Specialisation and Efficiency in World Production

The fundamental problem of activity analysis is the selection of productive processes which can be used to provide a maximum output from given resources. This emphasis on the selection of a limited number of productive processes and the suppression of others is rather new in the theory of general equilibrium systems. For example, the systems of Walras, Cassel and Leontiev assume a given set of productive processes which are always in use. There is one field of traditional economics, however, where explicit discrimination between processes to be used and processes to be suppressed has always been the fundamental object of analysis. The problem of specialisation in international trade according to comparative advantage is precisely the problem of selecting a group of productive processes to be used in the interest of maximum world output. The classical analysis, it is true, because of its dependence on bilateral comparisons, was inadequate. To overcome this defect, Frank D. Graham constructed general equilibrium models of world production with many countries and many commodities which he solved by trial and error in the sense of finding points of competitive equilibrium. Although the classical economists would no doubt expect these points to be points of maximum world output, they did not fully expose the relation of maximum output to the possibility of competitive equilibrium. In this paper I shall begin by presenting an elementary proof of the efficiency of competition and free trade in Graham's model.

The traditional theory of international trade is chiefly concerned with trade between competitive economies which have homogeneous production functions. In the long-run equilibrium of this system, prices must be such that no unused process of production is profitable and all the processes used earn zero profits. I shall refer to these conditions as the profit conditions of competitive equilibrium. Ultimate resources, however, are assumed to be immobile between countries, so that resources in different countries are essentially different goods and the processes of each country must be defined in terms of its own resources. Let output include ultimate productive services as negative goods. Then the profit conditions of competitive equilibrium are necessary and sufficient for a maximum value of world output at given prices, provided that the productive processes are linear and additive, that is, that the production functions of traditional analysis are homogeneous (of zero degree) and there are no external economies or dis-economies. On the other hand, the existence of a set of prices for which world output is a maximum value output is necessary and sufficient for the output to be in the boundary of the set of outputs which it is possible to produce with

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1 This article resulted from research initiated while a guest of the Cowles Commission and will be reprinted in Cowles Commission Papers, New Series, No. 72.
2 I wish to thank Professor Tjalling C. Koopmans for his encouragement and instruction. The debt to my old teacher, Frank D. Graham, will be apparent.
3 Throughout this paper "an output" means the net quantities of goods produced, by an economy or by an activity, as the case may be. A maximum output is an output in which the quantity of one good cannot be increased unless another is reduced. In activity analysis such an output is called efficient.
4 These being the prices used in the profit conditions. Of course, the maximum value of output must be zero by the profit conditions, since ultimate productive services are included in output.
the available productive processes. Thus, equilibrium points of a linear and additive competitive model always lie in the boundary of the set of possible outputs, and any point in this boundary is a possible equilibrium of the competitive model if the demand conditions can be satisfied.

However, it will not be necessary to consider the entire boundary of the set of possible outputs. Indeed, for most purposes it will suffice to consider maximum outputs, since these are the only outputs for which the profit conditions can be met with positive prices. But this is just the efficient point set used by Koopmans in his activity analysis.

The strong parallel between competitive theory and activity analysis suggests to me that the recent results achieved in activity analysis can contribute to the analysis of competition. I shall first describe the Graham model and then prove some propositions on patterns of specialisation in that model which achieve efficiency in the sense of Koopmans. Finally, I will discuss the major limitation of the Graham model.

THE GRAHAM MODEL

There are two extreme models of world production in which free trade is efficient. In one, the fundamental reason for trade is the presence of different productive processes in different countries. It is suggestive to impute these differences to a general factor, "climate," extending the meaning of climate to include the total environment. In the second model, the productive processes are identical between countries. There are no differences of "climate." But trade occurs because there are differences of tastes and factor endowments, so that countries are led to produce goods in quantities adapted to their factor supplies but not necessarily to their tastes. In this paper I shall be concerned with trade depending upon differences of "climate." It will be convenient to neglect the presence of transport costs and assume that no artificial trade barriers exist. However, we shall suppose that ultimate factors are completely immobile between countries. The analysis will be static. Finally, we will assume that production functions (in their implicit form) are homogeneous of zero degree and mutually independent.

The Graham model is a "climatic" model with productive processes differing between countries. He makes the Ricardian assumption that labour is the only ultimate factor and that the labour cost of each product is constant. However, it is not necessary to assume, to obtain Graham's model, that the activities are actually integrated, converting labour directly into final output, or that production coefficients are fixed. It is sufficient that intermediate products do not appear in international trade, or, should they be traded, that they do not contribute to outputs which re-enter trade, and that joint production is absent. As Samuelson and others have shown, if labour is the only unproduced factor for an economy in which each production function is homogeneous and has but one product; then, assuming that production is efficient, each good will be produced by a single process and the rate of transformation between

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1 The proof that the profit conditions are sufficient for this result is quite brief. The set of possible outputs is a cone $Y$ and the profit conditions are equivalent to $p'z \leq 0$ for $z \in Y$ and $p'y = 0$ for $z = y$, where $y$ is the actual output. But these are precisely the conditions for $p$ to be a normal to $Y$ at $y$. Thus $y$ must lie in the boundary of $Y$. If, in addition, the demand condition is met and $y$ is the vector of quantities demanded when $p$ is the price vector, $y$ is a competitive equilibrium by definition. ($p$ is a column vector, $p'$ is a row vector, and $p'y$ is the inner product of $p$ and $z$.) A rather lengthy mathematical argument is needed to prove, in the general case, that the profit conditions are also necessary.


