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On the Interpretation of Professor Leontief's System[1]

The widespread interest and development of input-output analysis introduced by Leontief and his followers raises basic questions of economic interpretation. Are the elements of input-output tables structural parameters of an economic system or are they merely ratios of two economic variables ? If input-output coefficients are truly structural parameters, can they be identified with well-defined *technological* parameters or are they mixtures of several types of parameters including some that are not purely technological ?

The standard assumptions of input-output analysis are that the coefficients of a *tableau économique* comprise a set of technological parameters in linear production functions with fixed proportions among the factors of production. In deriving a rationale for interpreting Leontief's system, the unrealistic assumption is made that each sector of the economy produces only a single type of output. Samuelson,[2] using this assumption and the additional assumption that the system has only one primary factor of production, has advanced our understanding of Leontief's system by showing that any set of input-output coefficients can be identified with two different underlying technological structures ; one the system of linear production functions with fixed proportions among factors, and the other a system of production functions homogeneous of the first degree in all the input variables. Particular interest attaches to the latter structure because, in contrast to the former, it permits substitution among factors of production.

The same assumptions have been used by Cameron[3] to establish a comparison between the systems of Leontief and classical general equilibrium. In these studies of the interpretation of Leontief's system it is assumed that conditions of market competition exist. The present paper is concerned with a more realistic interpretation of Leontief's system. In contrast with the other studies we shall try to adopt a theoretical model which corresponds to the practices of input-output analysis and which drops the assumption that each sector produces only a single output. We shall allow joint production in each producing sector. This is not to be regarded as a mere refinement since joint production is the rule and not the exception ; moreover, it has a strong influence on the results we shall derive. An examination will also be made of the effects of dropping the assumption of market competition.

Let us classify our results into two groups.

(1) Joint production and market competition : We are able to give a technological interpretation to input-output coefficients as structural parameters and extend Samuelson's findings on the possibility of substitution beyond the case in which each sector produces only one output.

(2) Joint production and market imperfection : *In general,* input-output coefficients cannot be interpreted as purely technological parameters ; they depend on the parameters of consumer demand and factor supply functions.

[1] This paper will be reprinted as Cowles Commission Paper, New Series No. 69. The research was carried out as part of the programme of the Research Seminar in Quantitative Economics, University of Michigan Helpful suggestions have been made by H S Houthakker, T C Koopmans, J. Marschak, H Rubin, P. A. Samuelson and D. B. Suits

[2] P. A. Samuelson, " Abstract of a Theorem Concerning Substitutability in Open Leontief Models," Chapter VII, *Activity Analysis of Production and Allocation,* edited by T. C. Koopmans, Wiley, 1951.

[3] Burgess Cameron, " The Construction of the Leontief System," THE REVIEW OF ECONOMIC STUDIES, Vol. XIX (1), 1950–51, pp. 19–27

Let us now consider a formal development of these propositions. In the customary exposition of the Leontief system[1] we have :

x_i = output of the i-th sector,

x_{ik} = output of the k-th sector used in the i-th sector.

The technical coefficients are defined as :

$$a_{ik} = \frac{x_{ik}}{x_i}. \tag{1}$$

The a_{ik} are the elements, assumed constant, of the input-output table.

One interpretation of (1) is that it defines a technological production function, with the condition that inputs are used in fixed proportions.

The actual expression for the production function would be :

$$x_i = \min\left(\frac{x_{i1}}{a_{i1}}, \frac{x_{i2}}{a_{i2}}, \ldots, \frac{x_{in}}{a_{in}}\right) \tag{2}$$

with (1) holding for all k as long as no inputs are free goods, thus insuring the equalities :

$$\frac{x_{i1}}{a_{i1}} = \frac{x_{i2}}{a_{i2}} = \ldots = \frac{x_{in}}{a_{in}}. \tag{3}$$

In order to measure input-output coefficients directly as in (1), it is necessary to assume that each sector produces only one type of output. However, even if input-output tables are refined to 1,000 × 1,000 classifications, the problem of joint production cannot be avoided. It is simply true, in most cases at least, that the a_{ik} are computed, not as the ratio of two physical quantities, but as the ratios of two values—the value of output of the k-th sector used in the i-th sector divided by the value of the output of the i-th sector. Joint production is the rule and not a special case.

