

GOODWIN'S NONLINEAR THEORY OF THE BUSINESS CYCLE: AN ELECTRO-ANALOG SOLUTION

BY R. H. STROTZ, J. C. McANULTY, AND J. B. NAINES, JR.¹

We present here an electro-analog solution of a nonlinear business cycle model proposed by R. M. Goodwin. The effects of variation in the parameters of the model are explored. The existence of an infinite number of limit cycle solutions to the model is indicated. Finally, the significance of this for economic theory construction is discussed. An appendix presents a solution by the method of successive integration.

I. INTRODUCTION

BOTH Professor J. R. Hicks and Dr. R. M. Goodwin recently presented models of the business cycle emphasizing the importance of a lagged nonlinear accelerator.² Although the models presented by both of these writers are strikingly similar, this article will be concerned primarily with the one proposed by Goodwin. The present authors investigated the formal properties of Goodwin's model by using an electro-analog computer,³ and it is the purpose of this article to present the results. We first compare the analog solution with the one that Goodwin obtained by graphic methods, as well as with the analytic solution of Dr. Frank Bothwell⁴ and an analytic solution of our own; secondly, we indicate how sensitive the analog solution is to changes in the values of the parameters; thirdly, we show that there is not a unique solution to the model, but a multiplicity of solutions depending upon initial conditions; and finally we consider the implications of the multiplicity

¹ We wish to acknowledge the helpful cooperation of Mr. N. F. Morehouse, of the Argonne National Laboratories; Dr. Frank Bothwell, of the U. S. Ordnance Test Station, China Lake, California; and Dr. J. F. Calvert, of the Department of Electrical Engineering, Northwestern University. This article will be reprinted as Cowles Commission Paper, New Series, No. 74.

² J. R. Hicks, *A Contribution to the Theory of the Trade Cycle*, Oxford: Clarendon Press, 1950, and R. M. Goodwin, "The Nonlinear Accelerator and the Persistence of Business Cycles," *ECONOMETRICA*, Vol. 19, January, 1951, pp. 1-17.

³ The computer used is known as the Aeracom and was made available through the cooperation of the Navy Department, Bureau of Aeronautics. The work was done in the Aerial Measurements Laboratory of the Technological Institute at Northwestern University. On the application of an analog computer to problems in economic dynamics, see N. F. Morehouse, R. H. Strotz, and S. J. Horwitz, "An Electro-Analog Method for Investigating Problems in Economic Dynamics: Inventory Oscillations," *ECONOMETRICA*, Vol. 18, October, 1950, pp. 313-328, and R. H. Strotz, J. F. Calvert, and N. F. Morehouse, "Analog Computing Techniques Applied to Economics," *Transactions of the American Institute of Electrical Engineers*, Vol. 70, Part I, 1951, pp. 557-563.

⁴ Frank E. Bothwell, "The Method of Equivalent Linearization," *ECONOMETRICA*, Vol. 20, April, 1952, pp. 269-283.

of solutions for economic theory construction. It is not our intention to consider the empirical validity of Goodwin's work. We begin by reviewing for the reader's convenience the nature of the model Goodwin proposed.

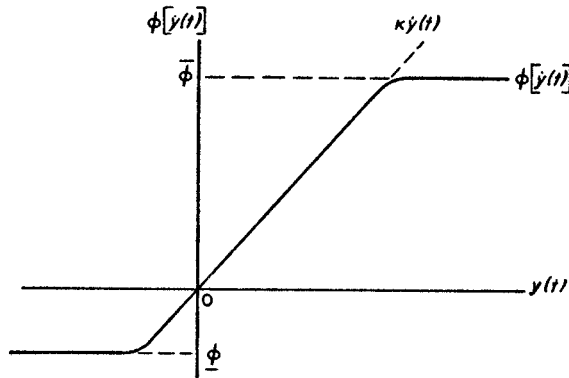


FIGURE 1.—The investment function.

II. THE GOODWIN MODEL

The Goodwin model consists of a consumption function, an investment function, and an accounting identity:

- (1) $c(t) = \alpha y(t) - \epsilon \dot{y}(t) + \beta(t),$
- (2) $\dot{k}(t) = \phi[\dot{y}(t - \theta)],$
- (3) $y(t) = c(t) + \dot{k}(t) + l(t).$

Here $c(t)$ is consumption; $y(t)$ is income; $\dot{k}(t)$, meaning $dk(t)/dt$, is induced investment; $l(t)$ is autonomous investment; and $\beta(t)$ is an autonomous component of consumption expenditure. All these values are flows and are expressed in billions of dollars per year. Since $\dot{k}(t)$ is induced investment, $k(t)$ is that portion of the stock of capital whose change is determined endogenously. The coefficient α is dimensionless; ϵ is a constant with the dimension of years; t is time in years; θ is a time lag in years. The model is in real terms and all money values are assumed to be adjusted for price-level changes.

The induced investment function is nonlinear, being linear only in its middle range and becoming completely inflexible at upper and lower levels. This function is shown in Figure 1. Its upper and lower limits are shown by $\bar{\phi}$ and ϕ , respectively. The slope of this function in the middle range is the acceleration coefficient, κ , which has the dimension of time. As this function approaches its limits it is rounded arbitrarily.

The model reduces to the equation

- (4) $\epsilon \dot{y}(t) + (1 - \alpha)y(t) = \phi[\dot{y}(t - \theta)] + \beta(t) + l(t).$

Goodwin obtained an analytical solution for this system after first assuming $\beta(t) + l(t)$ constant (a non-progressive economy) and then reinterpreting $y(t)$ as the deviation of income from its (unstable) equilibrium value, $[\beta(t) + l(t)]/(1 - \alpha)$. This gave the equation

$$(5) \quad \epsilon \dot{y}(t) + (1 - \alpha)y(t) = \phi[\dot{y}(t - \theta)],$$

or

$$(5') \quad \epsilon \dot{y}(t + \theta) + (1 - \alpha)y(t + \theta) = \phi[\dot{y}(t)].$$

This nonlinear differential-difference equation was then approximated by a second-order nonlinear differential equation upon replacing the difference term of equation (5') by the first two terms of their Taylor's series expansion. The approximating nonlinear differential equation is of the Lord Rayleigh type. Goodwin then integrated it graphically for the following values of the constants: $\epsilon = 0.5$, $\theta = 1.0$, $\alpha = 0.6$, $\kappa = 2.0$, $\bar{\phi} = +9.0$, and $\underline{\phi} = -3.0$.⁵ In this way he obtained a limit cycle with a period of slightly over nine years and with an income range of -5.0 to $+19.0$ billions of dollars per year below and above the unstable equilibrium value. The analysis was then extended to the case of a moderately progressive economy.