3 Paul A. Samuelson, Chapter V1 in *Activity Analysis of Production and Allocation*, ed. by Tjalling C. Koopmans.

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SPECIALISATION AND EFFICIENCY IN WORLD PRODUCTION

labour and any good will be constant. Thus, the economy may be treated as though its processes were completely integrated.

It is not clear that Graham appreciated the necessity for excluding intermediate products from international trade in his model. At least, he introduces both cloth and linen into his arithmetic examples, though linen presumably will be used largely in making cloth.

For the purpose of illustration, I will present one of Graham's simpler models. He summarises the productivity data in the following table:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B 10</th>
<th>C 10</th>
<th>D 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cloth</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Linen</td>
<td>19</td>
<td>20</td>
<td>15</td>
<td>28</td>
</tr>
<tr>
<td>Corn</td>
<td>42</td>
<td>24</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

A, B, C and D are four countries trading in the three goods—cloth, linen and corn. In each column of the table are recorded the quantities of linen and corn which can be produced in a certain country by the amount of labour which produces ten units of cloth. I shall take this quantity of labour to be the unit of labour in each country. We are free to do this, since labour does not move from one country to another. Graham also assumes that the labour supply is a given quantity which does not vary with the real wage. In our units of labour the labour supplies in the trading countries are given by Graham as follows:

<table>
<thead>
<tr>
<th></th>
<th>A 1,000</th>
<th>B 2,000</th>
<th>C 3,000</th>
<th>D 4,000</th>
</tr>
</thead>
</table>

These data can be introduced into the activities model:

\[
\begin{bmatrix}
A_{in} \\
A_{pri}
\end{bmatrix} x = \begin{bmatrix}
y_{in} \\
y_{pri}
\end{bmatrix},
\]

\[x \geq 0, y_{pri} \geq -\eta.\]

All of Graham's models have this form. Each column of the partitioned matrix represents the unit level of an integrated activity. \(A_{in}\) is the matrix of output coefficients, and \(A_{pri}\) the matrix of input coefficients. \(x\) is a column vector of activity levels. \(y_{in}\) is a column vector of outputs. \(y_{pri}\) is a column vector of inputs (negative outputs). \(\eta\) is the vector of labour supplies. Inserting the data of our illustration gives:

\[
\begin{bmatrix}
10 & 10 & 10 & 10 \\
19 & 20 & 15 & 28 \\
42 & 24 & 30 & 40 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1
\end{bmatrix} x = \begin{bmatrix}
y_{in} \\
y_{pri}
\end{bmatrix}
\]

\[x \geq 0, y_{pri} \geq \begin{bmatrix}
1,000 \\
2,000 \\
3,000 \\
4,000
\end{bmatrix}
\]

where the empty spaces are to be filled with zeroes.

This model can be used to define a problem in activity analysis, namely, to determine the set of vectors $x$ leading to boundary vectors $y$ which may become competitive equilibria and to determine the set of all such boundary vectors.\textsuperscript{1} If we assume that the supplies of labour are constant at positive prices and zero at zero or negative prices, the prices of any final outputs which are produced must be positive if the condition of zero profits is to be met for activities in use. Also, the price of a good which is not produced may be put at a low positive level without making its production profitable, since the cost is positive. Thus any equilibrium output must be a maximum value output for some set of positive prices, and we may confine attention to the boundary points for which this is so. As it happens, the set of such points is Koopmans' efficient point set.\textsuperscript{2} Thus our problem is the efficient allocation of resources as Koopmans defines it in his activity analysis. We may refer to a pattern of zeroes and non-zeroes appearing in an activity vector $x$ which generates an efficient point as an efficient specialisation. Then the set of efficient specialisations will be the set of all those specialisations which may appear in a competitive equilibrium when demand conditions, within the restrictions named, are appropriate.

THEOREM ON EFFICIENCY IN THE GRAHAM MODEL

I shall show that the basic theorem of activity analysis can be proved directly in Graham's model in an elementary way. First, we define an efficient point as a vector $y$ of primary resources consumed and goods produced such that the net output of intermediate products is zero and no final good can be increased in quantity within the resource limitations unless the condition on intermediate products is violated or the amount of some other final good is reduced. The basic theorem states that a necessary and sufficient condition for an output vector $y$ to be efficient is the existence of a price vector $\hat{p}$ which satisfies the following conditions. The prices of final goods are positive. The prices of intermediate products may be positive, negative or zero. The prices of primary resources are zero if those resources are in surplus supply and are positive or zero otherwise. At the prices $\hat{p}$ all activities in use yield zero profits, and all activities not in use would yield zero profits or losses were they put in use.\textsuperscript{3}

