and in particular to W. W. Leontief, the official discussant, to Cowles Commission staff members and their guests and in particular to L. Hurwicz, T. C. Koopmans and H. A. Simon.

The following papers are related to some of the points discussed below (in particular in Section 9): K. MAY: *Technological Change and Aggregation*, « Econometrica », vol. XV, January 1947, pp. 51-63; H. A. SIMON: *Effects of Increassed Productivity upon the Ratio of Urban to Rural Population*, « Econometrica », vol. XV, January 1947, pp. 31-42; H. A. SIMON: *The Effects of Atomic Power on National or Regional Economies*, Chapter 13 in *Economic Aspects of Atomic Power*, S. H. Schurr and J. Marschak eds., Princeton University Press, 1950, pp. 219-247; H. A. SIMON: *Effects of Technological Change in a Linear Model*, Chapter 15 in *Activity Analysis of Production and Allocation*, Cowles Commission Monograph 13, T. C. Koopmans ed., Wiley, New York, 1951, pp. 260-277. I have also had access to three unpublished Cowles Commission Discussion Papers by H. A. SIMON: *Economics*, 202-213-247.

whose coordinates are $x_{1i}, \ldots, x_{li}$. We assume that if $\mathbf{x}^1_i$ and $\mathbf{x}^2_i$ are two arbitrary points of this kind, the i-th consumption unit either « prefers $\mathbf{x}^1_i$ to $\mathbf{x}^2_i$ », « thinks $\mathbf{x}^1_i$ equivalent to $\mathbf{x}^2_i$ », or « prefers $\mathbf{x}^2_i$ to $\mathbf{x}^1_i$ » with the usual transitivity property. As a consequence (1) a number $s_i\ (\mathbf{x}_i)$ can be associated with each $\mathbf{x}_i$ in such a way that the three statements in quotation marks are respectively equivalent to « $s_i\ (\mathbf{x}^1_i) > s_i\ (\mathbf{x}^2_i)$ », « $s_i\ (\mathbf{x}^1_i) = s_i\ (\mathbf{x}^2_i)$ », « $s_i\ (\mathbf{x}^2_i) > s_i\ (\mathbf{x}^1_i)$ ». The number s_i can be called the satisfaction (or sometimes more appropriately in this context the standard of living) of the i-th consumption unit. Any monotonic increasing transformation on s_i provides an equally satisfactory function. A situation (2) is said to be « better » than a situation (1) if for all consumption units ($i = 1, \ldots, m$) $s^2_i \geqq s^1_i$ and for at least one of them say the i'-th $s^2_{i'} > s^1_{i'}$. A situation is said to be optimal if it admits no « better » situation.

y_{hj} will denote the quantity of the h-th commodity consumed by the j-th production unit; y_{hj} is positive if the h-th commodity is actually an input, negative if an output. The activity of the j-th production unit is thus characterized by the l numbers y_{hj} ($h = 1, \ldots, l$) or even better by the point $\mathbf{y}_j$ of the commodity space whose coordinates are $y_{1j}, \ldots, y_{lj}$. We assume that any such point $\mathbf{y}_j$, i. e., a complete specification of the quantities to be consumed and produced by the j-th production unit, can be classified as achievable or unachievable on the basis of constraints which we describe broadly as « technological knowledge ».

All achievable points taken together form in the commodity space the region Y_j of technological possibilities for the j-th production unit.

The n production units constitute the production sector or Industry; its net input (which may be positive or negative) of the h-th commodity is $y_h = \sum_{j=1}^{n} y_{hj}$. The net result of the production activity is therefore characterized by the point $\mathbf{y}$ of the commodity space whose coordinates are the numbers y_h ($h = 1, \ldots, l$). In the same way as above we define for the production sector taken as a whole the region Y of technological possibilities, the set of all points $\mathbf{y}$ technologically achievable. The region Y is clearly well determined when the n regions Y_j ($j = 1, \ldots, n$) are given.