III. THE ELECTRO-ANALOG SOLUTION

An electrical circuit equation that is analogous to (4) is⁶

$$(6) \quad RC\dot{q}(\tau) + (1 - \alpha)q(\tau) - \phi[\dot{q}(\tau - \Theta)] = q_0(\tau).$$

Here $q(\tau)$ represents charge in coulombs; $\dot{q}(\tau)$, rate of change of charge in coulombs/sec. or amperes; $\phi[\dot{q}(\tau - \Theta)]$, a time-lagged charge function in coulombs, analogous to the function shown in Figure 1; $q_0(\tau)$, charge in coulombs; R , resistance in ohms; C , capacitance in farads; RC , a time constant in seconds; and $\tau = t/k_1$ and $\Theta = \theta/k_1$, where t is time in years and k_1 is a time-scaling factor = 62.9 yrs./sec.

A schematic electrical circuit for this analog is shown in Figure 2. Summing the voltage drops around the outer circuit gives

$$(7) \quad R\dot{q}(\tau) + [(1 - \alpha)/C]q(\tau) - (1/C)\phi[\dot{q}(\tau - \Theta)] = (1/C)q_0(\tau),$$

which transforms to (6) when multiplied by C .

The actual representation of the circuit placed on the computer is

⁵ Goodwin is not at all specific about the base period to which the dollar values are adjusted, although they are based on "Kuznets' data for the great boom and depression" (Goodwin, *op. cit.*, p. 16).

⁶ Another analogous electrical circuit equation is one in which current corresponds to income, but the actual circuit presents greater operating difficulties.

shown in Figure 3.⁷ Throughout the study the arbitrary function $\beta(t) + l(t)$ was set equal to zero, except for a superficial observation noted

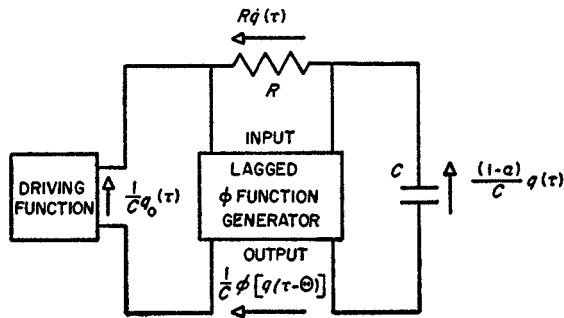
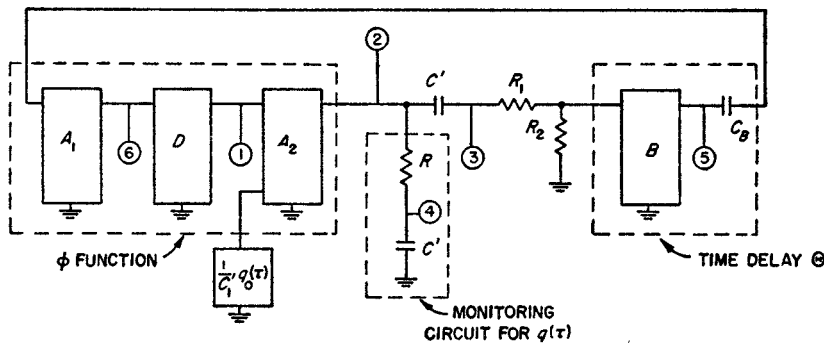


FIGURE 2.—Schematic electrical circuit of equation (7).



Code: D —diode device, varies $\bar{\phi}$; A_1 —amplifier, varies κ ; A_2 —amplifier, gain = 10; B —Brush time delay recorder; C' —condenser, varies α ; C_B —blocking condenser; R_1 —resistor, varies ϵ ; $R = R_1 + R_2$ —resistor, varies θ ; $q_0(\tau)$ —arbitrary function.

Monitoring points to ground: ①, ⑤, and ⑥; check points: ②— $\phi[\dot{y}(t - \theta)]$; ③— $\epsilon \dot{y}(t)$; ④— $(1 - \alpha) y(t)$.

FIGURE 3.—Electrical circuit used for investigation of Goodwin's model.

later on. The economic parameters were related to the electrical parameters in accordance with the transformation equations:

$$(8) \quad \epsilon = k_1(R_1 + R_2)(1/a),$$

⁷ Ignoring the blocking condenser C_B , which blocks a constant voltage generated in the time-delay unit B , the equation for this circuit is

$$\frac{1}{a} (R_1 + R_2)\dot{q}(\tau) + \frac{1}{a} \frac{1}{C'} q(\tau) - \frac{1}{a} R_2 A_1 A_2 D B \dot{q}(\tau - \Theta) = \frac{1}{a} \frac{A_2}{C_1} q_0(\tau).$$

That this equation conforms to (4), (6), and (7) may be seen from the transformation equations (8), (9), and (10), below.

$$(9) \quad \kappa = A_2[k_1 A_1 B D R_2 (1/a)],$$

and

$$(10) \quad \alpha = 1 - (1/a)(1/C'),$$

where a is a scale factor = $10^6/24$ ohms/sec., and $k_1 = \theta/\Theta$ is the time scaling = 1 yr./0.0159 sec. = 62.9 yrs./sec. The amplifier gains of the Brush recorder⁸ (B) and of the diode device (D) were $B = 1.4$ and $D = 0.93$. Values for the economic parameters were then chosen by selecting the appropriate values for the electrical parameters A_1 , C' , R_1 , and R_2 . The economic parameters were varied in accordance with Table I. The first row of values (italicized) are those that Goodwin used.

TABLE I

Case	α	κ	ϵ	θ	$\bar{\phi}:\phi$
<i>1(b)-5(b)</i>	<i>0.6</i>	<i>2.0</i>	<i>0.5</i>	<i>1.0</i>	<i>9:-3</i>
1(a)	0.400	2.00	0.500	1.00	9:-3
1(c)	0.733	2.00	0.500	1.00	9:-3
2(a)	0.600	1.58	0.500	1.00	9:-3
2(c)	0.600	8.42	0.500	1.00	9:-3
3(a)	0.600	2.00	0.349	1.00	9:-3
3(c)	0.600	2.00	0.802	1.00	9:-3
4(a)	0.600	2.00	0.500	0.50	9:-3
4(c)	0.600	2.00	0.500	1.50	9:-3
5(a)	0.600	2.00	0.500	1.00	6:-3
5(c)	0.600	2.00	0.500	1.00	15:-3

Solutions for each set or row in the table are shown in Photographs 1 through 5. Each photo shows the business cycle as a plot of y against t , and a closed loop, which is a phase plane plot of \dot{y} against y .⁹ The photos show some of the effects observed upon altering the values of Goodwin's constants. To facilitate comparison, the solution for Goodwin's values is repeated in the center column.

Not all of the pictures may be compared directly, however, because of variation in the scale factors. Nevertheless, the ratio of the scales can be used to adjust both amplitude and period where required. In Table II the periods and amplitudes obtained for the various sets of parameter values (as coded in Table I) are shown. Two values are given

⁸ Brush Transient Analyzer, loaned through the courtesy of Mr. H. Marsland and the Brush Development Company.

⁹ Because of operational limitations in the Brush recorder time-delay unit, a constant amount of noise was present in the signal, and this shows up as a random fluctuation on the photographic trace. It is most noticeable when the signal level is low as in Photo 5(a).