These conditions are much simplified, however, in the Graham model. Since we can use integrated activities in each country, intermediate products drop out. Moreover, we have only one primary resource in each country, which can be converted into any final good through an integrated activity. Therefore, it will never be efficient to have a primary resource in surplus supply, and since the prices of final goods are positive, the labour supplies must also bear positive prices in order to meet the profit condition. Thus all prices must exceed zero, and for this model the basic theorem takes the form:

\begin{equation}
\text{If } y = Ax \text{ with } x \geq 0, \text{" } y \text{ is efficient " is equivalent to } \text{" } y_{pr1} = \eta \\
\text{and there is a } \hat{p} \text{ such that } \hat{p} > 0 \text{ with } \hat{p}'A_j = 0 \text{ if } x_j > 0, \hat{p}'A_j \leq 0 \\
\text{if } x_j = 0. \text{"} \tag{3}
\end{equation}

Here $\hat{p}$ is the vector of prices, $A_j$ is the vector of input-output coefficients for the $j$th activity (the $j$th column of the activities matrix $A$) and $x_j$ is the level of the $j$th activity (the $j$th component of the vector $x$).

\textsuperscript{1} It is easily seen that the set of outputs attainable within the model form a closed and bounded convex set. Indeed, a convex cone truncated by hyperplanes expressing the limitations on the supplies of labour. See Koopmans, op. cit., p. 82.

\textsuperscript{2} Koopmans, op. cit., pp. 60-61.

\textsuperscript{3} Koopmans, op. cit., p. 82. These results are discussed in non-mathematical terms in Koopmans' "Efficient Allocation of Resources," Economica, October, 1937, pp. 455-65.
In words, the conditions are that prices are positive, labour supplies are fully used, and the profit conditions of competitive equilibrium are met. When an output is produced, it is efficient if, and only if, these conditions can be satisfied for some set of prices.

The proof of (3) is similar to the proof of an analogous result in a model of transportation by Koopmans. The essential fact seems to be, as Professor Koopmans pointed out to me, that each activity connects only two goods, a final good and a primary resource. Thus each primary resource is an independent origin of final goods and determines a possible ratio of substitution between final goods. This circumstance permits price relations and possibilities of substitution to be represented with a linear graph.

To construct the graph we may replace Graham’s table with a table of dots and circles, where dots represent activities at a positive level and circles activities which are unused. Then connect the dots along the rows and along the columns with lines which are regarded as joining only when they meet at a dot. We then have connections between the activities which produce the same good and between activities located in the same country.

Each possible pattern of specialisation will have an associated graph. For example, the graph:

![Graph](image)

corresponds to the specialisation that occurs in the solution to the model (2) when Graham’s demand conditions are introduced. Note that the $x_j$’s of (3), which represent the activity levels, have now been supplied with two subscripts.

Let us assume first that the conditions of (3) are satisfied. Then if we determine a price for one final good in one country, the condition of zero profits for activities in use will fix a price for labour and therefore for all other goods produced by that country. Also the price of the same good in different countries must be the same. Therefore, setting the price of one good will determine that of all other goods which lie on the same connected subgraph. I shall refer to maximal connected subgraphs as components of the graph. If it should be possible to perform a circuit on a component,

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1 Koopmans and Stanley Reiter, "A Model of Transportation," *Activity Analysis*, Chapter XIV, pp. 244-51.
2 A subgraph is a graph formed from some set of activities contained in the original graph. Thus every line segment of the subgraph is contained in the original graph, but not necessarily conversely. A subgraph is connected if there is a line contained in it between any two of its activities.
3 A set defined by stated conditions is maximal if it is not contained in a larger set satisfying those conditions. Thus a connected subgraph regarded as a connected set of activities is maximal if it is not contained in a larger connected subgraph.
that is, to return to the same output from which we departed, without retreating any link, the satisfaction of the conditions of the theorem clearly requires that we should not encounter contradiction by returning to a price different from the one from which we started. I will call these admissible circuits neutral. Thus if the conditions of (3) are met, all circuits, if any, must be neutral and no unused activity can be profitable. The result of selecting prices by this method for the specialisation of (4) is:

Cloth is the numéraire. The numbers beside dots represent the resulting prices of final goods. Numbers beside circles represent the costs of production in the unused activities. With a different specialisation, the prices and costs would, of course, be different. It is clear that the profit conditions of (3) are met in this case. Therefore, if the theorem is true, any output of this specialisation which uses the entire labour supply will be contained in the efficient point set, and we may term the specialisation itself efficient.

Let us now consider the graph from the standpoint of substitutions between final goods. We may speak of a move on a vertical branch of the graph when the production of one good is varied at the initial point and the production of a second good is varied at the end point as a result of shifting labour within the country between the two corresponding activities. Then the substitution ratio is, by the profit conditions, the reciprocal of the price ratio. A move on a horizontal branch will be taken to mean that a change in production of the good at the initial point is made up by an opposite change at the end point in another country. If, after a sequence of vertical and horizontal moves, we return to the original good, the initial variation in production may or may not be precisely offset. If it is precisely offset, I shall call the circuit neutral for substitution. This type of neutrality will turn out to be equivalent to the neutrality of price circuits.