We shall now show that even in the case of joint production in a competitive economy, there may exist a production function permitting substitution among inputs and yielding a set of constant elements of an input-output table. The approach used here and the results obtained are not strictly parallel to those of Samuelson in the one-commodity-per-industry case, but they do have a point in common, namely, to show that substitutability as an alternative to fixed proportions is consistent with Leontief's empirical findings and theoretical model.

Let $x_i^{(s)}$ = s-th output of sector i,

$x_{ik}^{(r)}$ = r-th output of sector k used in i.

The elements of an input-output table are defined by :

$$a_{ik} = \frac{\sum_r p_k^{(r)} x_{ik}^{(r)}}{\sum_s p_i^{(s)} x_i^{(s)}}. \tag{4}$$

The problem is to determine the characteristics of a technological relation, independent of market phenomena like prices :

$$F_i\,(\overset{(1)}{x_i}, \ldots, \overset{(S)}{x_i}, \overset{(1)}{x_{i1}}, \ldots, \overset{(R_1)}{x_{i1}}, \ldots, \overset{(1)}{x_{in}}, \ldots, \overset{(R_n)}{x_{in}}) = 0,\ i = 1, 2, \ldots, n, \tag{5}$$

[1] See, e.g. W. Leontief, *Structure of the American Economy*, Cambridge, Mass., 1941.

such that the quantities (4) are constants. In order to solve this problem some assumptions must be made to connect the pricing system to the technology. In a competitive market :

$$\frac{\partial F_i}{\partial x_i^{(s)}} = - \lambda_i \, p_i^{(s)}, \tag{6}$$

$$\frac{\partial F_i}{\partial x_{ik}^{(r)}} = \lambda_i \, p_k^{(r)} \, ; \tag{7}$$

hence the ratio in (4) can be written as :

$$\frac{\sum_r \dfrac{\partial F_i}{\partial x_{ik}^{(r)}} x_{ik}^{(r)}}{- \sum_s \dfrac{\partial F_i}{\partial x_i^{(s)}} x_i^{(s)}} = a_{ik} = \text{const.} \; ; \; k = 1, 2, \ldots, i-1, i+1, \ldots, n \; \ldots . \tag{8}$$

Equations (8) provide a set of partial differential equations of which the technological function in (5) is a solution.

H. Rubin suggests the transformation :

$$y_i^{(s)} = [x_i^{(s)}]^{\frac{1}{a_{ik}}} \tag{9}$$

in order to put (8) in the form of Euler's equations for homogeneous functions.

$$\sum_r \frac{\partial F_i}{\partial x_{ik}^{(r)}} x_{ik}^{(r)} = - \sum_s \frac{\partial F_i}{\partial y_i^{(s)}} y_i^{(s)} \; ; \; k = 1, 2, \ldots, i-1, i+1, \ldots, n \; \ldots \tag{10}$$

Since Euler's equation is a necessary and sufficient condition for the homogeneity of the function in question, whether for all or a subset of variables in the argument, we see that F_i must be homogeneous of degree zero in each of the subsets :

$$\overset{(1)}{x_{i1}}, \ldots, \overset{(R_1)}{x_{i1}}, \left[\overset{(1)}{x_i}\right]^{\frac{1}{a_{i1}}}, \ldots, \left[\overset{(S)}{x_i}\right]^{\frac{1}{a_{i1}}}$$

$$\cdots$$

$$\overset{(1)}{x_{in}}, \ldots, \overset{(R_n)}{x_{in}}, \left[\overset{(1)}{x_i}\right]^{\frac{1}{a_{in}}}, \ldots, \left[\overset{(S)}{x_i}\right]^{\frac{1}{a_{in}}}.$$