The following photos show graphically how a variation in the equation parameters from Goodwin's values affects the solution of the equation $y(t)$, and the phase plot. The solution for Goodwin's parameters is shown in the center column for rapid comparison with the variations.

Goodwin's Parameters

$$\alpha = 0.6 \quad \kappa = 2.0 \text{ yr.} \quad \epsilon = 0.5 \text{ yr.} \quad \theta = 1.0 \text{ yr.} \quad \bar{\phi}:\phi :: 3:1$$

These photos are coded for convenience in this form:

	<i>Phase plot</i>	<i>y(t) function</i>	
Photo (1 b)	$\alpha = 0.6$	Time 1 period = 8.12 yr.	Units: $y(t)$ in Billion \$/yr.
	x-axis: $y(t)$ 1 div. = 1.0	$y(t)$ 1 div. = 2.5	$\dot{y}(t)$ in Billion \$/yr./yr.
	y-axis: $\dot{y}(t)$ 1 div. = 2.0		1 div. = 0.054 inch

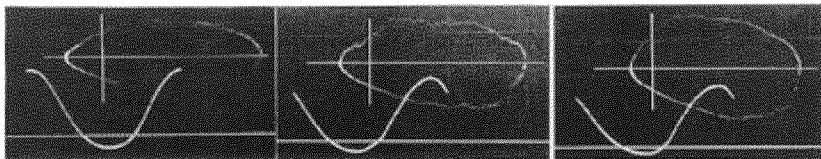


Photo (1a) $\alpha = 0.4$	0.667	6.97	Photo (1b) $\alpha = 0.6$	1.0	8.12	Photo (1c) $\alpha = 0.733$	1.5	9.61
	2.0	1.667		2.0	2.5		2.0	3.75

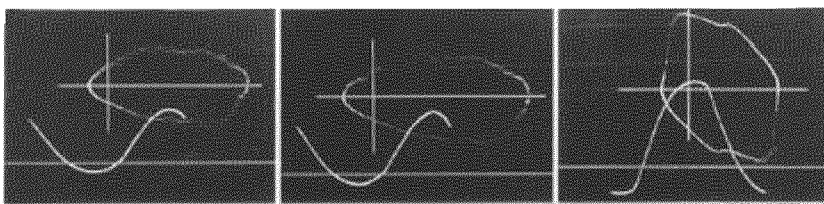


Photo (2a) $\kappa = 1.58$	1.0	8.04	Photo (2b) $\kappa = 2.0$	1.0	8.12	Photo (2c) $\kappa = 8.42$	2.5	8.43
	2.0	2.5		2.0	2.5		2.0	2.5

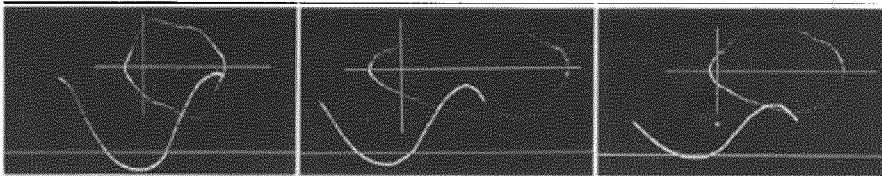


Photo (3a) $\epsilon = 0.349$ 2.5 7.21
2.87 2.5

Photo (3b) $\epsilon = 0.5$ 1.0 8.12
2.0 2.5

Photo (3c) $\epsilon = 0.802$ 1.0 9.53
1.25 2.5

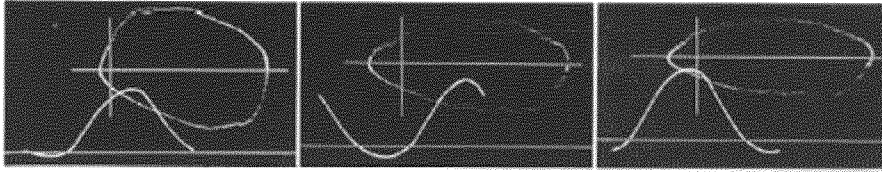


Photo (4a) $\theta = 0.5$ 1.0 10.66
2.0 2.5

Photo (4b) $\theta = 1.0$ 1.0 8.12
2.0 2.5

Photo (4c) $\theta = 1.5$ 1.0 7.05
2.0 2.5

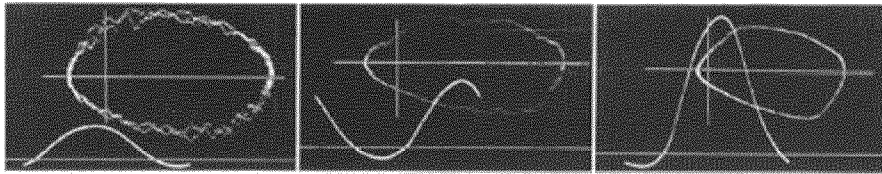


Photo (5a) $\bar{\phi}:\phi :: 2:1$ 0.5 7.86
0.8 2.5

Photo (5b) $\bar{\phi}:\phi :: 3:1$ 1.0 8.12
2.0 2.5

Photo (5c) $\bar{\phi}:\phi :: 5:1$ 2.5 8.62
4.0 2.5

for amplitude because amplitude may be measured from either the phase plot or the cycle itself. The close correspondence of these two measurements is a partial check on the accuracy of the oscillographic recording of the circuit's behavior. (The amplitudes should be the same, since they are separate recordings of the same signal.)

These results indicate that (for the parameter changes considered) the amplitude of the system is especially sensitive to changes in α , ϵ , and $\bar{\phi}:\phi$.¹⁰ It seems notably less sensitive to κ and to θ . This is as one would expect because it is mainly the multiplier and the upper and lower limits on the accelerator that should affect the amplitude of the cycle. By contrast, the period of the cycle appears to be much less sensitive

TABLE II

Case	Amplitude* (in billions of dollars/yr)						Period	Parameter Altered
	From Phase Plot			From Cycle Plot				
	Above origin	Below origin	Total	Right of origin	Left of origin	Total		
1(b)-5(b)	20.0	4.0	24.0	20.0	3.2	23.2	8.12	
1(a)	13.4	3.3	16.7	13.6	2.7	16.3	6.97	$\alpha = 0.4$
1(c)	28.5	4.5	33.0	29.6	3.0	32.6	9.61	$\alpha = 0.733$
2(a)	18.0	2.3	20.3	17.0	2.8	19.8	8.04	$\kappa = 1.58$
2(c)	28.7	8.8	37.5	27.8	9.0	36.8	8.43	$\kappa = 8.42$
3(a)	24.8	5.2	30.0	23.2	5.3	28.5	7.21	$\epsilon = 0.349$
3(c)	15.2	1.0	16.2	15.2	1.0	16.2	9.53	$\epsilon = 0.802$
4(a)	18.9	1.3	20.2	19.2	1.0	20.2	10.66	$\theta = 0.5$
4(c)	21.1	3.4	24.5	21.5	3.8	25.3	7.05	$\theta = 1.5$
5(a)	10.0	2.2	12.2	10.0	2.0	12.0	7.86	$\bar{\phi} = 6$
5(c)	41.2	3.3	44.5	41.5	3.5	45.0	8.62	$\phi = 15$