The quantity of the h-th commodity consumed by all consumption units is $x_h = \sum_i x_{hi}$ and thus the *net* quantity of this commodity consumed by all consumption units and all production units is $z_h = x_h + y_h$. It cannot exceed the available quantity z^0_h (for example a natural resource). The utilizable physical resources are characterized by the point $\mathbf{z}^0$ of the commodity space whose coordinates are z^0_h ($h = 1, \ldots, l$); for any commodity $z_h \leqq z^0_h$.

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(1) Points of rigor will not be discussed in this paper.

§ 2. Interpretations of the model

The formal properties of the model presented allow great freedom of interpretation.

A commodity can be any good or service, direct or indirect, playing a role in any production or consumption process. The subscript h can characterize the location of the commodity. If the economic activity extends over t successive time intervals, h can also characterize the time interval.

A consumption unit can be thought of as a classical consumer or household unit but one is *by no means* restricted to this narrow interpretation. For example if a preference ordering (of whatever nature) of the social product (the l quantities x_h, $h = 1, ..., l$) is given, the model is interpreted as containing only one consumption-unit having this very preference ordering.

The concept of production unit may coincide with that of industry, firm, plant,... The technological possibility region Y can be interpreted in many different ways. Its definitions can range from an abstract one, such as the set of possibilities which can be imagined on the basis of some ideal pooling of all existing knowledge, to concrete ones taking into account the innumerable obstacles which prevent such pooling; they can include research activity.

General as this model may appear, it still calls for a greater generality. Three directions in which generalizations are most desirable can be given as examples.

In our description of the economic system we implied that at any time the technological possibility region Y is perfectly known, i. e., the technological future is exactly foreseen. The extreme difficulty of any such foresight seems to require that uncertainty about the technological future be embodied in a model which purports to study technological change.

If the end of economic activity is to take place after t time intervals, certain quantities of the various commodities will then exist. No plan of action can be drawn at the initial time point if constraints imposed on these quantities are not specified. Shall one require that for each commodity the existing quantity has at least an *a priori* given value? Such a procedure is unsatisfactory since these quantities are in fact the result of optimizing processes taking into account the time intervals following the t-th one. The alternative is to consider an infinite sequence of time intervals.

Anticipating the subsequent exposition we finally remark that we are essentially concerned with the replacement in the model described of the possibility region Y^0 by a possibility region Y^1 which contains the first one. However the problem of technological change rather suggests the comparison of the possibility regions Y_τ and $Y_{\tau+1}$ refering to two successive time intervals but these are not readily comparable since Y_q corresponds to an economy with $t + 1 - q$ time intervals, and Y_{q+1} to an economy with $t - q$ time intervals. This difficulty would disappear if the sequence of time intervals were infinite.

The study of an economic system the activity of which extends over an infinite sequence of time intervals under uncertainty would naturally introduce notable mathematical difficulties which we wish to avoid.

§ 3. Definition of Technological Change

Technological change has sometimes been defined as a variation in the quantities consumed and produced by Industry (the l quantities $y_h, h = 1, \ldots, l$) but this definition is much too wide since this variation can result from a change in the individual preference orderings, in the set of technological possibilities Y, in physical resources z^0 or in economic organization (income distribution, tax system, etc.). Here technological change will be defined as the replacement of the technological possibilities region Y^0 by a larger region Y^1 containing the first one ($Y^1 \supset Y^0$). Such a change can clearly come from invention as well as from diffusion of technological knowledge.

The changes of the possibility set Y are considered as exogenous. It is not our purpose to investigate their important but complex and little known interrelations with changes of the other structural data.

§ 4. Numerical representations

Our purpose is to associate one or a few numbers with the replacement of the possibility set Y^0 by the set Y^1 that occurs in the economic system. An entirely satisfactory definition of such a numerical representation should come from the uses to which it is put. It should, in other words, be a by-product of an integrated theory in which technological change would be a significant feature; in this theory one or several parameters or variables by dint of the functional relationships into which they enter might warrant the name « technological change ».

In the absence of a specific theory of this kind it may still be useful to represent a situation too complex to be grasped at a glance by one or a few representative numbers. The precise definition of the numerical representation will in this case result from intuitive requirements. This natural procedure is indeed general and two well-known examples borrowed from economics (the definition of a price index or of any index) and from statistics (the definition of an estimator of which one requires that it be unbiased, of least variance) will probably be sufficient as illustrations.

