ALTERNATIVE CONDITIONS FOR SOCIAL ORDERINGS

By Clifford Hildreth

A set of conditions for social orderings is developed. It is shown that, under stated assumptions, orderings derived by any of a class of methods based on von Neumann-Morgenstern utilities will satisfy these conditions. It is noted that the present conditions imply all but one of the conditions previously imposed by Kenneth Arrow. Objections to this condition are stated. It is argued that any procedure that extends the partial ordering of the new welfare economics must involve interpersonal comparisons of preferences.

I. INTRODUCTION

In a recent monograph, Arrow [1] has considered methods for obtaining group preferences among alternative situations given the preferences of the individual members of the group. After developing a set of conditions to be imposed on such a method, he obtained the result that no method could be found to satisfy these conditions and considered possible modifications of the conditions.

In Section II of the present paper, an alternative set of conditions for group preferences is presented and it is shown that many methods for obtaining group preferences satisfy these conditions. Many of the concepts and symbols used are taken from Arrow’s study. Section III contains some discussion of the difference between the conditions of Section II and those suggested by Arrow. In the last section, a proof needed in the development of Section II is given.

The remainder of this section contains some general remarks on the relation of the present discussion to other work in welfare economics. The interpretations presented are not regarded as original contributions or as final dicta but do summarize the author’s current views and are intended to help explain the motivation for the material which follows.

Welfare economics may be regarded as the study of the implications of ethical propositions for the choice of economic policies. In what is called “old” welfare economics the implications of the precept that total utility should be maximized were studied. Much of the “new” welfare economics consists of seeking implications of the proposition that one situation is to be regarded as better than an alternative if in the former situation someone is more satisfied and no one is less satis-

1 The author is indebted to I. N. Herstein for mathematical advice and to Leonid Hurwicz, Gerard Debreu, and Milton Friedman for criticism of an earlier paper. This article will be reprinted as Cowles Commission Paper, New Series, No. 68.

2 In fact, these introductory remarks are mainly a restatement of some previous observations of Samuelson [9].
fied than in the alternative. Let us call a set of ethical propositions together with its economic implications a system of welfare economics. It is clear that the economist's interest in any system of welfare economics will be strongly influenced by (1) the acceptability of the ethical propositions used, (2) the extent to which the propositions can be related to observable or potentially observable phenomena, and (3) the range of policy questions to which the system gives answers.

Criticisms of proposed systems of welfare economics reflect these desiderata. The principal criticisms of old welfare economics were either that utility was nonmeasurable (nonobservable) or that, if measurable, the measure was arbitrary within wide limits and could therefore not have the ethical significance attributed to it. The new welfare economics has been primarily criticized for the narrowness of the range of questions to which it provides answers. In general, the range of application of a system can be extended by adding to the set of ethical propositions on which it is based. However, this will typically involve more controversial propositions. Construction of a useful system of welfare economics thus involves a rather delicate compromise between the desire to keep the underlying ethical judgments generally acceptable and the desire to provide criteria for a wide range of policy choices.

From the point of view outlined above, the contribution of Bergson [2] is not a new system of welfare economics but an analytical and expository device for ascertaining and explaining the implications of various sets of ethical propositions. As an indicator of social preferences he postulated a social welfare function—a function of the amounts of commodities used and contributed by various individuals in society. He did not suggest a specific form for the function but showed how various welfare propositions could be interpreted as restrictions on the form of the function and their implications studied in this fashion. While Arrow's approach is fundamentally similar to that of Bergson, there are several differences [1, pp. 22, 23]. Both consider methods for ordering social situations. Arrow makes the social ordering dependent on individual preferences in the first instance; Bergson treats this dependence as one of a number of alternative assumptions to be analyzed [2, pp. 311–319]. Bergson's ordering is given by the relative magnitudes assigned to different situations by a real-valued function; Arrow does not assume the existence of such a real-valued function and in this respect his approach is more general. The latter difference makes the mathematical techniques employed by Bergson inappropriate for Arrow's study and leads to the use of symbolic logic rather than calculus.

The approach used in this paper is similar to that of Arrow. It is assumed at the outset that social choices are to be based on individual
preferences. The alternative situations among which a choice is to be made will be called social states and will be defined more precisely, along with other concepts, in the next section.