* Goodwin's result for Cases 1(b)-5(b) amplitude 24 (-5 to +19), period, slightly over nine years

to parameter changes. We are, as a matter of fact, impressed by how little change occurred in the period of the cycle even for substantial percentage changes in the parameters. The importance of this for the theory is apparent when one realizes that the economic parameters cannot be expected to hold steady through time. In particular, substantial changes in $\bar{\phi}:\phi$ seem to have little effect on the length of the cycle. One might expect that altering the limits of the maximum permissible rates of investment and disinvestment would affect considerably the length

¹⁰ The reader may notice that only changes in $\bar{\phi}$ are indicated in Table II. This is, of course, sufficient to change the ratio $\bar{\phi}:\phi$, and this is all that is required because the behavior of the system is independent of the money values of these limits.

of time required for a boom or a depression to work itself out. However, altering these limits also affects the magnitude of the induced investment *desired* during the boom and of the induced disinvestment *desired* during the depression (as may be inferred from the effects on amplitude). This means that the more or less rapidly the actual stock of capital *can* be changed, the more or less it *need* be changed. The resulting effect on the period of the cycle is, therefore, quite moderate.

The several photographs also seem to indicate that the general shape of the cycle is not very sensitive to changes in the parameters.¹¹

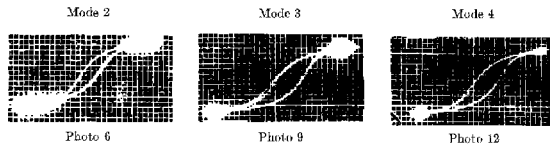
IV. THE MULTIPLICITY OF SOLUTIONS

In addition to the limit cycle that corresponds to Goodwin's, we found on the analog computer at least twenty-five other limit cycles that are also solutions to the same equation, indicating that there are an infinite number of additional solutions. These other solutions have periods that do not differ much from one year, one-half year, one-third year, etc. The cycle of 8.12 years may be called the first mode of oscillation; the cycle with period close to one year may be called the second mode of oscillation, etc. Modes 2, 3, and 4 are then those shown in Photographs 6 through 14. Whereas Photos 1 through 5 show variation in the first mode for different parameter values, Photos 6 through 14 show variation among the higher (viz., 2nd, 3rd, and 4th) modes of oscillation for the *same* parameter values. The parameter values are those given by Goodwin. It should be understood that these higher modes are not simply superimposed upon the first mode, but are each an *alternative* to it.

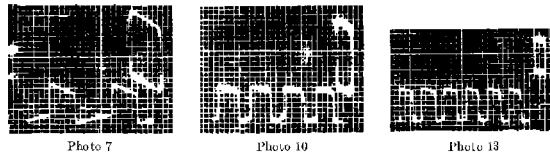
There are several characteristics of the higher-mode solutions which are peculiar to the apparatus used and which introduce an error in the quantitative analysis of the circuit. The first of these is the loop-like effect of the ϕ function. This is caused by stray capacitances in the ϕ function generator resulting in a slight damping of $\dot{y}(t)$ when it undergoes sudden changes in magnitude. This results in a lateral shift in the ϕ function, to the right when $\dot{y}(t - \theta)$ is increasing and to the left when $\dot{y}(t - \theta)$ is decreasing. Because of the nature of the ϕ function, this affects the system only when $\dot{y}(t - \theta)$ is changing rapidly. This affects $y(t)$ only at the peak and trough, and then only with respect to the sharpness of the point.

¹¹ However, by deviating considerably from Goodwin's constants, other effects were observed. An example was the appearance on the phase-plane plot of an egg-shaped loop which would swing back and forth across the face of the oscilloscope. The amplitude of the swing would grow until the loop appeared to strike an invisible barrier (amplifier saturation), which caused it to distort and break-up. This effect was reproducible but was not recordable with a still camera.

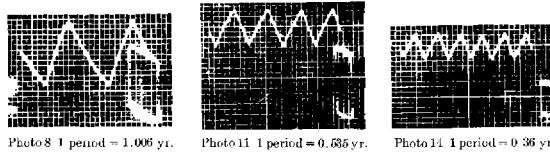
Photos 6, 9, 12 ϕ Function
 z-axis: $\hat{y}(t-\theta)^*$
 y-axis: $\hat{y}(t-\theta)^*$
 *not to be scaled
 $\alpha = 0.6$ $\epsilon = 0.5$ yr.
 $\kappa = 2.0$ yr. $\delta = 1.0$ yr.



Photos 7, 10, 13 $\hat{y}(t)$ and Phase Plot
 Phase Plot
 z-axis: $y(t)$ 1 div. = 0.50 Bil.\$/yr.
 y-axis: $\hat{y}(t)$ 1 div. = 1.0 Bil.\$/yr/yr.
 Except photo 13 where
 y-axis: $\hat{y}(t)$ 1 div. = 2.0 Bil.\$/yr/yr.
 Time Plot
 z-axis: Time
 y-axis: $y(t)$ 1 div. = 2.0 Bil.\$/yr/yr.



Photos 8, 11, 14 $y(t)$ and Phase Plot
 z-axis: Time
 y-axis: $y(t)$ 1 div. = 0.25 Bil.\$/yr
 Note: Add 20 divisions to scaling of y axis
 in Photo 8, and 30 divisions in Photos 11
 and 14



There is also a "burned-out" effect on the limits of these ϕ functions, and a "washed-out" effect in the middle region. These effects are due to the method used to record $y(t)$ and $\dot{y}(t)$. In these cases an oscilloscope and camera were used as the monitoring and recording devices. The traces were outlined by an electron beam on a phosphorescent screen whose intensity of illumination was a function of the number of electrons striking the screen. Hence, if the photograph shows a faint trace over a given region, that is so because the tracing beam passed rapidly over that region, the time interval for the region being short. Conversely, if the trace is heavy, the time interval was long. This also applies to Photos 1 through 5.

Because of a technical difficulty, it was impossible to obtain an entirely satisfactory *scalable* presentation of the ϕ functions of Photos 6, 9, and 12. The lower limit is put at -3 billion dollars/yr. in all cases, and the upper limit is then always $+9$ billion dollars/yr. This enables one to construct the horizontal axis. The vertical axis, however, can then be drawn only approximately, intersecting the horizontal axis in such a way that the origin would be (say) halfway between the upper and lower values shown for ϕ . For the other functions, however, scale factors are the same for modes 2, 3, and 4, and the photos may be compared directly. It is nevertheless necessary to add twenty divisions to the scaling of $y(t)$ in Photo 8, and thirty divisions to the scaling of $y(t)$ in Photos 11 and 14. This correction is required because, as the mode order increased, the picture shifted above the zero level by the number of units indicated.

An examination of Photographs 8, 11, and 14 shows that for these higher modes $y(t)$ always remained positive, fluctuations occurring beneath the "full employment" level of national income but above its unstable equilibrium level. This means that the phase plots are entirely to the right of the origin. In those presented in Table III, all values for national income are measured as deviations from the unstable equilibrium value.