In the Graham model every substitution of different final goods is composed of substitutions of final goods within single countries, and if production is not to fall for any good, every decrease in production of a good in one country must be offset by an increase in production of the same good in another. Thus every good must appear in two variations, which creates a horizontal link, and, to keep the use of labour constant, every variation must lie in a country containing another variation, which creates a vertical link. Therefore, if efficiency is preserved, every attempt to increase the production of one or more goods without reducing that of other goods must involve one

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1. Consider a change in \( x_{1C} \) of the amount \( \Delta x_{1C} \). Efficiency demands that the condition \( x_{1C} + x_{2C} = y_C \) be preserved. Then \( \Delta x_{1C} \) must equal \( -\Delta x_{2C} \). Thus every variation must lie in a vertical move.

2. Consider again the change \( \Delta x_{1C} \). Let \( a_{1C} \) be the product per unit level of the activity. Then the change in production of corn is \( a_{1C} \Delta x_{1C} \). If total production of corn is to be constant, there must be a change in \( x_{1D} \) so that \( a_{1C} \Delta x_{1C} + a_{1D} \Delta x_{1D} = 0 \).
or more circuits, and if there is no circuit which can cause the production of a final good to increase, the output vector is efficient.

Consider diagram (5) once more. Price and cost in the diagram are proportionate in each country to the quantity of labour required to produce a unit of output, and their reciprocals are proportionate to the quantity of output got from a unit of labour. Thus the reciprocals of prices represent quantities which may be substituted without destroying efficiency when they lie on a vertical segment. It is, on the other hand, equal quantities of a good which are substituted along a horizontal segment, so that total production of the good does not suffer. We may consider circuits which contain activities not in use, but then the variations in output must be such that only positive variations are called for in the levels of heretofore unused activities. In the general case, the result \( \Delta y_s \) of traversing a circuit leading from a variation \( v_{sj} \) in the production of the \( s \)th good in the \( j \)th country back to a compensating change in production of the same good in the \( q \)th country, when the levels of activities outside the circuit are constant, is given by:

\[
\Delta y_s = -v_{sj} \frac{\pi_{sj}}{\pi_{mj}} \frac{\pi_{mk}}{\pi_{nk}} \ldots \frac{\pi_{pq}}{\pi_{qj}} + v_{sj}
\]  

(6)

In this formula \( \pi_{sj} \) equals the price \( p_j \) of the \( s \)th good if the \( j \)th country produces it. In any case, \( \pi_{sj} \) equals the cost of producing \( y_s \) in the \( j \)th country. Each fraction \( \frac{\pi_{mk}}{\pi_{nk}} \) is equal to the substitution ratio in the vertical move from a variation of the \( m \)th good, which has the sign of the original variation \( v_{sj} \), to a variation of the \( n \)th good, of opposite sign, in the output of the \( k \)th country. Since the variation of output in an unused activity must be positive, if \( v_{sj} \) is positive, \( \pi \)'s can differ from price only in the numerators, and if \( v_{sj} \) is negative, only in the denominators. But by the profit condition \( \pi_{mk} \) is greater than or equal to \( p_m \). Hence, if any \( \pi \)'s exceed price when the circuit introduces unused activities, \( \Delta y_s \) is negative. On the other hand, if no unused activities are included, all the \( \pi \)'s cancel and \( \Delta y_s \) is zero.\(^1\) This proves that if the profit conditions can be satisfied in \( S \) with positive prices, no circuit can show a gain and hence the specialisation \( S \) is efficient.

To prove the converse, note that if \( y \) is efficient, then one component of \( y \) cannot increase unless another falls. Consequently, every circuit must lead to unchanged production of all goods or must reduce the production of the initial good. Selecting an arbitrary good as \textit{numéraire}, prices may be assigned within a component of the graph by following its subgraph as in (5). Inconsistency can only arise at the completion of a circuit, that is, by (6) if \( \pi_{sj} \neq \pi_{sj} \), since the prices associated with goods at intermediate steps in the circuit are carried over from denominator to numerator at each step. But again by (6) in this case the circuit is non-neutral for substitutions and, therefore, by the appropriate choice of \( v_{sj} \), \( \Delta y_s \) can be made positive and \( y \) is not efficient.

\(^1\) The ratios \( \pi_{sj} \) are equal to the ratios \( a_{sj} \) of elements of \( A_{pm} \) when the unit levels of the activities are defined as using a unit of labour. If the determinant of the sub-matrix of (2) which contains the activities appearing in the circuit is expanded, it will be found that only two terms are non-zero. One term is equal to the product of the numerators of the ratios \( a_{sj} \). The other is equal to the product of the denominators of these ratios. But they have opposite signs. Thus \( \Delta y_s \) is zero if, and only if, this determinant is zero, and, therefore, some activity can be expressed as a linear combination of the other activities in the circuit. This is the general form of the problem in activity analysis.
Suppose, however, that the component is completely traversed without contradiction, but an activity \( A_{ij} \) is unused and profitable, where \( A_{ij} \) represents that activity of the \( j \)th country which can produce the \( i \)th good and where the \( j \)th country and the \( i \)th good are represented, separately, in the component. Then, if we introduce \( A_{ij} \) into the component, a circuit is created, and if we give \( v_{ij} \) a positive value, \( y_3 = \frac{\Delta y_3}{\Delta y_1} \) will be positive according to (6), since \( \pi_{ij} < p_y \) by the assumption that \( A_{ij} \) is profitable. Therefore, \( y \) is not efficient.