§ 5. Invariance requirement

It seems that a satisfactory representation of the change from the possibility region Y^0 to the larger region Y^1 should be invariant with respect to changes of the other characteristics of the economic system. In other words the representation should be defined only in terms of the two regions, Y^0 and Y^1, of the commodity space. The possibility of such a definition was discussed at the Princeton Conference in other papers, in particular by C. Kaysen (Topic 1: *The Limitations of Measurement in Describing Changes in the Store of Technological Knowledge*). The discussion can perhaps be summed up by the remark that no representation of this type was suggested; none which would not be entirely artificial occurs to us.

The representation of technological change will therefore be defined in connection with other economic characteristics. Our basic invariance requirement is that these additional characteristics be as few as possible.

§ 6. Intrinsic Price Systems

The constraints imposed on the economic system are

a) the global input-output point $\mathbf{y}$ belongs to the technological possibility region Y^0;

b) the utilized quantity of each commodity is at most equal to the available quantity: $z_h \leq z^0_h$ $(h = 1, \ldots, l)$.

Let $(s^0_1, \ldots, s^0_i, \ldots, s^0_m)$ be a set of values of the satisfaction levels of the m consumption units corresponding to an optimal situation (See Section 1) under these constraints. Now impose on the economic system a new set of constraints

α) the input-output point $\mathbf{y}$ belongs to the region Y^0;

β) the satisfaction level of each consumption-unit is at least equal to its value in the above optimal situation: $s_i \geqq s^0_i$ $(i = 1, \ldots, m)$.

The utilizable physical resources point $\mathbf{z}$ can clearly not be chosen arbitrarily in the commodity space. If the available quantities of commodities are too small the m satisfaction levels s^0_i $(i = 1, \ldots, m)$ cannot be reached on the basis of the technological possibilities Y^0. We shall call $Z^0 = Z^0\ (s^0_1, \ldots, s^0_m)$ the region of the commodity space to which $\mathbf{z}$ must belong. Moreover a resource point $\mathbf{z}^2$ shall be said to be more economical than another point $\mathbf{z}^1$ if for all coordinates $z^2_h \leq z^1_h$ and for at least one $z^2_{h'} < z^1_{h'}$. The most economical points of $\bar{Z}^0$ are the points of the boundary Z^0. It is intuitive that $\mathbf{z}^0$ is one of them since $(s^0_1, \ldots, s^0_m)$ corresponds to an optimal situation. The basic theorem of the New Welfare Economics (1) asserts (under

(1) A non-calculus proof of this theorem and a detailed presentation of many of the concepts used in this paper are given in G. Debreu: *The Coefficient of Resource Utilization*, « Econometrica », vol. XIX, July 1951, pp. 273-292.

suitable convexity assumptions) the existence of a system of intrinsic prices $(p_1, \ldots, p_l)$ associated with any optimal situation. The intrinsic price vector $\boldsymbol{p}$ is nothing else than the normal to Z^0 at the point z^0.

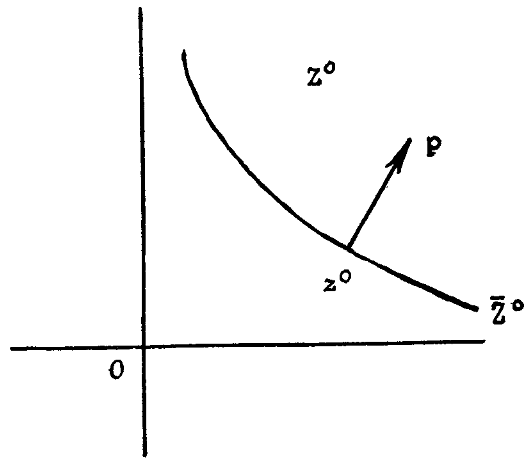

Fig 1

§ 7. A Numerical Representation of Technological Change

When the economy undergoes technological progress the technological possibility region Y^0 is replaced by a new region Y^1 containing the first one.

