NONNEGATIVE SQUARE MATRICES

BY GERARD DEBREU AND I. N. HERSTEIN

1. INTRODUCTION

Square matrices, all of whose elements are nonnegative, have played an important role in the probabilistic theory of finite Markov chains (See [6] and the references there given) and, more recently, in the study of linear models in economics [2] to [6], [10] to [12], [15] to [20], and [24].

The properties of such matrices were first investigated by Perron [22], [23], and then very thoroughly by Frobenius [7], [8], [9]. Lately Wielandt [26] has given notably more simple proofs for the results of Frobenius.

In Section 2 we study nonnegative indecomposable matrices from a different point of view (that of the Brouwer fixed point theorem); a concise proof of their basic properties is thus obtained. In Section 3 properties of a general nonnegative square matrix \( A \) are derived from those of nonnegative indecomposable matrices. In Section 4 theorems about the matrix \( sI - A \) are proved; they cover in a unified manner a number of results recurrently used in economics. In Section 5 a systematic study of the convergence of \( A^p \) when \( p \) tends to infinity (\( A \) is a general complex matrix) is linked to combinatorial properties of nonnegative square matrices.

Unless otherwise specified, all matrices considered will have real elements. We define for \( A = (a_{ij}) \), \( B = (b_{ij}) \):

\[
A \leq B \quad \text{if} \quad a_{ij} \leq b_{ij} \quad \text{for all} \quad i, j,
\]

\[
A \not\leq B \quad \text{if} \quad A \leq B \quad \text{and} \quad A \neq B,
\]

\[
A < B \quad \text{if} \quad a_{ij} < b_{ij} \quad \text{for all} \quad ij.
\]

Primed letters denote transposes.

When \( A \) is an \( n \times n \) matrix, \( A_\tau = TAT^{-1} \) denotes the transform of \( A \) by the nonsingular \( n \times n \) matrix \( T \).

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2. NONNEGATIVE INDECOMPOSABLE MATRICES

An n \times n matrix $A$ ($n \geq 2$) is said to be indecomposable if for no permutation matrix $\Pi$ does $A = \Pi A \Pi^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where $A_{11}, A_{22}$ are square.

**Theorem I:** Let $A \geq 0$ be indecomposable. Then
1. $A$ has a characteristic root $r > 0$ such that
2. $r$ can be associated an eigen-vector $x_0 > 0$;
3. if $\alpha$ is any characteristic root of $A$, $|\alpha| \leq r$;
4. $r$ increases when any element of $A$ increases;
5. $r$ is a simple root.

**Proof:**
1. (a) If $x \geq 0$, then $Ax \geq 0$. For if $Ax = 0$, $A$ would have a column of zeros, and so would not be indecomposable.

1. (b) $A$ has a characteristic root $r > 0$.

Let $S = \{ x \in R^n | x \geq 0, \sum x_i = 1 \}$ be the fundamental simplex in the Euclidean $n$-space, $R^n$. If $x \in S$, we define $T(x) = [1/\rho(x)]Ax$ where $\rho(x) > 0$ is so determined that $T(x) \in S$ by (1.a) such a $\rho$ exists for every $x \in S$. Clearly $T(x)$ is a continuous transformation of $S$ into itself, so, by the Brouwer fixed-point theorem (see for example [14]), there is an $x_0 \in S$ with $x_0 = T(x_0) = [1/\rho(x_0)]Ax_0$. Put $r = \rho(x_0)$.

2. $x_0 > 0$. Suppose that after applying a proper $\Pi$, $\tilde{x}_0 = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $\xi > 0$.

Partition $A_{1}$ accordingly. $A_{2} \tilde{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{22} \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} r\xi \\ 0 \end{pmatrix}$, thus $A_{2} = 0$, so $A_{11} = 0$, violating the indecomposability of $A$.

If $M = (m_{ij})$ is a matrix, we henceforth denote by $M^*$ the matrix $M^* = (|m_{ij}|)$.