We do not, however, have much confidence in the measurements of the turning points. This is so because there was a DC shift of the ϕ function along the y axis, and this shift affects the values of the turning points. It does not, however, affect the amplitude estimates that may be obtained from them. As observed, the phase loops of the second and third modes intersect.¹² We have no confidence that this should actually be the case.

¹² The intersection of phase loops is, of course, impossible in a first-order differential system, but it involves no logical contradiction in the system under study because the phase space is actually three-dimensional. The phase plots shown in Photographs 7, 10, and 13 might be regarded as projections of twisted loops running about the sides of a possibly irregularly shaped cylinder.

No provision was made to control the initial conditions of $y(t)$, $\dot{y}(t)$, and $\phi[\dot{y}(t - \theta)]$ at $t = 0$; instead, the system was set into operation by opening and closing the R, C loop, point ③ of Figure 3. Whenever this was done the solution took the form of some mode, ranging from the first up to the twenty-fifth or higher, the distinguishing feature being frequency. Since modes higher than the fourth were generally unstable on the computer, degenerating into lower modes, the probability of obtaining any one of the first four modes was good.

So far in the solution to Goodwin's equation (4), the autonomous function $\beta(t) + l(t) =$, say, $O_A(t)$ has been set equal to zero, although it may be any function. No serious attempt was made to investigate the general role of this function, although some superficial observations were made. Referring to Figure 3, one may see that if amplifier A_2 is a difference amplifier, i.e., if $e_{\text{out}} = (e_1 - e_2)A_2$, the new function may be introduced into the circuit as a $(-)$ signal in channel B of the amplifier.

TABLE III

Mode	Lower Turning Point	Upper Turning Point
2	\$2\frac{3}{4}\$ billion/yr.	\$5\frac{3}{4}\$ billion/yr.
3	\$6\frac{3}{4}\$- billion/yr	\$8\frac{1}{4}\$ billion/yr.
4	\$6\frac{3}{4}\$+ billion/yr	\$8 billion/yr

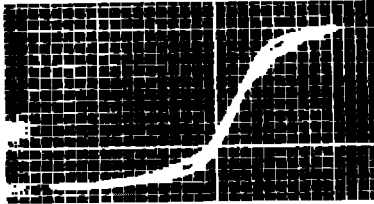
This was done, and $O_A(t)$ was represented by a sine function of variable frequency. The result was that, for a voltage level below a certain critical value, $O_A(t)$ merely modulated (added to) $y(t)$ and $\dot{y}(t)$ and did not appear to alter the period of the cycle. If the critical voltage level was exceeded, $O_A(t)$ became a driving function and took over control of the cycle, and the period of the cycle became the period of $O_A(t)$, while $y(t)$ and $\dot{y}(t)$ became modified functions of $O_A(t)$.

V. THE VALIDITY OF THE RESULTS

A. Internal checks on the computer. In order to determine the validity of the results several check methods were devised. These methods consisted of checking individual units of the circuit, setting up the circuits in different forms with different elements, and comparing results with analytical solutions. The amplifiers, A_1 and A_2 , were checked for overdrive by viewing the input versus output signals of each amplifier (during operation) on an oscilloscope. In order to operate near the maximum signal-to-noise ratio of the Brush recorder (B , in Figure 3), it was necessary to use it near full gain (unity). To check that the recorder was not overloaded and consequently introducing a signal distortion, a scope with overdrive limits marked was used to monitor the output continuously (at point ⑤, Figure 3).

After Photographs 6 through 14 were taken, the circuit was torn down,

the Brush recorder was overhauled, and the circuit was reconstructed in order to facilitate further investigation of parameter variations. In some cases different pieces of equipment were used. Goodwin's solution



Mode 1
Goodwin's Parameters
 Photo 15 ϕ Function
 x -axis: $\dot{y}(t-\phi)^*$
 y -axis: $\phi[\dot{y}(t-\theta)]^*$
 *not to be scaled
 $\alpha = 0.6$ $\epsilon = 0.5$ yr.
 $\kappa = 2.0$ yr. $\theta = 1.0$ yr.

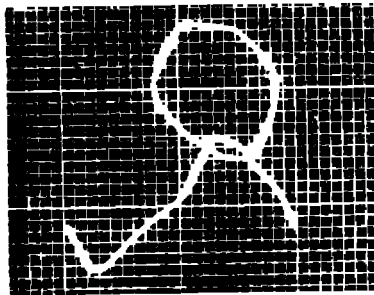


Photo 16 $\dot{y}(t)$ and Phase Plot
 x -axis: Time 1 Period = 8.28 yr.
 y -axis: $\dot{y}(t)$ 1 div = 2.00 Bil.\$/yr/yr.
 Phase Plot
 x -axis: $y(t)$ 1 div. = 2.50 Bil. \$/yr.
 y -axis: $\dot{y}(t)$ 1 div. = 2.00 Bil \$/yr./yr.

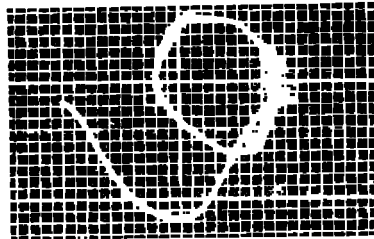


Photo 17 $y(t)$ and Phase Plot
 x -axis: Time 1 Period = 8.28 yr.
 y -axis: $y(t)$ 1 div. = 2.5 Bil.\$/yr.

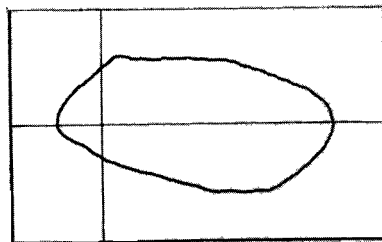


Figure 4. Rescaled Sketch of Photo 17
 x -axis: $y(t)$ 0.1 in. = 1.03 Bil.\$/yr.
 y -axis: $\dot{y}(t)$ 0.1 in. = 2.00 Bil.\$/yr/yr.

had also been photographed from the earlier circuit, however, and then appeared as shown in Photos 16 and 17. A rescaled drawing of the phase plot is shown in Figure 4. This may be compared directly with Photo 1(b), and $y(t)$ in Photo 17 can be compared directly with $y(t)$ in Photo

1(b). This comparison indicates consistency in the computer results. The ϕ function for the first mode had been taken earlier and is presented in Photo 15.