If we impose on the economic system the constraints

α) the input-output point y belongs to the region Y^1;

β) the satisfaction level of each consumption unit is at least equal to its value in the given situation which was optimal in relation to Y^0:

$$s_i \geqq s^0_i \quad (i = 1, \ldots, m),$$

the utilizable physical resources point z must belong to a new region Z^1. Since all the old productive combinations are still available, every point of the old region Z^0 belongs to the new region Z^1. Thus the resource point z^0 is inside the region Z^1 and the effect of technological change on the economic system can be described by the relative positions of the point z^0 and the new set Z^1 of most economical resource points. This constellation of a point and a set is however still complicated and we wish to summarize this description by one number.

Let z be an arbitrary resource point belonging to the boundary Z^1 and $\boldsymbol{p}$ its associated price-vector (i. e., the normal to Z^1 at z). The economy which previously could not use a resource point more economical than z^0 can now use z; the value of the saving thus realized

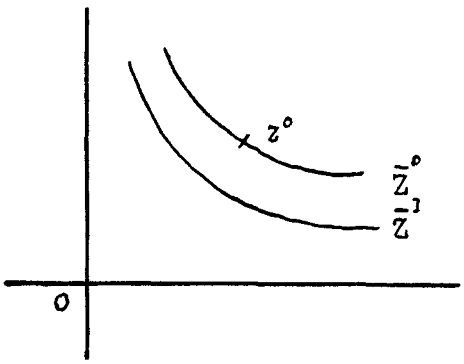

Fig.2

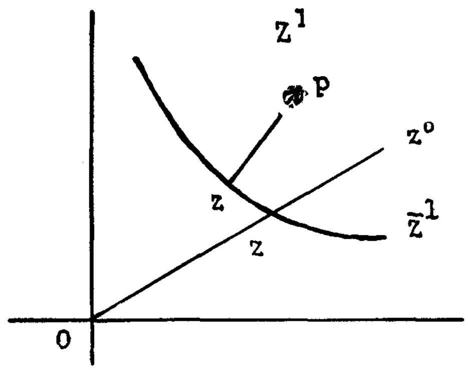

Fig.3

is, on the basis of the *intrinsic* price system $\boldsymbol{p}$, $\sum_{h=1}^{l} p_h (z^0_h - z_h)$. However $\boldsymbol{p}$ is determined but for a multiplication by an arbitrary positive scalar; to eliminate this arbitrariness we divide $\boldsymbol{p}$ by $\sum_{h=1}^{l} p_h z_h$ and adopt finally for the value of the saving

$$v = \frac{\sum_{h=1}^{l} p_h (z_{0h} - z_h)}{\sum_{h=1}^{l} p_h z_h}$$

The expression chosen for the normalization factor gives the same value to all the points $\boldsymbol{z}$ of the set Z^1 of most economical resource points and is thus intuitively justified.

To every point $\boldsymbol{z}$ of Z^1 corresponds a value v; the minimum of this value is attained when $\boldsymbol{z}$ coincides with the point $\boldsymbol{z}^*$ resulting from $\boldsymbol{z}^0$ by a reduction of all its coordinates *by the same proportion* ρ. In other words, for all $h = 1, \ldots, l$, $z^*_h = \rho z^0_h$ (1).

A possible numerical representation of technological change is $1 - \rho$, where ρ is the smallest fraction of all the available resources which would allow the economy to attain the old (optimal) standards of living when the new technology is available.

The resource saving could also be described by the quantities of commodities $z^0_h - z^*_h = z^0_h (1 - \rho)$, or by a money value based on some fixed price system $(p^0_1, \ldots, p^0_l)$, $\sum_{h=1}^{l} p^0_h z^0_h (1 - \rho)$.

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(1) The proof of this fact and a more sophisticated, but still « naive », justification of the choice of z^* can be found in the paper quoted in footnote (2) where this cho ice is pesented as the outcome of antagonistic activities of all consumption-units and production-units on one side and the price selecting agency on the other.