3-4. If $0 \leq B \leq A$, and if $\beta$ is a characteristic root of $B$, then $|\beta| \leq r$. Moreover, $|\beta| = r$ implies $B = A$.

$A^*$ is indecomposable and therefore has a characteristic root $r_1 > 0$ with an eigen-vector $x_1 > 0$: $A^*x_1 = r_1x_1$. Moreover $\beta^* = By$. Taking absolute values and using the triangle inequality, we obtain

(i) $|\beta| y^* \leq By^* \leq Ay^*$. So
(ii) $|\beta| x_1^* y^* \leq x_1^* Ay^* = r_1 x_1^* y^*$.

Since $x_1 > 0$, $x_1^* y^* > 0$, thus $|\beta| \leq r_1$.

Putting $B = A$ one obtains $|\alpha| \leq r_1$. In particular $r \leq r_1$ and since, similarly, $r_1 \leq r$, $r_1$ is equal to $r$.

* A permutation matrix is obtained by permuting the columns of an identity matrix. $\Pi A \Pi^{-1}$ is obtained by performing the same permutation on the rows and on the columns of $A$. 
Going back to the comparison of $B$ and $A$ and assuming that $|\beta| = r$ one gets from (i) and (ii)

$$ry^* = By^* = Ay^*.$$ 

From $ry^* = Ay^*$, application of 2 gives $y^* > 0$. Thus $By^* = Ay^*$ together with $B \leq A$ yields $B = A$.

5(a) If $B$ is a principal submatrix of $A$ and $\beta$ a characteristic root of $B$, $|\beta| < r$.

$\beta$ is also a characteristic root of the $n \cdot n$ matrix $B = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. Since $A$ is indecomposable, $B \leq A$, for a proper $A$ and $|\beta| < r$ (by 3–4).

5(b) $r$ is a simple root of $\Phi(t) = \det(tI - A) = 0$.

$\Phi'(r)$ is the sum of the principal $(n - 1) \cdot (n - 1)$ minors of $\det(rI - A)$. Let $A_i$ be one of the principal $(n - 1) \cdot (n - 1)$ submatrices of $A$. By 5(a) $\det(tI - A_i)$ cannot vanish for $t \geq r$, whence $\det(rI - A_i) > 0$ and $\Phi'(r) > 0$.

With a proof practically identical to that of 3–4, one obtains the more general result:

If $B$ is a complex matrix such that $B^* \leq A$, $A$ indecomposable, and if $\beta$ is a characteristic root of $B$, then $|\beta| \leq r$. Moreover $|\beta| = r$ implies $B^* = A$.

More precisely if $\beta = re^{i\omega}, B = e^{i\omega}DA^{-1}$ where $D$ is a diagonal matrix such that $D^* = I$. A proof of this last fact is given in ([26] p. 646 lines 4–11).

From this can be derived

**Theorem II:** Let $A \geq 0$ be indecomposable. If the characteristic equation $\det(tI - A) = 0$ has altogether $k$ roots of absolute value $r$, the set of $n$ roots (with their orders of multiplicity) is invariant under a rotation about the origin through an angle of $2\pi/k$, but not under rotations through smaller angles. Moreover there is a permutation matrix $P$ such that

$$PAP^{-1} = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with square submatrices on the diagonal.

As an immediate consequence of 4 one obtains:

$$\min \{ \Sigma_i a_{ii} \leq r \leq \max \{ \Sigma_i a_{ii} \}$$

and one equality holds only if all row sums are equal (then they both hold).

This is proved by increasing (resp. decreasing) some elements of $A$ so as to
Again the reader is referred to the excellent proof of Wielandt [26, p. 646–647].

If \( k = 1 \), the indecomposable matrix \( A \geq 0 \) is said to be primitive.

3. NONNEGATIVE SQUARE MATRICES

If \( A \) is an \( n \times n \) matrix, there clearly exists a permutation matrix \( \Pi \)

\[
\Pi A \Pi^{-1} = \begin{bmatrix}
A_1 & * \\
& A_2 \\
& & \ddots \\
& & & \ddots \\
& & & & A_n
\end{bmatrix}
\]

where the \( A_k \) are square submatrices on the diagonal and every \( A_k \) is

either indecomposable or a 1 \( \times \) 1 matrix.