II. ASSUMPTIONS, CONDITIONS, AND A CLASS OF ORDERINGS

Consider a group of \( n \) individuals. Let \( X_i \) \((i = 1, 2, \cdots, n)\) be the prospect of the \( i \)th individual in social state \( X \). A prospect is either a specification of amounts of commodities to be received and furnished by an individual over a given period of time or a probability combination of such specifications. \( Z_i = pX_i + (1 - p)Y_i \) is the prospect that specification \( X_i \) will be realized by the \( i \)th individual if a chance event with probability \( p \) occurs and \( Y_i \), will be realized otherwise. \( Z_i \) is then a probability combination of \( X_i \) and \( Y_i \).

A social state is an assignment of a prospect to each individual. \( Z = pX + (1 - p)Y \) means \( Z_i = pX_i + (1 - p)Y_i \) for every \( i \). If this holds, the state \( Z \) is said to be a probability combination of states \( X \) and \( Y \). The set of all conceivable social states is denoted by \( S \).

It is assumed that social states are ordered according to the preferences of each individual. The following notation (taken from Arrow [1]) will be used in discussing the individual orderings, that is:

\[
XR_i Y \text{ means "social state } X \text{ is preferred or indifferent to } Y \text{ in the preferences of the } i \text{th individual."}
\]

\[
XP_i Y \text{ means "} XR_i Y \text{ and } YR_i X \text{" where } R_i \text{ means "is not preferred or indifferent to."}
\]

\[
XP_i Y \text{ can be read, "} X \text{ is preferred to } Y \text{ in the preferences of the } i \text{th individual."}
\]

\[
XI_i Y \text{ means "} XR_i Y \text{ and } YR_i X \text{."}
\]

This can be read, "\( X \) is indifferent to \( Y \) in the preferences of the \( i \)th individual."

Let \( \mathfrak{a}_i(S) \) be the ordering of elements of \( S \) according to the preferences of the \( i \)th individual. We are interested in methods for obtaining social orderings of \( S \) that will satisfy certain conditions. Let \( \mathfrak{a}(S) \) be a social ordering of \( S \). \( XRY \) means \( X \) is preferred or indifferent to \( Y \) in the social

\(^2\) From the observations of Knight [4] and Clark [5] on the nature of individual preferences, one might easily question the desirability of this aspect of the approach. While most writers on welfare economies have used a similar approach, this question should be recognized as one of the unsettled problems beyond the scope of the present paper.

\(^3\) For a fuller discussion of individual prospects, see Marschak [5, pp. 113, 114].
ordering \( \rho(s) \); \( X \rho Y \) and \( X \rho I \) express, respectively, preference and indifference in the social ordering.

It was indicated in the previous section that the social ordering was to depend on individual orderings. We may indicate this symbolically as follows:

\[
(2.4) \quad \rho(s) = M[\rho_1(s), \rho_2(s), \ldots, \rho_n(s)],
\]

where \( M \) stands for the method of obtaining a social ordering from a set of individual orderings. The interesting aspects of any discussion using this approach will lie in the assumptions made about the \( \rho_i(s) \) and the conditions imposed on \( \rho(s) \). \( M \) may be regarded as a function in a very general sense. Assumptions about the \( \rho_i(s) \) limit the domain of \( M \), conditions on \( \rho(s) \) restrict its range. Together the assumptions and conditions impose requirements on \( M \).

The conditions to be imposed in this section will be required to hold under the following assumptions about the individual orderings. These are assumed to hold for all \( i \).

**Assumption 1:** Each \( \rho_i(s) \) is a complete ordering. This means:

(a) For all \( X, Y \), either \( XR_Y \) or \( YR_X \) or both;

(b) \( XR_Y, YR_Z \Rightarrow XR_Z \).

**Assumption 2:** \( XP, Y \Rightarrow XP, \{pX + (1 - p)Y\}P, Y \) (for \( 0 < p < 1 \)).

**Assumption 3:** If \( XP, YP, Z \) then there exists a \( p \) \((0 < p < 1)\) such that \( YI, \{pX + (1 - p)Z\} \).

**Assumption 4:** There exist two states, say \( \bar{X}, \bar{Y} \), for which the following hold:

(a) \( \bar{X}_i = \bar{X}_i, \bar{Y}_i = \bar{Y}_i \) (all \( i, j \));

(b) \( \bar{X}P, \bar{Y} \) (all \( i \)).