If some good is not producible in the specialisation \( S \), it may be assigned any positive price which is less than the lowest cost of production in any country. This is possible, since the lowest cost of production must be positive. Thus we have proved that if \( y \) is efficient, the profit condition can be met in the corresponding specialisation \( S \), with prices which are all positive, when \( S \) has a single component.

Suppose, then, that more than one component is present. The components are maximal connected subgraphs and, therefore, the set of countries in one component is disjoint\(^1\) from the set in other components and the set of goods produced in one component is disjoint from the set of goods produced in other components. Since efficiency is assumed, it must be possible to set price ratios within each component which are determined by the substitution ratios between the goods produced and which satisfy the profit conditions in this component. However, the relative price levels between components are not thereby determined. Let us select a numéraire for each component \( k_i \) and assign it a price \( \lambda_i \). If the conditions of the theorem on efficiency are to be met, it must be possible to choose the \( \lambda_i \) so that all unused activities remain non-profitable.

To simplify the exposition, I shall consider a graph with two components in the pattern:

\[\text{(7)}\]

We may fix \( \lambda_1 \) equal to 1. Suppose that it is not possible to find a value of \( \lambda_2 \) greater than zero such that all unused activities remain profitable. Select the maximum value of \( \lambda_2 \) that leaves the unused activities of \( A \) and \( B \) profitable. There must then be an unused activity in \( A \) or \( B \), say the production of 4 in \( B \), which would earn zero profits, if used, and an unused activity in \( C \) or \( D \), say the production of 2 in \( D \), which remains profitable, if used. These two activities may be introduced into the graph to create

\(^1\) Two sets are disjoint if they contain no common elements.
the circuit which is completed in the figure with dashed lines. According to (6), the effect of a positive variation \( v_{2D} \) in the circuit is:

\[
\Delta y_2 = -v_{2D} \frac{\pi_{5D}}{p_5} \cdot \frac{\pi_{1D}}{p_5} + v_{2D}
\]  

(8)

Since \( \pi_{SB} = \frac{p_5}{p_2} \) and \( \pi_{2D} < \frac{p_5}{p_2} \), according to the assumptions made, \( \Delta y_2 \) is positive and no output of the specialisation \( S = K_1 + K_2 \) is efficient.

It should be clear that this argument could be extended to any number of components without difficulty. Therefore, we have proved that if \( y \) is an efficient output, the profit conditions can be met in the specialisation which produces \( y \). Thus, "\( S \) is efficient" is equivalent to the possibility of satisfying the profit conditions, with positive prices, and if \( S \) is efficient, \( y \) is an efficient output of \( S \) if, and only if, the labour supplies are fully used. This last condition can be slightly relaxed if some countries cannot produce some goods. However, the exception has little interest.

EFFICIENT FACETS

Since every good on a component of the graph is directly or indirectly connected to every other good by vertical links along which substitutions can be made, we see that any good on a given component can be substituted for any other, provided the amount of the substitution is small enough not to require negative production anywhere. Moreover, since the substitutions may be added (that is, may be imposed successively) if \( r \) goods are present in the component, the quantities produced of \( r - 1 \) of the goods may be altered in any proportions so long as the quantity produced of the \( r \)th good is properly adjusted, and the variation is sufficiently small. Thus the set of outputs which may be efficiently produced when \( r \) goods are present in a connected graph is \( r - 1 \) dimensional. The theory of activity analysis assures us that when the prices are unique up to a common factor, their ratios are the reciprocals of substitution ratios. In the case of a connected graph, however, this is directly apparent from the profit conditions. Indeed, the prices were assigned by use of the substitution ratios. It is also clear that any efficient output must lie in the boundary of the set of all possible outputs since otherwise all components of the output could be increased simultaneously.

Substitution between goods produced on different components of the graph is not possible, unless the specialisation pattern is changed, since no country has activities in both components. Therefore, the dimension of the set of efficient outputs for a specialisation \( S \) is equal to the number \( r \) of goods in \( S \) diminished by the number \( t \) of components. I shall refer to the set of outputs consistent with a specialisation as a facet of the efficient point set. In the extreme case where there is but one good in each component, the dimension \( r - t \) of the corresponding set of efficient points is zero, that is, the facet is made up of one efficient point.

It is possible to represent the set of efficient outputs schematically for the model given by (2). This is conveniently done in Fig. 1 by the subdivision of a triangle whose vertices represent points in which each country is producing the same good. Moving from one vertex to another along a side of the triangle, one country after another is shifting over from the output of the first vertex to that of the second. The other points at which each country is producing only one good may be located symmetrically within the triangle, so that the nearer the point to a vertex the more countries are producing the corresponding good. These points all represent zero-dimensional facets. In our example, there are fifteen such facets. This is the number of ways, consistent with efficiency, in which each of four countries can specialise completely in one of three goods when the rates of substitution differ between countries, so that the order in which countries enter the production of a good is determined. If another good were
added, the triangle would have to be replaced by a pyramid, and, in general, the presence of an additional good increases the dimension of the figure by one.