The properties of \( A \) will therefore be easily derived from those of the

\( A_k \). For example \( \det(U - A) = \prod_{k=1}^{n} \det(U - A_k) \) and Theorem I

gives

**Theorem I**: If \( A \geq 0 \) is a square matrix, then

1. \( A \) has a characteristic root \( r \geq 0 \) such that
2. to \( r \) can be associated an eigen-vector \( x_0 \geq 0 \);
3. if \( \alpha \) is any characteristic root of \( A \), \( |\alpha| \leq r \);
4. \( r \) does not decrease when an element of \( A \) increases.

Let \( r_k \) be the maximal nonnegative characteristic root of \( A_k \), we take

\( r = \max_k r_k \); 1–3–4 are then immediate. To prove 2 we consider a se-

quence \( A_i \) of \( n \times n \) matrices converging to \( A \) such that for all \( i \), \( A_i > 0 \).

Let \( r_i \), be the maximal positive characteristic root of \( A_i \), \( x_i > 0 \) its as-

sociated eigen-vector so chosen that \( x_i \in S_i \), the fundamental simplex of

\( \mathbb{R}^n \). Clearly \( r_i \) tends to \( r \). Let us then select \( x_0 \in S \) a limit point of the

set \( (x_i) \); thus there is a subsequence \( x_{i'} \), converging to \( x_0 \geq 0 \) and for

every \( i' \), \( A_i' x_{i'} = r_{i'} x_{i'} \), therefore \( A x_0 = r x_0 \).

Statement 5 of Theorem I no longer holds, but 5.(a) becomes:

If \( B \) is a principal submatrix of \( A \) and \( \beta \) a characteristic root of \( B \),

\( |\beta| \leq r \).

make all row sums equal to

\[ \max_i \sum a_{ii}, \ (\text{resp. } \min_i \sum a_{ii}). \]

A similar result naturally holds for column sums.
The proof, almost identical, now rests on 4 of Theorem 1*.

3. Properties of $sI - A$ for $s > r$

In this section $A \geq 0$ is an $n \times n$ matrix, and $r$ is its maximal nonnegative characteristic root.

**Lemma**: If for an $x > 0$, $Ax \leq sx$ (resp. $\leq$), then $r \leq s$ (resp. $\geq$).

If for an $x \geq 0$, $Ax < sx$ (resp. $>$), then $r < s$ (resp. $>$).

The proofs of the four statements being practically identical, we present only the first one. Let $x_0 \geq 0$ be a characteristic vector of $A'$ associated with $r$ (2 of Theorem 1*): $A'x_0 = rx_0$. $Ax \leq sx$ with $x > 0$, therefore $x_0^tAx \leq x_0^tsx$ i.e., $x_0^trx \leq x_0^tsx$ and, since $x_0^tx > 0$, $r \leq s$.

We now derive two theorems (III* and III) from the study of the equation

$$(sI - A)x = y$$

**Theorem III**: $(sI - A)^{-1} \geq 0$ if and only if $s > r$.

**Sufficiency.** Since $s > r$, (2) has a unique solution $x = (sI - A)^{-1}y$ for every $y$; we show that $y \geq 0$ implies $x \geq 0$.

If $x$ had negative components (2) could be given the form [by proper (identical) permutations of the rows and columns and partition]

$$\begin{bmatrix}
  sI - A_1 & -A_{12} \\
  -A_{12} & sI - A_{22}
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = y$$