Assumptions 1–3 restate the axioms of von Neumann and Morgenstern [8]. Assumption 4 assumes at least a certain amount of similarity of individual preferences—i.e., there are two individual prospects such that every one agrees that the state which assigns one of the prospects to every individual is preferable to the state that assigns the other prospect to every individual.

The symbol \( S \) will be used to denote a subset of \( S \) consisting of the social states that are regarded as achievable when technical, economic, and possibly other limitations are taken into account.\(^*\) Much of economic analysis is concerned with finding out which social states are achievable and which states will result from particular policies. In considering methods of obtaining social orderings, we are concerned with the problem of choosing from an achievable set that is regarded as

\(^*\) Possible other limitations are briefly mentioned on p. 91.
given. The achievable set may vary from one application to another. It will be left unspecified in the present discussion except that it will be assumed that any achievable set with which we may be concerned will have the Properties 1 and 2 below.

**Property 1**: \( X \in S, \ Y \in S \Rightarrow \{ pX + (1 - p)Y \} \in S \).

For any \( i \) and any \( Y \in S \), let \( S_{iY} \) be the set of elements, \( X_i \), in \( S \) for which \( X_i \leq Y \). Let \( \alpha \) be a subset (null set not excluded) of the \( n \) individuals. Then \( S_{\alpha Y} = \bigcap_{\alpha} S_{iY} \) is an indifference set for the group of individuals included in \( \alpha \), i.e., \( X \in S_{\alpha Y} \Rightarrow X \in S \) and \( X_i \leq Y \) for all \( i \in \alpha \). If \( \alpha \) is null then \( S_{\alpha Y} = S \) for any \( Y \).

**Property 2**: Any \( S_{\alpha Y} \) has a maximal element with respect to \( R, \) for \( j \neq \alpha \).

The role of Properties 1 and 2 is somewhat different from that of the previous assumptions. Assumptions 1–4 state the nature of the individual orderings to which the analysis applies and may be regarded as restrictions on the domain of \( M \) in (2.4). Since the concept of the achievable set \( S \) is used later in the statement of conditions on the social ordering, Properties 1 and 2 help specify these conditions. Property 1 says that if two states are achievable, probability combinations of the two states are also achievable. Property 2 says that if we consider only achievable states that assign given levels of satisfaction to certain individuals, there exists an optimal state (or states) in the preferences of any other individual. The motive for this assumption is to rule out sets of achievable states in which, whatever state is chosen, it is possible to offer a preferred prospect to some individual without forcing any of a previously selected set of individuals to take a less desirable prospect. Suppose for example, that we had a technical restriction which stated that, with given resources, any amount of some commodity, say beef, less than a given upper limit, say one hundred thousand tons, could be produced. If the decision as to allocation of resources is regarded as already determined, there is clearly no best amount of beef to be produced. For any stated amount that is achievable, there is a larger (and presumably better) amount that is also achievable. This seems an unlikely kind of technical restriction and one could hardly expect to make reasonable choices from such sets of alternatives. Property 2 is a statement of the exclusion of such possibilities from the present discussion.

One more concept should be defined before the conditions to be imposed on \( \alpha \) are stated. Let \( X^{(i)} \) be the social state obtained from \( X \) by interchanging the \( i \)th and \( j \)th elements of \( X \)—i.e., by having individuals \( i \) and \( j \) trade prospects. Thus \( i \) and \( j \) will be said to be similar individuals if

\[
(2.5) \quad XR,Y \iff X^{(i\alpha)} R_{ij} X^{(i\alpha)} \quad \text{(for all X, Y)},
\]
\[(2.6) \quad XI_iX^{i(j)} \quad (\text{for all } X \text{ and all } k \neq i \text{ or } j),\]
\[(2.7) \quad X \in S \Rightarrow X^{i(j)} \in S \quad (\text{for all } X).\]

(2.5) means that the two individuals have similar preferences. (2.6) states that they are similar with respect to the preferences of others. When (2.7) holds, they may be said to have similar possibilities.

The conditions on the social ordering can now be specified.