When rates of substitution differ between countries, there is an easy inductive formula for the number of zero-dimensional facets. It is derived from the fact that, when another good is introduced, the number of complete specialisations equals the sum of the number when no country produces the new good, the number when one does, on up to the number with all countries producing the new good. But the number of specialisations in the other goods when an assigned number of countries produce

![Diagram](image_url)

**Fig. 1.**

The country indicated beside a one-dimensional facet is changing over in that facet from the output of one vertex to that of another. In general, within a facet a country produces every good that it produces at some vertex of that facet. Equal distances do not represent equal quantities in different facets. However, within a facet distances parallel to a side may be interpreted in terms of quantities of the goods being substituted along that side. Then each point of the diagram corresponds to a definite efficient output.

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1 Failure of this condition does not vitiate our methods. See the footnote at the end of the next paragraph.

2 That is, specialisations in which each country produces one good.
the new good is a problem in specialisation with the smaller number of goods, and by
induction is assumed to be known. Thus the formula may be written:

\[ C(r, n) = \sum_{i=0}^{n} C(r - 1, i) \] (9)

where \( C(r, n) \) is the number of complete specialisations when there are \( r \) goods and \( n \)
countries. The induction begins with \( C(2, n) = n + 1 \). Then \( C(3, n) = \sum_{i=0}^{n} C(2, n) \)

\[ = \sum_{i=0}^{n} (i + 1) = \frac{(n + 1)^2 + n + 1}{2}, \] which is 15 if \( n = 4 \).\(^1\)

The one-dimensional facets are represented by line segments drawn between those pairs of zero-dimensional facets for which only one country specialises differently. Each facet of given dimension \( s \) is a polygon whose vertices are a maximal set of zero-dimensional facets, between each pair of which \( s \) countries or less specialise differently. Moreover, the points of every such maximal set form the vertices of an \( s \)-dimensional facet. Within the facet each country produces every good which it produces at some vertex of the facet. In our example, the two-dimensional facets are either quadrangles or triangles, and it is clear that no other possibilities exist for two-dimensional facets. This may also be seen from the graphic representation of the facets. In a two-dimensional facet substitutions may occur between three goods and thus there must be two countries each producing two of the goods, or one producing three. In the former case, the facet is a quadrangle; in the latter case, a triangle. If a neutral circuit exists, however, certain adjacent facets will present the same substitution ratios and will, by the definition usual in activity analysis, form a single facet.\(^2\)

It is also possible to derive a simple relation for the number of activities in a graph and thus in the corresponding facet. Suppose \( r \) goods are produced and \( n \) countries are present. If there is only one component and no circuits, the number of activities in use must be \( r + n - 1 \). There must be \( r \) activities to ensure that each of the \( r \) goods are produced. These activities leave the countries entirely unconnected. Then \( n - 1 \) additional activities are needed to connect them. If the number of components is larger than one, say equal to \( k \), one activity must drop out to slough off each additional component. Therefore, with no circuits present, the number of activities is \( r + n - k \) and the dimension of the facet is \( r - k \). If \( c \) independent

\(^1\) It is easily shown that \( C(r, n) \) is the binomial coefficient \( \binom{n + r - 1}{n} \), which gives
the number of distinguishable arrangements of \( n \) indistinguishable objects in \( r \) cells. Of course, the
countries are distinguishable, but when a certain number of countries are assigned to each good, the
assignment of individual countries is unique because of comparative advantage. See Wm. Feller, Prob-
ability Theory, New York, 1950, p. 52.

\(^2\) Our definition of facet is easily modified to conform with the standard definition. The facets of
activity analysis correspond to maximal efficient specialisations (or maximal efficient graphs). This means
that any activity which retains zero profitability for all sets of prices at which the specialisation satisfies (3)
must be used. If no circuits ever appear, the definitions are equivalent.

If the substitution ratios between two goods are the same in two countries, the zero-dimensional facets
may initially be determined by slightly altering the ratios to make them differ. Then recognising that the
ratios are actually the same will eliminate the zero-dimensional facet in which each country produces a
different one of the two goods, and some adjacent facets will merge. The final result is independent of the
particular perturbation of the ratios in the intermediate step, but it should be small enough to leave other
comparative advantages unchanged.
circuits\textsuperscript{2} are then introduced, the number of activities must rise by $c$, since one activity is needed to generate a new circuit in an existing component, and more than one must either alter the number of components or lead to another new circuit.\textsuperscript{3} Therefore, the number of activities will be $r + n - k + c$.

In the facet whose linear graph was exhibited in (4), $C$ produces cloth and corn, $D$ produces linen and corn, $A$ produces corn, and $B$ produces cloth. Thus six activities are used. In this example, $r = 3$, $n = 4$, $k = 1$, $c = 0$, and $r + n - k + c = 6$, as it should. The facet is number 8 in Fig. 1, and since there is only one component, the facet is of maximal dimension and the prices must be in fixed proportions. A reference to the substitutions made on the sides of this facet, which is a quadrangle, show it to be comparatively large. Indeed, within the facet, the change in production in terms of the maximum potential world production of each good is about one-half for corn and linen and one-third for cloth, despite the presence of nine other two-dimensional facets.

THE EQUILIBRIUM POSITION

Graham makes three remarks on the equilibrium position of his model which I will examine briefly. He argues that "limbo" prices corresponding to facets of dimension less than $r - 1$ are improbable. He also argues that even large changes in demand are not likely to affect price ratios very much. Finally, he conjectures that with his demand functions the equilibrium of supply and demand is unique.