An obvious example of a production function with these properties is :

$$\sum_s \overset{(s)}{\beta_i} \left[\overset{(s)}{x_i}\right]^{\beta} - A \prod_k \prod_r \left[\overset{(r)}{x_{ik}}\right]^{\overset{(r)}{a_{ik}}} = 0, \tag{11}$$

where the parameters of (11) must be subject to :

$$\frac{\sum_r \overset{(r)}{a_{ik}}}{\beta} = a_{ik} \tag{12}$$

in order that (8) be satisfied by (11). Other functions satisfying the system of partial differential equations can easily be constructed.

In studying a model of a competitive economy we have not assumed zero profits. Zero profits are not synonymous with the assumptions of competition in (6)–(7). It is sometimes argued that the existence of positive profits will lead to entry of new firms into a sector and indefinite expansion of production by all firms until profits vanish. A reverse process is assumed to take place when negative profits exist. The model of this paper implicitly rules out such arguments. By a competitive market, we mean simply one in which entrepreneurial decisions do not influence market prices, that entrepreneurs adjust their operations, optimally, to prevailing market prices viewed as parameters by them. Production will not necessarily be expanded indefinitely as a result of the existence of positive profits because entrepreneurs cannot vary fixed capital at will in a short period. The Leontief input-output tables we are considering in this paper do not have fixed-capital coefficients. The capital structure is taken as given and factors of production, in any particular sector, are purely current flows of output from other (inputting) sectors. We follow Leontief in assuming non-zero profits in each sector,[1] and do not view this as denying the applicability of equations (6)–(7) to our model.

In general the elements of Leontief's input-output table may thus be interpreted as parameters of a class of production functions, all of which permit substitution among factors of production and types of output. Moreover, the a_{ik} can be interpreted as technological parameters.

This interpretation is applicable only if equations (4), (6) and (7) hold. In so far as the a_{ik} are determined purely from engineering information *without any requirement that different products be weighted by relative prices*, equations (4) need not hold, and the above results about substitutability do not apply. In earlier publications of Leontief, equations of the type in (4) were used to determine the a_{ik}. Even engineering information, however, must make assumptions about the proportions in which different types of product or input are combined. In many or most cases, these proportions are current relative prices or time averages of them. For example, in studying railroad traffic in the United States, one might use :

$$\text{Traffic unit} = \text{Freight ton miles} + 2.4 \text{ passenger miles}$$

as a composite output variable since it is not possible to make a complete separation of operations into those dealing exclusively with freight and exclusively with passenger service. The coefficient 2.4 is not a technological parameter ; it is the mean price ratio of the two types of service averaged over many years.

The production function (5) and associated marginal productivity equations (6)–(7) are written as though there is only one entrepreneur in each sector. This would be strictly true only if a sector were a firm instead of a competitive industry. We shall now state, without proof, a result applicable to a model in which each sector contains several entrepreneurs. At the same time we retain our earlier assumptions of joint production under market competition.

Trivial results follow if we require that each entrepreneur within a sector has an identical production function or that the value of output be distributed in a known way among firms in the same sector. A less restrictive approach is to assume that the production function of each entrepreneur depends only on variables directly produced or used by his firm with no exchange of products among firms in the same sector. Under these conditions we can show the existence of a set of input-output coefficients as constants and parameters of production functions, the sum of which satisfy homogeneity conditions analogous to those derived above for individual functions. The homogeneity conditions can be extended to the individual production functions of each entrepreneur provided each function vanishes for zero values of all arguments. Essentially, our main conclusions of this paper are not dependent on the fact that we have

[1] W. Leontief, " Wages, Profit and Prices," *The Quarterly Journal of Economics*, Vol. LXI, November, 1946, pp. 26–39

simplified our models by treating them as though there were only one entrepreneur in each sector.