**Condition 1:** $\mathfrak{B}(S)$ is a complete ordering. This means:
(a) For all $X, Y$ either $XRY$ or $YRX$;
(b) $XRY, YRZ \Rightarrow XZR$.

**Condition 2:** For any $S$ having Properties 1 and 2, there exists an element $X^S$ in $S$ such that $X^SRX$ for all $X$ in $S$.

**Condition 3:** $\mathfrak{B}$ agrees with the partial ordering of the new welfare economics. This means:
(a) if $X_RiY$ for all $i$ then $XRY$;
(b) if $X_RiY$ for some $j$ then $XPY$.

**Condition 4:** The social ordering is independent of the way in which indices are assigned to individuals.\(^6\)

**Condition 5:** If the $ith$ and $jth$ individuals are similar then $X^S_{i}X^{i(j)}$. It may be noted that if $i$ and $j$ are similar and $X^S_I, X^{i(j)}$ then
\[X^S_I, X^{i(j)}\]
also holds.

Conditions 1 and 2 might be called operational conditions. They are not motivated by ethical considerations but by the desire for power and convenience of application. Condition 1 guarantees that any two

\(^6\) This condition can be stated more simply in words than in the symbols employed in this section. I believe the verbal statement is clear, but, for the sake of formal completeness, it may be worthwhile to state an equivalent condition symbolically.

**Condition 4':** If there are two sets of individual orderings $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ and $\mathfrak{R}_1^*, \mathfrak{R}_2^*, \ldots, \mathfrak{R}_n^*$ such that:
(a) $X_RY \Leftrightarrow X^{(1)R_1Y^{(1)}}$ and $XR_1Y \Leftrightarrow X^{(1)R_1Y^{(1)}}$ for some $i, j, i \neq j$ and for all $X, Y$;
(b) $XR_1Y \Leftrightarrow X^{(1)R_1Y^{(1)}}$ for all $k \neq i$ or $j$; then
(c) $XR_1Y \Leftrightarrow X^{(1)R_1Y^{(1)}}$

where $\mathfrak{R}, \mathfrak{R}^*$ are the social orderings corresponding to $\mathfrak{R}_1, \ldots, \mathfrak{R}_n; \mathfrak{R}_1^*, \ldots, \mathfrak{R}_n^*$ respectively.

(a) and (b) hold if the orderings $\mathfrak{R}_1^*, \ldots, \mathfrak{R}_n^*$ are obtained from $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ by interchanging the subscripts identifying the $ith$ and $jth$ individuals with no change in individual preferences. Since any permutation of subscripts can be obtained by a finite number of such interchanges, Condition 4' $\Rightarrow$ Condition 4. The converse is obvious, hence the two are equivalent.
states can be compared and that the comparisons will be transitive.\footnote{While the requirements of a complete ordering seem to me to have intuitive desirability, one might consider relaxing this condition if it were found to conflict with the realization of strongly held ethical values. It might suffice to require a method for making a choice from any of a suitably general class of achievable sets. The formulation of the bargaining problem by Nash [7] is an example of the latter approach.} Condition 2 requires that the ordering specify an optimal state (or a set of optimal states) for each achievable set considered. Condition 3 expresses the value judgment underlying the new welfare economics—that satisfying individual preferences insofar as they do not conflict with each other is socially desirable. Conditions 4 and 5 prescribe, in addition, a kind of equality of treatment of individuals. Condition 4 insures that no accidental elements in arranging individuals or assigning them numbers will affect the social ordering. In the present context, this means that social ordering depends only on the set of individual orderings from which it is derived. If one wished to make the social ordering a function of other variables as well (education, ancestry, or other characteristics) it would probably be convenient to introduce these explicitly (enlarge the domain of \( M \)) and still retain a condition analogous to Condition 4. Condition 5 might be described as requiring similar treatment for similar people, similar treatment being realized when the individuals would be indifferent to an exchange of prospects.

A class of methods that yield orderings satisfying the above conditions is described below. There is no reason to believe that these are the only methods; though others would probably be more difficult to investigate. The fact that a large class of methods exists means that value judgments in addition to those embodied in the above conditions would be needed to choose a unique social ordering.