A stochastic $n \times n$ matrix $P$ is defined by $p_{ij} > 0$ for all $i, j$ and $\sum_j p_{ij} = 1$ for all $i$. Clearly 1 is a characteristic root of $P$ (take an eigen-vector with all components equal). 1 is therefore a root of some of the indecomposable matrices $P_1, P_2, \cdots, P_N$. Suppose that 1 is a root of $P_1$, it follows from footnote (3) that all row sums of $P_1$ are equal to 1, i.e.,

$$\Pi P \Pi^{-1} =$$

This remark makes many properties of stochastic matrices (the subject of the theory of finite Markov chains; see [6] and its references) ready consequences of the results of this article.
where \( x_1 > 0, x_2 \geq 0, y \geq 0 \). Therefore \(- (sI - A_1) x_1 - A_{12} x_2 \geq 0\),
i.e., \(- (sI - A_1) x_2 \geq 0\). From the Lemma* the maximal nonnegative characteristic root of \( A_1, r_1 \geq s \), a contradiction to the fact that \( r \geq r_1 \) (see end of Section 3) and \( s > r \).

**Necessity.** Since \((sI - A)^{-1} \geq 0\), to a \( y > 0 \) corresponds an \( x \geq 0 \).
Therefore from \( sx - Ax = y \) follows \( Ax < sx \) and, by the Lemma*, \( r < s \).

If \( A \) is indecomposable these results can be sharpened to the

**Lemma:** Let \( A \) be indecomposable.
If for an \( x \geq 0, Ax \leq sx \) (resp. \( \geq \)), then \( r \leq s \) (resp. \( \geq \)).
If for an \( x \geq 0, Ax \leq sx \) (resp. \( \geq \)), then \( r < s \) (resp. \( > \)).

The proofs, practically identical to those of the Lemma*, use a positive characteristic vector of \( A' \) associated with \( r \). One of these statements indeed has already been proved in 3–4 of Theorem 1.

**Theorem III:** Let \( A \) be indecomposable. \((sI - A)^{-1} > 0\) if and only if \( s > r \).

**Sufficiency.** We show that \( y \geq 0 \) implies \( x > 0 \). It is already known (from the proof of sufficiency of Theorem III*) that \( x \geq 0 \). If \( x \) had zero components, (2) could be given the form

\[
\begin{bmatrix}
  sI - A_1 & - A_{12} \\
  - A_{21} & sI - A_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = y
\]

where \( x_1 = 0, x_2 > 0, y \geq 0 \). Therefore \(-A_{12} x_2 \geq 0\), and, since \( x_2 > 0 \),
\( A_{12} = 0 \) violating the indecomposability of \( A \).

The **Necessity** has already been proved since \((sI - A)^{-1} > 0\) implies \((sI - A)^{-1} \geq 0\).

**Theorem IV:** The principal minors of \( sI - A \) of orders 1, \( \cdots \), \( n \) are all positive if and only if \( s > r \).

**Sufficiency.** \( \det(sI - A) \) cannot vanish for \( t > r \), thus \( \det(sI - A) > 0 \) for \( s > r \). Similarly, the maximal nonnegative characteristic root of a principal submatrix of \( A \) is not larger than \( r \) (see end of Section 3); it is therefore smaller than \( s \), and the corresponding minor of \( sI - A \) is positive.

**Necessity.** The derivative of order \( m(<n) \) of \( \det(sI - A) \) with respect to \( t, \) for \( t = s \), is a sum of principal minors of order \( (n - m) \cdot (n - m) \)

* It is worth [12] emphasizing a result obtained in the proof of necessity of Theorem III*.

**Remark.** Let \( A \geq 0 \) (resp. \( A \geq 0 \) indecomposable) be a square matrix. If for a \( y > 0 \) (resp. \( y > 0 \)), \( x \geq 0 \), then \((sI - A)^{-1} \geq 0 \) (resp. \((sI - A)^{-1} > 0\)).

The proof for indecomposable matrices uses the Lemma instead of the Lemma*. 
of \( sI - A \) and thus is positive. As its derivatives of all orders (0, 1, \( \cdots \), \( n-1 \), \( n \)) are positive for \( t = s \), the polynomial \( \det((I - A)^{t}) \) can vanish for no \( t \geq s \) i.e., \( s > r \).^6,^7

Since a square matrix with nonpositive (resp. negative) off-diagonal elements can always be given the form \( sI - A \) where \( A \preceq 0 \) (resp. \( >0 \)), many of the results of Arrow [2], Bray [3], Chipman [4], [5], Georgescu-Roegen [10], Goodwin [11], Hawkins and Simon [12], Metzler [15] to [18], Morishima^8 [19], Mosak [20], Solow [24] are contained in the above.