Returning to the model (4)–(7), we might consider the consequences of dropping the assumption of competitive markets. In this event, equations (6) and (7) do not hold. Corresponding equations for a non-competitive market can be written as :

$$\frac{\partial F_i}{\partial x_i^{(s)}} = -\ \lambda_i\ p_i^{(s)}\ (1 + \eta_i^{(s)}), \tag{13}$$

$$\frac{\partial F_i}{\partial x_{ik}^{(r)}} = \lambda_i\ p_k^{(r)}\ (1 + \epsilon_{ik}^{(r)}), \tag{14}$$

in which $\eta_i^{(s)}$ and $\epsilon_{ik}^{(r)}$ are entrepreneurs' estimates of parameters of demand and supply functions respectively.

In addition to equations (4), (13) and (14) we have in the system :

$$\overset{(s)}{p_i} = \overset{(s)}{g_i}\left(\overset{(1)}{x_i}, \ldots, \overset{(S)}{x_i}\right), \tag{15}$$

$$\overset{(r)}{p_k} = \overset{(r)}{h_{ik}}\left(\overset{(1)}{x_{i1}}, \ldots, \overset{(R_n)}{x_{in}}\right), \tag{16}$$

the demand and supply functions, respectively, facing the i-th sector. The parameters used in (13) and (14) are :

$$\eta_i^{(s)} = \sum_u \frac{x_i^{(u)}}{p_i^{(s)}} \frac{\partial p_i^{(u)}}{\partial x_i^{(s)}},$$

$$\epsilon_{ik}^{(r)} = \sum_t \sum_v \frac{x_{it}^{(v)}}{p_k^{(r)}} \frac{\partial p_t^{(v)}}{\partial x_{ik}^{(r)}}.$$

In order to give a purely technological interpretation to the so-called technical coefficients, a_{ik}, we should be able to show that they depend exclusively on parameters of the production functions, F_i. This was the case in the corresponding competitive model as shown in equations (12) for a particular production function. Except under special circumstances, the a_{ik} will also depend on the parameters of $g_i^{(s)}$ and $h_{ik}^{(r)}$; therefore, they cannot strictly be regarded as technical coefficients.

This can be shown by reference to a particular solution of the present model. Let the production function be given by[1] :

$$\prod_s \left[\overset{(s)}{x_i}\right]^{\beta_i^{(s)}} - A \prod_k \prod_s \left[\overset{(r)}{x_{ik}}\right]^{a_{ik}^{(r)}} = 0, \tag{17}$$

and let the demand and supply equations be also of the constant elasticity type.

$$\overset{(s)}{p_i} = B\left[\overset{(s)}{x_i}\right]^{\eta_i^{(s)}} \tag{18}$$

$$\overset{(r)}{p_k} = C\left[\overset{(r)}{x_{ik}}\right]^{\epsilon_{ik}^{(r)}} \tag{19}$$

[1] This production function is also apparently a particular solution of the earlier system of differential equations (8), yet it does not satisfy the second order conditions for profit maximisation under competition. The same difficulty does not occur under conditions of imperfect markets.

The technical coefficients must then satisfy :

$$\frac{\sum_r \frac{1}{1 + \epsilon_{ik}^{(r)}} a_{ik}^{(r)}}{\sum_s \frac{1}{1 + \eta_i^{(s)}} \beta_i^{(s)}} = a_{ik} = \text{const.} \qquad (20)$$

The coefficients are thus defined in terms of constants in the production, demand and supply functions. Non-technological behavioral changes in the economy ; changes in tastes, for example ; would cause the a_{ik} to change.

Leontief's model is estimated from data that are the complex result of many more *economic* interrelationships than are expressed by the *technical* relationships usually assumed to underlie input-output tables. Although a Walrasian type system is often cited in theoretical justification for input-output analysis, it is seldom recognised that numerous complex economic processes in addition to technological relations are involved in the interpretation of the a_{ik}. Perhaps the most realistic interpretation of Leontief's system is that of the imperfectly competitive economy in which the coefficients of input-output tables are not easily identified with any single set of basic structural parameters.

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