Let

\[
(2.8) \quad u_i = f_i(X) \quad (i = 1, 2, \cdots, n),
\]

be a numerical utility function for the \( i \)th individual. Assumptions 1–3 imply that individual utility functions exist which have the property,

\[
(2.9) \quad Z = pX + (1 - p)Y \Rightarrow f_i(Z) = pf_i(X) + (1 - p)f_i(Y).
\]

Functions that satisfy (2.9) are unique up to a linear transformation. A unique function is found for each individual by imposing (2.9) and

\[
(2.10) \quad f_i(\bar{X}) = a, \quad f_i(\bar{Y}) = b
\]

\((a, b \text{ constants}, \ a > b, \ i = 1, 2, \cdots, n),\)

where \( \bar{X}, \bar{Y} \) are the states assumed to exist in Assumption 4. Let \( g(u_i) \)
be continuous, monotonic increasing, and strictly concave\(^8\) in the domain \(-\infty < u_i < \infty\). We now define

\[
(2.11) \quad v = \sum_{i=1}^{n} g(u_i) = \sum_{i=1}^{n} g[f_i(X)] = h(X).
\]

The principal result of this section is that a social ordering given by

\[
(2.12) \quad XRY \iff h(X) \geq h(Y),
\]

satisfies Conditions 1–5.

There are, of course, many ways to define \(v\). Thus \(g\) can be chosen from a wide class of functions, \(a\) and \(b\) are arbitrary except for order, and in general there will be alternative pairs \(X, \bar{X}\) to choose from. It is obvious that any of these definitions leads to an ordering which satisfies Conditions 1, 3, and 4. The proof that Condition 2 is satisfied is rather long and will be given in Section IV. Condition 5 presupposes Condition 2 and is proved below on the assumption that Condition 2 is satisfied.

Suppose \(i, j\) represent similar individuals and that \(X^s\) is a maximal element of \(S\), i.e., \(X^sRX, h(X^s) \geq h(X)\) for all \(X \in S\). To avoid lengthy superscripts let \(X^{i(i)} = X^s\). Condition 5 requires that \(X^sI_sX^*\) or in terms of the utility measure \(f_i(X^s) = f_i(X^*)\). We shall show that to suppose the contrary leads to a contradiction. Let \(Y = \frac{1}{2}X^s + \frac{1}{2}X^*, X^* \in S\) by (2.7), and \(Y \in S\) by Property 1.

From similarity of \(i, j\), and (2.9) we have

\[
(2.13) \quad f_i(X^s) = f_i(X^*),
\]

\[
f_j(X^s) = f_j(X^*),
\]

\[
f_i(Y) = f_i(Y) = \frac{f_i(X^s) + f_i(X^*)}{2},
\]

\[
f_j(X^s) = f_j(X^*) = f_j(Y) \quad \text{(for all \(k \neq i\) or \(j\)).}
\]

From (2.11), we have

\[
(2.14) \quad h(Y) - h(X^s) = 2g\left[\frac{f_i(X^s) + f_i(X^*)}{2}\right] - g[f_i(X^s)] - g[f_i(X^*)].
\]

If \(f_i(X^s) \neq f_i(X^*)\), the quantity on the right is positive by the concavity of \(g\) and \(h(Y) > h(X^s)\) which contradicts the assertion that \(X^s\) is maximal.

\(^8\) \(g(u_i)\) strictly concave means that \(g[a u'_i + (1 - a) u''_i] > a g(u_i) + (1 - a) g(u''_i)\) for all distinct \(u'_i, u''_i\) and for \(0 < a < 1\).
III. COMPARISON WITH ARROW'S CONDITIONS

It should be emphasized that no ethical priority has been established for the orderings given by (2.12) or for orderings based on any von Neumann-Morgenstern utility measure. The measure served as a mathematical device for showing that Conditions 1–5 are not mutually inconsistent. To choose one of the family of orderings indicated or a different ordering that also satisfied the conditions, one would have to accept or develop criteria in addition to those expressed by Conditions 1–5. The choice of additional criteria is primarily an ethical problem and, in the view taken in this paper, is the central problem of social choice.

The previous section differs from Arrow's development of the problem both in the assumptions made and the conditions imposed. I believe that it is useful to consider these differences separately and that the differences in conditions are more important. The assumptions of Section II appear reasonable to the author and have been used in showing consistency of the conditions. If some should regard the assumptions as unreasonable, then the possibility of satisfying the conditions under alternative assumptions would need to be considered. Taking such a position with respect to the assumptions would not, however, need to affect one's view of the desirability of the conditions.