5. Convergence^9 of \( A^{p} \)

**Theorem V:** Let \( A \) be a \( n \times n \) complex matrix. The sequence \( A, A^{2}, \cdots, A^{p}, \cdots \) of its powers converges if and only if

^4 Georgescu-Roegen [10] stated a result whose counterpart here would be the following theorem (stronger than IV): The \( n \) northwest principal minors of \( sI - A \) of orders 1, \( \cdots \), \( n \) are all positive if and only if \( s > r \).

^6 We give a last property useful in economics [17], [18].

**Theorem.** Let \( A > 0 \) be a square matrix and let \( C_{ij} \) be the cofactor of the \( i^{th} \) row, \( j^{th} \) column element of \( sI - A \). If \( s > \sum_{j} a_{ij} \) for all \( i, \) then \( i \neq j \) implies \( C_{ii} > C_{ij} \).

Let us define the matrix \( B = (b_{pq}) \) as follows:

\[
b_{pq} = a_{pq} \text{ if } p \neq i; \quad b_{pq} = 0 \text{ if } i \neq q \neq j; \quad b_{pq} = s/2 = b_{ij}.
\]

\( B \) is indecomposable, moreover \( \sum_{q} b_{iq} = s, \sum_{p} b_{pq} < s \) for \( p \neq i \). Therefore (see footnote 3) the maximal positive characteristic root of \( B, r(B) < s \). Thus \( \det (sI - B) > 0 \); a development according to the \( i^{th} \) row yields:

\[
s/2C_{ii} - s/2C_{ij} > 0.
\]

^8 Morishima studies square matrices \( A \) such that for a permutation matrix \( \Pi, \)

\[
\Pi A \Pi^{-1} = A_{e} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where \( A_{11} \geq 0 \) and \( A_{22} \geq 0 \) are square, \( A_{12} \leq 0, A_{21} \leq 0 \). The relation

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
A_{11} & -A_{12} \\
-A_{21} & A_{22}
\end{bmatrix}
\]

shows how properties of \( A_{e} \) can be immediately derived from those of the non-negative matrix

\[
A_{e}^{2} = \begin{bmatrix}
A_{11} & -A_{12} \\
-A_{21} & A_{22}
\end{bmatrix}
\]

In particular \( A_{e} \) and \( A_{e}^{2} \) have the same characteristic roots.

^9 The Cesaro convergence of \( A^{p} \) i.e., the convergence of \( \frac{1}{p} (A + A^{1} + \cdots + A^{p}) \) can be studied in exactly the same fashion.
1. each characteristic root $\alpha$ of $A$ satisfies either $|\alpha| < 1$ or $\alpha = 1$;
2. when the second case occurs the order of multiplicity of the root 1 equals the dimension of the eigen-vector space associated with that root.

There is a nonsingular complex matrix $T$ such that

$$A_T = T A T^{-1} = \begin{bmatrix}
J_1 & & & 0 \\
& \ddots & & \vdots \\
& & \ddots & J_s \\
0 & \cdots & & J_k
\end{bmatrix}$$

where

$$J_i = \begin{bmatrix}
\alpha_i & 1 & & 0 \\
& \ddots & & \vdots \\
& & \ddots & 1 \\
0 & \cdots & & \alpha_i
\end{bmatrix}$$

is a square matrix on the diagonal and $\alpha$, a characteristic root of $A$. To every root $\alpha$, corresponds at least one $J_i$ (for this reduction of $A$ to its Jordan canonical form see for example [25]).

Since

$$T A^p T^{-1} = \begin{bmatrix}
J_1^p & & & 0 \\
& \ddots & & \vdots \\
& & \ddots & J_s^p \\
0 & \cdots & & J_k^p
\end{bmatrix},$$

$A^p$ converges if and only if every one of the $J_i^p$ converges. Let us therefore study one of them; for this purpose we drop the subscripts $i$ and $i$.