Another partial check was made by reducing the time delay θ to zero. When this was done, the system, as expected, took on the characteristics of a relaxation oscillator, having only one limit cycle.¹³

B. Analytic solutions. Three analytic solutions are available for comparison with the computer results. The first of these is Goodwin's, which has already been discussed. Another is a solution obtained by our former colleague, Dr. Frank Bothwell, who used an approximation method of equivalent linearization.¹⁴ Bothwell obtained a period of slightly less than eight years for the first mode. The equation for the periods of the higher-order modes is of the form

$$T = \frac{1}{n} \left(1 - \frac{\mu}{\pi^2 n^2} \right) \text{years} \quad (n = 1, 2, 3, \dots, \text{for } |\mu| < 2)$$

for Goodwin's constants. These results tend to substantiate the computer findings, namely, that, for modes higher than the first, $1/n$ ($n = 1, 2, 3, \dots$) is an approximation to the observed periods of oscillation. Bothwell's results differ most sharply from ours with regard to the turning points of the subsidiary cycles.¹⁵

As we stated above, we have no confidence in our own observations of the magnitudes of these turning points because of the DC shift that occurred for these modes along the y axis. Bothwell's phase loops for

¹³ There are some difficulties in the mathematical treatment of relaxation oscillators. These involve the behavior of the system at points of discontinuity (critical points). On this, we quote N. Minorsky, *Introduction to Non-Linear Mechanics*, Ann Arbor: J. W. Edwards, 1947, pp. 392-393:

... as a result of certain idealizations, discontinuities appear in the mathematical treatment of physical phenomena which exhibit rapid changes at certain points of their cycles. The use of discontinuities is convenient in some respects but inevitably introduces certain complications. . . . In some particularly simple circuits in which the effect is known to exist, one succeeds in "explaining" it by a more or less elementary physical argument. In more complicated circuits it is impossible to give an account of what actually happens and, still less, to predict theoretically the existence, or non-existence, of such effects. There exists no analytical theory of these oscillations which would permit a treatment of these phenomena on a uniform basis. . . . In order to be able to find a solution and to correlate the numerous experimental phenomena on a common basis, it becomes necessary to define terms and to introduce some kind of basic assumption, the value of which is to be justified by its agreement with the observed facts."

Minorsky then proceeds to analyze a relaxation oscillator similar to the one here under study (pp. 402-404). Economic and electrical considerations both justify the type of solution here assumed and observed.

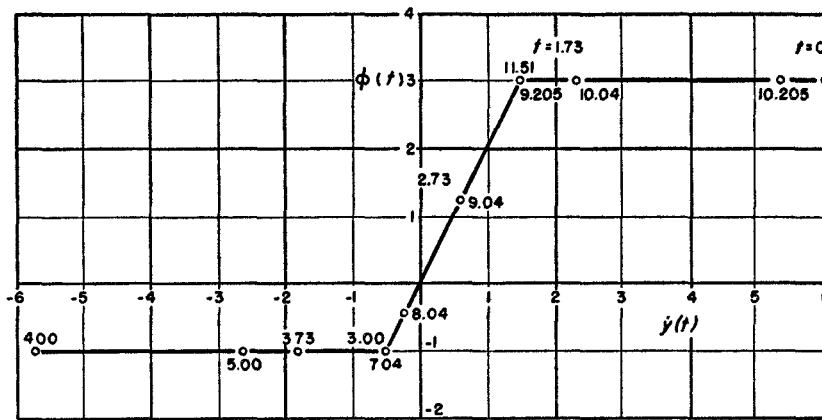
¹⁴ Bothwell, *op. cit.*

¹⁵ *Ibid.*, p. 278.

the higher-order modes are nested one in another. We have no reason to believe that this is incorrect. Our amplitude figures for the higher-order modes are not subject to error resulting from a DC shift, however; and our amplitude values run somewhat less than Bothwell's. The comparison is given in Table IV. The discrepancies must be ascribed either to the inaccuracy of solutions by electrical analog or to the approximating assumptions that Bothwell was forced to make in his method of equivalent linearization.

TABLE IV

Mode	Computer Amplitude	Bothwell's Amplitude
2	\$3.0 billion	\$4.8 billion
3	\$1.5 billion	\$2.4 billion
4	\$1.25 billion	\$1.6 billion

FIGURE 5.—Simplified ϕ function.

Using another analytical approach, the Goodwin equation was solved by a method of successive integration. The solution holds for the simplified ϕ function of Figure 5 rather than for the rounded-off ϕ function used on the analog computer. The details of this solution are presented in the Appendix. The results are shown in Figure 6.

C. Comparison of results. Four possible solutions for Goodwin's equation have been mentioned in this paper: Goodwin's based on a method of graphical integration, Bothwell's based on equivalent linearization, our analytical solution based on successive integration, and that of the analog computer. Among these solutions there are differences in the nature of the ϕ function that was assumed. Goodwin assumes that the acceleration principle holds over some middle range, but passes *gradually* to complete inflexibility at the extremities. Bothwell replaces the ϕ function (with complete inflexibility at the extremities) by an equivalent

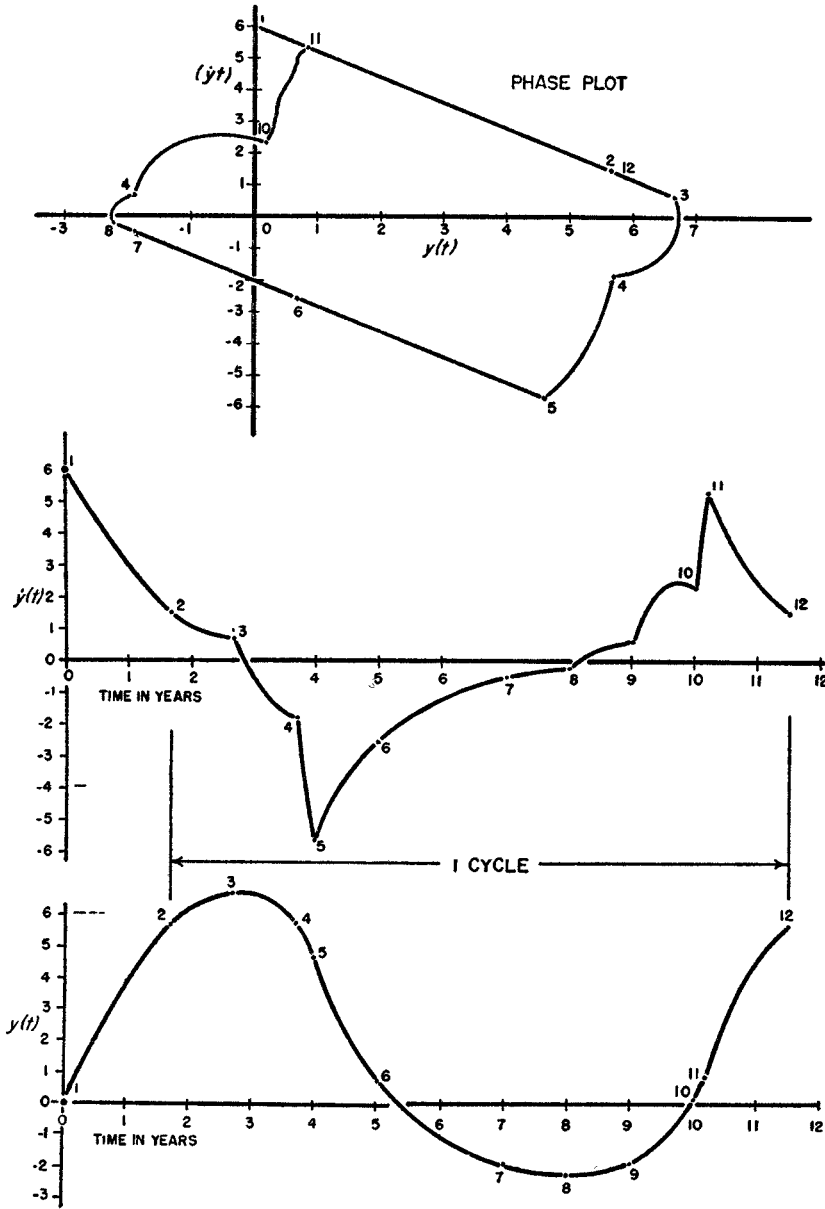


FIGURE 6.—Analytical solution.