If the orderings given by (2.12) are checked against Arrow's conditions, it is found that his Condition 3 is the only one to be violated. For reference, Condition 3 is quoted.⁹

**Condition 3 (Arrow):** Let \( \alpha_1, \ldots, \alpha_n \) and \( \alpha'_1, \ldots, \alpha'_n \) be two sets of individual orderings and let \( C(S) \) and \( C'(S) \) be the corresponding social choice functions. If, for all individuals \( i \) and all \( X \) and \( Y \) in a given environment \( S \), \( XR_iY \) if and only if \( X'R_iY \), then \( C(S) \) and \( C'(S) \) are the same (independence of irrelevant alternatives).

\( C(S) \) in this statement stands for the set of all \( X \in S \) such that \( XRY \) for all \( Y \in S \) [1, p. 15]. In the notation of Section II, it could be defined as the subset of \( S \) that includes \( X^a \) and all \( X \) for which \( XIX^a \). In Arrow's context this condition requires that the social ordering of alternatives in an achievable set, \( S \), depend only on the individual orderings of alternatives in \( S \). If this were to hold, (2.4) could be rewritten

\[
(3.1) \quad \alpha(S) = M\{\alpha_1(S), \alpha_2(S), \ldots, \alpha_n(S)\}.
\]

This seems to me to be rather extreme and, on the whole, undesirable. Consider two cases involving two individuals \( i, j \) and two states

⁹ See Arrow [1, p. 27]. Minor changes in the notation have been made to agree with that used in Section II. The reader should also note that \( S \) in Arrow's book is not subject to Properties 1 and 2.
X, Y. Suppose that in Case I, i barely prefers X to Y and j desperately prefers Y to X. In Case II let i desperately prefer X to Y and j barely prefer Y to X. By Arrow's Condition 3, the social ordering between X and Y must be the same in Case I as in Case II. Admittedly our facilities for distinguishing bare preferences from desperate preferences may often be questionable, but we have to decide whether or not this justifies us in excluding all variations in degrees of preference from consideration.10

It may be noted that, while the methods described in Section II do not satisfy Condition 3, the difficulties cited by Arrow ([1, pp. 26, 27]) as evidence for the need of such a condition would not arise with these methods. That is, a change in the achievable set S would not affect the relations among those elements of S not involved in the change. In the approach of Section II this is achieved by defining a basic ordering for an all-inclusive set S and maintaining this ordering regardless of which elements are or are not achievable in any particular instance. This is analogous to the way in which the matter is treated in the theory of individual choice ([1, pp. 11, 12]).

Condition 3 was undoubtedly motivated by Arrow's determination to exclude any possibility of interpersonal comparison of utilities [1, pp. 9–11]. It is one thing to admit that no intuitively acceptable basis for interpersonal comparisons exists and quite a different thing to say that all such comparisons are fundamentally impossible. If we interpret utility as an ordinal preference indicator and if social orderings are to be based on individual preferences, then the latter position really excludes the possibility of ordering social states any more completely than is done by the new welfare economics. For as soon as we say that state X is socially preferred to state Y for two states such that some individuals prefer X to Y and others prefer Y to X, we are thereby saying that the gains to those who prefer X are socially more important than the losses of those who prefer Y. This implies that we have some basis for comparing the relevant gains and losses. Such a comparison is fundamentally an interpersonal comparison of utilities. For example, Arrow has shown ([1, Chapter VII]) that under suitable assumptions about individual preferences (which involve modifying his Condition 1) the method of

10 L. J. Savage has made the interesting observation that many of the comparisons of degrees of preference which people are commonly willing to make contain an implicit assumption that certain corporal states have similar significance for different individuals. Marshall's observation that a clerk with a lower salary will generally choose to walk to business through a heavier rain than a clerk with a higher salary (Marshall [6, p. 19, p. 95]) seems to involve a comparison of this kind. Willingness to risk extreme physical hardships such as hunger, exposure, torture, or death for an objective is almost universally accepted as evidence of the subjective importance of the objective.
majority decisions satisfies his conditions. Does the method of majority decisions involve interpersonal comparisons of utilities? I believe so; it chooses between $X$ and $Y$ by comparing the number who would gain utility in passing from $X$ to $Y$ with the number who would lose. Thus anyone's gain is exactly the equivalent of any other's gain and exactly the inverse of anyone's loss.