$J$ is a $k \times k$ matrix of the form $J = \alpha I + M$ where $M = (m_{st})$: $m_{st} = 1$ if $t = s + 1$, $m_{st} = 0$ otherwise.

$$J^p = \alpha^p I + \left( \frac{p}{1} \right) \alpha^{p-1} M + \cdots + \left( \frac{p}{k-1} \right) \alpha^{p-k+1} M^{k-1}. $$

It is easily seen that for $M^k$, $m_{st}^{(k)} = 1$ if $t = s + k$ and $m_{st}^{(k)} = 0$ otherwise. Thus $M^h = 0$ if $h \geq k$; also the nonzero elements of $M^h$ and $M^{h'}$
be a root of \( A \) of order \( \mu \). Thus \( x \) (resp. \( y \)), an eigen-vector of \( A \) (resp. \( A' \)) associated with the root 1, has the form \( x = X\xi \) (resp. \( y = Y\eta \)) where \( X \) (resp. \( Y \)) is an \( n \times \mu \) matrix of rank \( \mu \) and \( \xi \) (resp. \( \eta \)) is a \( \mu \times 1 \) matrix. For an arbitrary \( x \) the relation \( AA'x = A'x \) gives in the limit \( ACx = Cx \) i.e., \( Cx = X\xi(x) \). To determine \( \xi(x) \) we remark that \( Y' = Y'A \) i.e., by iteration \( Y' = Y' \) \( A' \), and therefore \( Y' = Y'C \); thus \( Y'x = Y'Cx = Y'X\xi(x) \).

Finally for all \( x, Cx = X(Y'X)^{-1}Y'x \) i.e., \( C = X(Y'X)^{-1}Y' \).

**Corollary:** Let \( A \geq 0 \) be indecomposable and 1 be its maximal positive characteristic root. The sequence \( A^p \) converges if and only if \( A \) is primitive.

The necessity is obvious. The sufficiency follows from the fact that 1 is a simple root.

Let then \( x_0 > 0 \) (resp. \( y_0 > 0 \)) be an eigen-vector of \( A \) (resp. \( A' \))

\( X_T = TX \) (resp. \( Y_T = Y'T^{-1} \)) plays for \( A_T \) the same role as \( X \) (resp. \( Y' \)) does for \( A \). Moreover \( Y'X = Y'X_T \). The right-hand matrix is nonsingular for the form taken by the Jordan matrix \( A_T \) in the convergence case implies that the eigen-vector space \( U \) generated by \( X_T \) is identical with the eigen-vector space \( V \) generated by \( Y_T \). Thus \( Y'X_T \xi = 0 \) implies \( X_T \xi = 0 \) (there is no vector different from zero in \( U \) perpendicular to \( V \) i.e., to \( U \)) therefore \( \xi = 0 \) since the rank of \( X_T \) is \( \mu \).
associated with the root 1, the limit \( C \) of \( A^p \) has the simple expression
\[
C = x_0 y_0 / y_0 x_0.
\]
Clearly \( C > 0 \), thus if the indecomposable matrix \( A \geq 0 \) is primitive, there is a positive integer \( m \) such that \( A^m > 0 \) when \( p \geq m \). The converse is an immediate consequence of the decomposition (1) of Theorem II.\(^{11}\)

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REFERENCES\(^{12}\)


\(^{11}\) This characterization of a primitive matrix, due to Frobenius, is typical of the purely combinatorial properties of the nonnegative square matrix \( A \) (used for example in the theory of communication networks): the smallest \( m \) satisfying the above condition is independent of the values of the nonzero elements of \( A \) as long as they stay positive.

The development of combinatorial techniques adapted to the treatment of such properties is the subject of [13].

\(^{12}\) In these references, we have tried to cover the economic literature with reasonable completeness. No such attempt has been made for the mathematical literature of which only a few essential papers have been quoted.