linear function the slope of which is determined by the amplitude of the assumed form of the argument of the ϕ function. In our analytical solution we assume an abrupt passage into complete inflexibility at the extremities. The computer solution assumes a gradual passing of the

ϕ function into ranges of nearly, but not perfect, inflexibility. Although the ϕ function in the computer solution is not specified algebraically, it is of the form that Goodwin ascribed to reality.

The solutions may be compared in several ways. All four produced the same form and the same order of magnitude for the period of the first mode: Goodwin's, 9+ years; Bothwell's, 8- years; our analytic solution, $9\frac{3}{4}$ years; and the analog computer solution, 8+ years. The periods of the higher modes as determined by Bothwell and the analog computer were comparable. In addition, the computer showed the *form* of the higher-mode solutions and made possible the easy investigation of the effects of parameter variations.

VI. THE STABILITY OF THE CYCLES

The question of the stability of the various limit cycles has to this point been neglected. We shall say that a limit cycle is stable if, for any sufficiently small (non-zero) displacement of the system, it returns to that limit cycle. Proof of stability can be inferred on an analog computer from the observed behavior of the system when one takes into account the constant "noise" that is present in the circuit. An unstable limit cycle could in principle be observed only on an ideal, noise-free computer because otherwise the slightest disturbance would cause the system to depart immediately from any unstable cycle that it chanced to attain. It should be realized, on the other hand, that on an "imperfect" analog computer, where noise is ever present, a stable limit cycle may not persist indefinitely. This is so because the electrical disturbances may move the system beyond the limits of its stable region. If there are two or more stable limit cycle solutions to the equation, the system may from time to time move from one to another in response to the electrical disturbances impinging upon the circuit. How often a movement from one stable cycle to another will be observed depends upon the size of the region in the phase space in which initial conditions will lead to the particular cycles considered and upon the magnitude of the noise that constantly disturbs the circuit.

It was difficult to obtain any given mode higher than the fourth and, whenever once obtained, it would not persist for long. On the other hand, the first mode was easily obtained and would persist for hours. This implies that the stability regions for the lower-order modes were larger than the regions for the higher modes. Since no provision was made to control the initial conditions, however, it was not feasible to explore the stability regions of the various cycles.

VII. IMPLICATIONS FOR ECONOMIC THEORY CONSTRUCTION

The multiplicity of cycles that has been observed can be ascribed to the presence of the difference term. Had Goodwin approximated his

nonlinear difference-differential equation by using the first four terms of the Taylor's series expansion of the lagged ϕ function, the resulting approximating equation would have been a nonlinear differential equation of the fourth order, which we believe would have had two limit cycle solutions rather than one, both dependent on initial conditions. Improving the approximation by retaining more terms of the Taylor's expansion would increase the order of the differential equation and this would increase the number of solutions provided by the approximation. To the extent that this is generally true of nonlinear mixed systems, economic theory encounters a methodological dilemma. Goodwin has insisted on the necessity of nonlinear theory and limit cycle solutions for business cycle analysis. He has emphasized furthermore—and quite properly—that economic models must take account of finite time lags. This means that either nonlinear difference or nonlinear mixed systems must be used. If mixed systems seem to be required, this implies that we must in general expect a multiplicity of solutions. The resulting indeterminacy must then be overcome by specifying the initial conditions of the model. But prominent in Goodwin's case favoring nonlinear theory is the argument (which we now believe to be spurious for mixed systems) that a single cycle is obtained independently of the "initial conditions." He writes:¹⁶

Along with explaining the maintenance of oscillation, nonlinear theory does away with the necessity for "initial conditions." No matter how the mechanism is started, it tends to a certain type of cycle. Otherwise we are involved in believing that the magnitude and turning points, for example, of a cycle now are completely determined by events which took place many years ago. The absurdity of such an assumption is obvious.

The reappearance of the need for initial conditions therefore introduces a contradiction in his work.¹⁷

We do not feel, however, that the importance of initial conditions ought to invalidate this approach. We should emphasize first that "initial conditions" can be understood to refer not only to "events which took place many years ago," but also to any exogenous disturbance that subsequently jars the system out of its equilibrium path. Starting anew from a disequilibrium position the system moves once again to a stable limit cycle. The cycle to which it moves need not be the same as

¹⁶ Goodwin, *op. cit.*, p. 3.

¹⁷ The observation that a mixed system in macrodynamic economics may have an infinite number of solutions should be no surprise to readers of this journal. At the Leyden meeting of the Econometric Society in 1933, this observation was made concerning Kalecki's equation, $\dot{y}(t) = ay(t) - cy(t - \theta)$, and the characteristic solutions were presented by Ragnar Frisch and Harold Holme in "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," *ECONOMETRICA*, Vol. 3, April, 1935, pp. 225-239. In the Kalecki equation, as in Goodwin's, all higher modes have periods $\leq \theta$.

that from which it just departed. Subjected to exogenous forces, national income may move back and forth from cycle to cycle. This may actually enrich the explanatory value of the theory.

If certain cycles that are theoretically possible are in fact not observed, this may be explained in terms of the sorts of "initial conditions" (disturbances or exogenous forces) required to produce them. In order for certain cycles to be realized, "initial conditions" may be required that do not in fact take place. The problem is to determine what kinds of "initial conditions" lead to the various possible cycles, and then to determine whether these conditions can occur. This presents an analytical problem of great complexity, but one that must be solved if non-linear mixed models are to provide unambiguous answers to problems in economic theory.

APPENDIX

SOLUTION BY SUCCESSIVE INTEGRATION

Referring to Figure 5, it is seen that there are three distinct regions. The function equations for the three regions are:

- (a) $\phi[\dot{y}(t)] = 3n,$
- (b) $\phi[\dot{y}(t)] = \kappa\dot{y}(t),$
- (c) $\phi[\dot{y}(t)] = -n.$

(n is a convenient scaling number, and may be set equal to 3 to compare with Goodwin's and computer results.)