The number ρ depends on the old set of technological possibilities Y^0, the new set Y^1, the available resources z^0, the preference orderings of the m consumption units, and the particular optimal situation $(s^0_1, \ldots, s^0_m)$ selected. A higher degree of invariance may be achieved. If, for example, instead of a particular optimal situation $(s^0_1, \ldots, s^0_m)$ one considers a range of relevant situations optimal with respect to z^0 and Y^0 (see Section 6) one obtains for ρ a minimum ρ_m and a maximum ρ_M. The two bounds $(1 - \rho_M, 1 - \rho_m)$ now form the numerical representation of technological change. In this case, the pair (ρ_m, ρ_M) depends only on the old set Y^0, the new set Y^1, the available resources z^0, and the preference orderings of the m consumption units.

§ 8. Concrete Evaluation

Currently suggested representations of technological change are far from satisfying the invariance requirement as well. One might however argue that such representations are at least more readily accessible than the representation proposed in Section 7. It seems therefore worthwhile to bring the issue into sharper focus: a choice must be made between a definition conceptually acceptable which does not lead to an immediate numerical evaluation but for which approximation processes can be devised and a definition leading to an easy evaluation but conceptually unsatisfactory.

It must be remarked next that the representation of Section 7 is only as intricate as one makes it. It applies for example to the very simple case where the technology of the economy consists of all positive linear combinations of a finite set of activities characterized by fixed inputs and outputs and an ordering of the bundle of final commodities is given.

§ 9. A Calculus Study (1)

In this section we shall assume that all the conditions for the use of the calculus are fulfilled. The formulas that we will obtain for first differentials of the variables can of course be looked upon as approximation formulas for small finite variations. This calculus study will serve other purposes: it will show how in the small.

i) the gain attributable to technological change is obtained directly from technological data;

ii) the variation of the standard of living can be imputed to the different factors of economic evolution.

(1) This last section, which is more technical and more concise than the other ones, may be considered an appendix.

Let $e_j(\boldsymbol{y}_j, \alpha_j) = 0$ be the equation of the production surface (the efficient boundary of Y') of the j-th production unit; α_j is a numerical parameter whose variation corresponds to technological change. $(s^0_1, \ldots, s^0_m)$ is a situation optimal on the basis of $\boldsymbol{z}^0$ and $(\alpha^0_1, \ldots, \alpha^0_n)$, the values of the α_j before the change. $\boldsymbol{p}^0$ is the price system actually observed in this situation under competitive market conditions; it coincides with the intrinsic price system.

Given $\boldsymbol{z}^0$, $(\alpha_1, \ldots, \alpha_n)$ and $(s^0_1, \ldots, s^0_m)$, ρ is determined by the condition that $(s^0_1, \ldots, s^0_m)$ is an optimal situation on the basis of $\rho\, \boldsymbol{z}^0$, and $(\alpha_1, \ldots, \alpha_n)$ i. e., that there exists an intrinsic price-vector $\boldsymbol{p}$ satisfying the equations

(1) $s_i(\boldsymbol{x}_i) = s^0_i$ (2) $e_j(\boldsymbol{y}_j, \alpha_j) = 0$

(3) $\dfrac{ds_i}{d\boldsymbol{x}_i} = \sigma_i \boldsymbol{p}$ (4) $\dfrac{\partial e_i}{\partial \boldsymbol{y}_j} = \varepsilon_j \boldsymbol{p}$

(5) $\sum_i \boldsymbol{x}_i + \sum_j \boldsymbol{y}_j = \rho\, \boldsymbol{z}^0.$

The σ_i and the ε_j are numerical coefficients expressing collinearity of the left-hand and right-hand vectors in (3) and (4), Clearly ρ^0 (the value of ρ before the change) is equal to one.

A variation of the α_j induces a variation of ρ whose differential is readily computed. Differentiate (1) (2) and (5)

$$\frac{ds_i}{d\boldsymbol{x}_i} \cdot d\boldsymbol{x}_i = 0 \qquad \frac{\partial e_j}{\partial \boldsymbol{y}_j} \cdot d\boldsymbol{y}_j + \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = 0$$

$$\sum_i d\boldsymbol{x}_i + \sum_j d\boldsymbol{y}_j = \boldsymbol{z}^0 d\rho.$$