The demand assumption which Graham uses is that the same proportion of each country's expenditure is devoted to each product regardless of price and that these proportions are equal between countries.\textsuperscript{8} Thus redistributions of income between countries will not matter. It is assumed that all income is earned in the production of the goods included in the model. In other words, the proportion of world income devoted to each product is constant.

I have shown elsewhere that Graham's model always has a solution and, indeed, that this solution is unique.\textsuperscript{4}

Let us consider the probability that large changes in demand will leave relative prices unaffected. This will depend on the size of the facet within which the original output vector lies. The sizes of the facets with Graham's assumptions of constant costs are determined essentially by the sizes of the countries, and there is little else to be said. If the original point of equilibrium lies in a facet like 8 in Fig. 1, large changes may occur in demand without leaving this facet and consequently without changing relative prices in the competitive equilibrium. On the other hand, if the original equilibrium were in facet 4, a much smaller change would suffice to remove the equilibrium to another facet and different relative prices. Graham's own initial solution for the example we have used did lie in facet 8, as we have noted.

Finally, Graham regarded "limbo" prices as very improbable. That is, he thought it quite improbable that an equilibrium point would be found in a facet of dimension

\textsuperscript{1} That is, no one of the $c$ circuits can be expressed as a sum of the other $c - 1$ circuits. In adding two circuits, give each an orientation. Then branches which appear twice with opposite orientations cancel. Different orientations yield different sums. See D. König, Theorie der endlichen und unendlichen Graphen, Leipzig, 1936.

\textsuperscript{2} It can be shown that the rank of the matrix of input-output coefficients for the activities of a specialisation equals the number of activities reduced by the number of independent circuits, i.e., $r + n - k$.

\textsuperscript{3} The precise assumption made for the model (4) is that each country spends one-third of its income on each good. Then the solution is in facet 8 and total outputs are approximately 111,302 units of cloth, 239,664 units of linen and 379,714 units of corn. The solution must be found by trial and error.

\textsuperscript{4} In a paper read before the Econometric Society in December, 1952, and published in Econometrica, April, 1954.
less than the maximum, \( r - 1 \), and in a facet of maximum dimension relative prices are precisely fixed. However, this presumption does not appear to be justified. The dimension of the set of relative prices associated with a facet is \( r - f - 1 \), where \( f \) is the dimension of the facet, for price ratios are fixed just between those outputs which can be substituted in the facet. But then the dimension of the set of outputs, demanded at some set of prices associated with the facet, is also \( r - f - 1 \). The equilibrium lies in the facet if the facet contains an output with goods in the proportions demanded. There seem no a priori grounds for preferring the probability that an \( r - 1 \) dimensional facet will contain an output with goods in given proportions over the probability that the unique output of a zero-dimensional facet will have goods in the same proportions as some output in the \( r - 1 \) dimensional set associated with the facet through the demand functions.

**EFFECT OF TRADE IN INTERMEDIATE PRODUCTS**

If intermediate products are traded, the analysis of efficiency in terms of substitution circuits is no longer adequate. The difficulty is that changes in world output can no longer be decomposed into elementary substitutions within single countries, for even substitutions between the products of a single country will typically involve changes in the supplies of intermediate products from other countries. As a consequence, the effectiveness of one country in the production of a certain good will often depend on the ability of other countries to supply some of the intermediate products needed in its production. The patterns of efficient specialisation may have little relation to those which would appear with integrated activities.

In the competitive model the effect of trade in intermediate products will be reflected in their prices. Clearly, some activities may now be profitable which would have been definitely excluded earlier because of the great cost of producing a necessary intermediate product within the country.

The introduction of trade in intermediate products can be expected, in general, to push the production frontier outwards throughout its entire extent. The effect is most simply illustrated for the case of two countries and two goods where each good enters into the production of the other. In Fig. 2 the outer frontier combines the activities of the two countries in a completely general way, so that country \( A \) produces no \( y_1 \) until country \( B \) has become entirely committed to \( y_2 \). With the inner frontier, however, each country produces its own intermediate products. In other words, it is as though integrated production functions were used.

The appearance of new directions of efficient specialisation may be illustrated in a simple arithmetic model. Suppose the cloth is made of linen, but the linen also has direct uses, and the activities are:

\[
\begin{array}{ccc}
A & B & C \\
\text{l.} & \text{cl.} & \text{w.} & \text{l.} & \text{cl.} & \text{w.} & \text{l.} & \text{cl.} & \text{w.} \\
L_A & -1 & -1 & -1 & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
L_B & \text{ } & -1 & -1 & -1 & \text{ } & \text{ } & \text{ } & \text{ } \\
L_C & \text{ } & \text{ } & \text{ } & \text{ } & -1 & -1 & -1 & \text{ } \\
\end{array}
\]

\[\text{Linen} \quad 2 -10 \quad 5 -10 \quad 10 -10 \]

\[\text{Cloth} \quad \text{10} \quad \text{10} \quad \text{10} \]

\[\text{Wheat} \quad \text{7} \quad \text{10} \quad \text{10} \quad \text{10} \]
Then the table of comparative costs on the basis of integrated activities will be:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cloth</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Linen</td>
<td>12</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Wheat</td>
<td>42</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

Fig. 2.