The Goodwin equation (5) can then be written for these three distinct regions after a lag θ as:

- (d) $\epsilon\dot{y}(t) + (1 - \alpha)y(t) = 3n,$
- (e) $\epsilon\dot{y}(t) + (1 - \alpha)y(t) = \kappa\dot{y}(t - \theta),$
- (f) $\epsilon\dot{y}(t) + (1 - \alpha)y(t) = -n,$

assuming that $O_A(t) = 0$. Equations (d) and (f) are readily solved and the solutions are of the form

$$(g) \quad y(t) = \frac{1}{1 - \alpha} (Ke^{-t(1-\alpha)/\epsilon} + d),$$

where the constant d represents the right-hand side of equations (d) and (f) and K is the constant of integration.

Equation (e) is not so easily solved because of the time delay term, θ . One may note that the left-hand side of this equation is equal to the solution of the section of the function preceding in time by an amount θ . Hence the problem can be solved by treating bounded sections of the ϕ curve in consecutive order starting at some known initial point.

As long as the initial point lies somewhere on the phase plot along the line given by equation (a) it is immaterial what value $y(t)$ has for time $t = 0$. So we

arbitrarily choose $y(t) = 0$. Then

$$(h) \quad K = 3n,$$

$$(i) \quad y = \frac{3n}{1-\alpha} (1 - e^{-[(1-\alpha)/\epsilon]t})$$

$$(j) \quad \dot{y} = \frac{3n}{\epsilon} e^{-[(1-\alpha)/\epsilon]t}.$$

At the upper corner of the ϕ function from equations (a) and (b),

$$(k) \quad \kappa \dot{y}(t) = 3n.$$

Solving for time t_1 , using equations (j) and (k),

$$(l) \quad t_1 = \frac{\epsilon}{1-\alpha} \ln(\kappa/\epsilon) = 1.25 \ln 4 = 1.7329.$$

At this time equations (i) and (j) show

$$y_1 = 5.63n \text{ and } \dot{y}_1 = 1.5n.$$

Now since we are in the region of the function given by equation (b), we use equation (e). In this region the term $\phi[\dot{y}(t - \theta)]$ is equal to $3n$ for a period of $(t_1 + \theta)$, so we determine y and \dot{y} at the point $t_2 = t_1 + \theta = 1.7329 + 1 = 2.7329$, using equations (i) and (j) again. Here

$$y_2 = 6.66n \text{ and } \dot{y}_2 = 0.6756n.$$

In the region from t_2 to $t_3 = t_2 + \theta$ we use equation (e):

$$(m) \quad \epsilon \dot{y}(t_3) + (1-\alpha)y(t_3) = \kappa \dot{y}(t_2) = \kappa(3n/\epsilon)e^{-[(1-\alpha)/\epsilon]t_2} = \kappa(3n/\epsilon)e^{-[(1-\alpha)/\epsilon](t_2-\theta)}.$$

This is a differential equation of the general form

$$(n) \quad \dot{y} + Py = Q,$$

where $Q = f(t)$, whose solution is

$$(o) \quad y = e^{\int -Pdt} \left(\int Q e^{\int Pdt} dt + K_1 \right).$$

By substitution from (m) we have

$$(p) \quad y_3 = e^{-\int [(1-\alpha)/\epsilon] dt} \left(\int \frac{3n\kappa}{\epsilon^2} e^{-[(1-\alpha)/\epsilon](t_2-\theta)} e^{\int [(1-\alpha)/\epsilon] dt} dt + K_1 \right) \\ = e^{-[(1-\alpha)/\epsilon]t_3} \left(\frac{3n\kappa}{\epsilon^2} t_3 e^{[(1-\alpha)/\epsilon]\theta} + K_1 \right),$$

$$(q) \quad \dot{y}_3 = \frac{n}{\epsilon} e^{-[(1-\alpha)/\epsilon]t_3} \left[\frac{3n}{\epsilon} e^{[(1-\alpha)/\epsilon]\theta} \left(1 - \frac{1-\alpha}{\epsilon} t_3 \right) - (1-\alpha)K_1 \right].$$

K_1 is determined by substituting the value of y at the initial time t_2 for this region. $K_1 = -86.700n$ and

$$t_3 = t_2 + \theta = 2.7329 + 1 = 3.7329.$$

Substituting into (p) and (q), we obtain

$$y_3 = 5.693n \text{ and } \dot{y}_3 = -1.852n.$$

TABLE V
COEFFICIENTS FOR FORMALIZED EQUATIONS (r) AND (s) USED TO PLOT CURVES OF FIGURE 6

Points Constants*	1	3	4	5	8	9	10	11	12
$a \begin{cases} y \\ \dot{y} \end{cases}$	0	0	0	0	0	0	0	-10,527.5	0
	0	0	0	0	0	0	0	8,422.00	0
$b \begin{cases} y \\ \dot{y} \end{cases}$	0	0	-190.182	0	0	0	4,434.76	361,085.6	0
	0	0	152.146	0	0	0	-3,547.80	-320,451.0	0
$c \begin{cases} y \\ \dot{y} \end{cases}$	0	53.41298	1,473.300	0	-1,245.43	-81,449.32	-4,059,625.	-4,059,625.	0
	0	-42.73039	-1,559.00	0	996.344	74,028.98	3,969,871.5	3,969,871.5	0
$d \begin{cases} y \\ \dot{y} \end{cases}$	-7.500	-86.700	-2,736.75	174.877	8,634.70	371,261	15,015,518.7	15,015,518.7	-23,411.83
	6.000	122.773	3,662.70	-139.902	-8,153.19	378,458	-16,072,040.	-16,072,040.	18,729.46
$g y$	7.500	0	0	-2.500	0	0	0	0	7.500

* $h = -0.8$ for all regions of both equations.

Solving equations (b) and (c) shows $\dot{y} = -n/\kappa = -0.5n$ for the lower corner. Thus point t_3 is below the corner so that t_3 lies out on the lower line of the ϕ function. Solving for t at the lower corner, using $y = -0.5n$ in equation (q), $t_c = 3.002$.

Equations (p) and (q) will be used from here to the point $t_4 = t_c + \theta = 3.002 + 1 = 4.002$. Beyond the point t_4 equation (f) will be used and the solutions are similar to equations (i) and (j) except that $-n$ is substituted for $3n$. This equation gives values of \dot{y} that are increasing, so that the point moves back along the function, eventually reaching the lower corner and moving up. At a time $t'_c + \theta$ the equations (n) and (o) again apply, and the whole process is repeated until the upper corner is again reached for decreasing \dot{y} , ending the cycle.

The results are graphed¹⁸ in Figure 6. The equations that apply between the labeled points of Figure 6 may be written by substituting the appropriate numbers from Table V for the constants in these formalized equations:

$$(r) \quad y(t) = ne^{at} (at^3 + bt^2 + ct + d) + g,$$

$$(s) \quad \dot{y}(t) = ne^{at} (at^3 + bt^2 + ct + d).$$

Investigation of the computer results shows that the solution of the equation stays in the "saturated" regions of the ϕ function for a period of time greater than θ ; hence the solution always moves out of the "saturated" regions with the same information from time $t - \theta$ that it had during the preceding cycle. This is a necessary and sufficient condition for the first cycle to be a stable cycle.

Northwestern University

¹⁸ The results need to be scaled. This may be done by assigning to n the value of \$3 billion/yr.