Using (3) and (4) one finds

(6) $\boldsymbol{p}^0 \cdot d\boldsymbol{x}_i = 0$ (7) $\boldsymbol{p}_0 \cdot d\boldsymbol{y}_j = -\dfrac{1}{\varepsilon_j} \dfrac{\partial e_j}{\partial \alpha_j} d\alpha_j$

and therefore

(8) $\boldsymbol{p}^0 \cdot \boldsymbol{z}^0 d\rho = -\sum_j \dfrac{1}{\varepsilon_j} \dfrac{\partial e_j}{\partial \alpha_j} d\alpha_j.$

The resource saving (more precisely its first differential; this tedious verbal specification will not be repeated) due to the change is $-\boldsymbol{z}^0 d\rho$; its value is $-\boldsymbol{p}^0 \cdot \boldsymbol{z}^0 d\rho$. The expression

(7) $-\dfrac{1}{\varepsilon_j} \dfrac{\partial e_j}{\partial \alpha_j} d\alpha_j = \boldsymbol{p}^0 \cdot d\boldsymbol{y}_j$

suggests the following evaluation of the left-hand member: take any input output combination $\boldsymbol{y}^1_j$ on the new production surface in a neighborhood of $\boldsymbol{y}^0_j$ (the initial combination, which was on the old production surface) and compare it to $\boldsymbol{y}^0_j$ by forming the difference $\boldsymbol{y}^1_j - \boldsymbol{y}^0_j$ the value of which is $-\boldsymbol{p}^0 \cdot (\boldsymbol{y}^1_j - \boldsymbol{y}^0_j)$. This value can be called the gain due to technological change for the j-th production unit. The total gain is obtained by summation over all production units. The essential point is that any new efficient combination $\boldsymbol{y}^1$, in a neighborhood of $\boldsymbol{y}^0$, can be used in this evaluation and not necessarily the combination which will be observed in the new economic equilibrium (¹). This is what we summarized in statement *i*) (²).

If we take up the more general case where *a*) the standard of living, *b*) the technological possibilities, *c*) the degree of efficiency ρ of the economic organization, and *d*) the physical resources all vary, the differentials of (1), (2), (5) take the form

$$\frac{ds_i}{d\boldsymbol{x}_i} \cdot d\boldsymbol{x}_i = ds^0_i \qquad \frac{\partial e_j}{\partial \boldsymbol{y}_j} d\boldsymbol{y}_j + \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = 0$$

$$\sum_i d\boldsymbol{x}_i + \sum_j d\boldsymbol{y}_i = \boldsymbol{z}^0 \, d\rho + \rho \, d\boldsymbol{z}^0.$$

The last relation can be transformed into

$$\sum_i \boldsymbol{p}^0 \cdot d\boldsymbol{x}_i + \sum_j \boldsymbol{p}^0 \cdot d\boldsymbol{y}_j = \boldsymbol{p}^0 \cdot \boldsymbol{z}^0 \, d\rho + \boldsymbol{p}^0 \cdot d\boldsymbol{z}^0$$

i. e. by use of (3) and (4) into

$$\sum_i \frac{ds^0_i}{\sigma_i} - \sum_j \frac{1}{\varepsilon_j} \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = \boldsymbol{p}^0 \cdot \boldsymbol{z}^0 \, d\rho + \boldsymbol{p}^0 \cdot d\boldsymbol{z}^0$$

which gives a relation between the variations of the four factors *a*)-*d*).

(¹) A diagram makes this point intuitive Let y'_j, y''_j be two points on the new production surface Y^1_j in a neighborhood of y^0_j (which is on the old

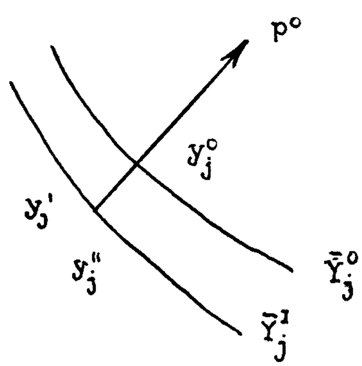

Fig 4

production surface Y^0_j). Since p^0 is normal to Y^0_j at y^0_j, $p^0 . (y'_j - y''_j)$ is clearly of the second order while $p^0 . (y'_j - y^0_j)$ is of the first order.

(²) A similar idea has been stressed by H. A. Simon in the unpublished papers listed in the title footnote