$B_1$ is the input-output vector (exclusive of labour) for the first activity in $B$ when all the labour supply of $B$ is devoted to it, and correspondingly for the other activities. Integrated activities are indicated with a bar. In deriving the set of possible outputs in the free case, draw the vectors representing the activities of $A$ with the point $T$ as origin and slide $A_1$ along the line $TP$. In the integrated case draw the vectors representing the integrated activities of $A$ with $S$ as origin and slide $A_2$ along $SR$.

It is clear from (11) that if one country specialised to each product, and intermediate products are not traded, the specialisation will have to be $A$ in wheat, $B$ in cloth,
and C in linen. However, it is not difficult to show that this specialisation is never efficient when trade in intermediate products occurs. A cannot profitably produce wheat unless \( 7p_w \geq 10p_{cl} - 10p_l \) or \( 10p_{cl} \leq 10p_l + 7p_w \). But B cannot profitably produce cloth unless \( 10p_{cl} \geq 10p_l + 10p_w > 10p_l + 7p_w \). Thus the profit conditions cannot be satisfied in this specialisation and the specialisation must be inefficient by the general theorem on efficiency in finite linear models.

On the other hand, it is efficient to have A produce only cloth, B only wheat, and C only linen. Any price vector for which \( 10p_l > 10p_w \) and \( 10p_l + 10p_w > 10p_{cl} \) makes the production of cloth and wheat unprofitable in C. Then if \( 10p_w > 5p_l \) and \( 10p_w > 10p_{cl} - 10p_l \), B cannot profitably produce cloth or linen. Finally, if \( 10p_{cl} > 12p_l \) and \( 10p_{cl} > 7p_w + 10p_l \), A cannot profitably produce either linen or wheat. These inequalities can be satisfied, for example, by the prices, \( p_l = 12 \), \( p_w = 10 \), \( p_{cl} = 20 \). Consequently, by setting the price of labour in each country just high enough to give zero profits in the one activity which is used, the profit condition is met, and the specialisation is proved to be efficient.

There is nothing shocking to common sense in these results. A moment's consideration will convince one that Lancashire would be unlikely to produce cotton cloth if the cotton had to be grown in England. On the other hand, the production of raw materials, especially the mining of metallic ores, might not be profitable in backward areas without machinery supplied by the industrial countries. The patterns of interdependence are already very complex, but with free trade they would undoubtedly be far more complex than they now are. The low cost of ocean transport would reduce the advantage of internal supply in continental countries.

In discussing the models with trade in intermediate goods, we have implicitly assumed that each country always used the same productive process to produce a given good. However, there is no longer the same justification for this assumption. If the coefficients of production are variable, they will now depend on the prices of intermediate products which will vary with final demand. In other words, what activity it is efficient for a country to use in producing a good will depend on what outputs are being produced in other countries. We have lost the use of Samuelson's theorem, since there are several ultimate factors in the whole interdependent system of world production. Or, looked at differently, imports of intermediate products are like supplies of ultimate factors to a country. Thus, as soon as trade in intermediate products is allowed, the problem loses its special simplicity, and we may as well allow joint production and many factors in each country. The problem of efficient specialisation then assumes the proportions of general activity analysis.

**Efficiency in World Production**

It has long been a claim of orthodox economists that, if we assume pure competition and the absence of external economies, free trade will lead to efficiency in world production. The notion of efficiency intended in this claim is no doubt broader than the definition we have used here, but it would surely include efficiency in our sense. Free trade, through promoting proper specialisations in production, would make possible larger outputs of whatever desired composition. However, the methods of

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1. This may be seen by putting all countries initially in cloth. Then if wheat is to be produced, the largest returns are got only if A switches over, and if linen is desired C provides larger returns for the sacrifice of cloth than does B.

2. If the profit conditions cannot be met, even with negative prices, there is no normal at the output and the output is therefore interior to the cone of possible outputs. Then there is another point interior to the cone using no more resources and providing larger supplies of all outputs.

3. This claim is, at least, implicit in classical discussions of comparative advantage and gains from trade.
the classical economists are entirely appropriate only to the analysis of trade between two countries. Of course, one of these countries may be taken to be "the rest of the world," but this is largely a subterfuge, for we are still left with a problem of specialisation within "the rest of the world."

The deficiency of the classical methods, the method of comparative labour costs in early days, the method of alternative costs as displayed in production possibility curves in later times, is their dependence on bilateral comparison. They are admirably adapted to demonstrating that there are gains to be made from trade when countries begin with different rates of transformation between outputs. But it is not possible through merely bilateral comparison to develop a complete theory of efficient multilateral specialisation.

Graham detected the error of classical ways and introduced a genuinely multilateral method into the study of international trade. We have seen that the efficiency of free trade in his model can be proved through an analysis of the role of substitution circuits. Indeed, the basic theorem on efficiency in linear models can be demonstrated for his model with elementary techniques.

However, we have found that this simplicity is bought at the expense of prohibiting all trade in intermediate products (with a slight exception), which is indeed a heavy price. When the further degree of interdependence which results from trade in intermediate products is recognised, there is no recourse but to use the general methods of activity analysis.

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