Thus, if we wish to go beyond the comparisons that are possible using only the principle of new welfare economics, the issue is not whether we can do so without making interpersonal comparisons of satisfactions. It is rather, what sorts of interpersonal comparisons are we willing to make. Unless the comparisons allowed by Arrow's Condition 3 could be shown to have some ethical priority, there seems to be no reason for confining consideration to this group.

The conditions of Section II do not answer this question. They seem to me to serve two purposes—the results of applying these conditions may help explain the nature of the difficulty encountered in Arrow's initial formulation, and to the extent that the conditions are regarded as reasonable they may serve as a start for the development of stronger and more interesting sets. This development would presumably take place through the formulation of new conditions reflecting additional or stronger value judgments.

While this paper is primarily concerned with formal aspects of the problem, it may be worth noting that it is possible to vary somewhat the interpretations of the concepts employed without essentially altering the formal analysis. For example, one could include noneconomic factors in the definition of a social state without changing the formal analysis if he were willing to retain unchanged the assumptions about individual preferences and about sets of achievable states. It has sometimes been assumed that each individual is indifferent to the prospects of others and evaluates each social state solely with reference to his own prospect in that state. If this assumption had been made in the previous section, a few simplifications would have been possible but the results would have been unchanged.

It is possible that some ethical values which should be recognized in making social choices apply independently of individual preferences. In the present approach at least some of these values might be expressed as restrictions on the set of achievable alternatives and entered along with technical restrictions in the determination of $S$. Thus, if certain social states were judged to be ethically undesirable independently of individual preferences, these states could be regarded as not achievable and could be excluded before the ordering based on individual preferences was applied.
IV. PROOF OF CONDITION 2

In this section we show that Condition 2 is satisfied by a method given by (2.12). Let \( F = (f_1, f_2, \cdots, f_n) \) represent the individual utility functions of (2.8). Let \( U = (u_1, u_2, \cdots, u_n) \) be a point in an \( n \)-dimensional Euclidean space \( E \). The \( n \) equations (2.8) can also be indicated by

\[
U = F(X).
\]

Let \( T \) be the image in \( E \) of the achievable set \( S \), i.e.,

\[
T = F(S),
\]

and define \( \varphi(U) \) by

\[
\varphi(U) = \sum_{j=1}^{2} g(u_j),
\]

where \( g \) is continuous, monotonic increasing, and strictly concave as in (2.11).

We see from Property 1 and (2.9) that

\[
T \text{ is convex.}
\]

As in Section II, let \( \alpha \) be a subset of the indices that represent individuals. Equivalently, \( \alpha \) denotes a subset of the coordinates of \( E \). For any \( U^0 \in T \), let \( T_{\alpha \cup \beta} \) contain those elements of \( T \) that agree with \( U^0 \) in the coordinates denoted by \( \alpha \). From Property 2 and the continuity of \( g \), we can say,

\[
\text{given any } T_{\alpha \cup \beta} \text{ and any } j \text{ not in } \alpha, \text{ there exists a } U^{(j)} \in T_{\alpha \cup \beta} \text{ such that } u^{(j)}_j \geq u_j \text{ for all } U \in T_{\alpha \cup \beta}.
\]

To show that Condition 2 holds we must prove the following assertion.

For any \( T \) satisfying (4.4) and (4.5) there exists a

\[
U^T \in T \text{ such that } \varphi(U^T) \geq \varphi(U) \text{ for all } U \in T.
\]

When \( \alpha \) is null, \( T_{\alpha \cup \beta} = T \) and (4.5) says that there exists a maximum for the values of any particular coordinate of points in \( T \). Let \( u_i^T \) (\( i = 1, 2, \cdots, n \)) be the maximum of \( u_i \) for points of \( T \). \( U^m \) is the point (not necessarily in \( T \)) whose coordinates are equal to these maxima.

Let \( \overline{T} \) be the closure of \( T \). \( \overline{T} \) contains an element \( U^+ \) such that \( \varphi(U^+) \geq \varphi(U) \) for all \( U \in T \). This follows from the closure of \( \overline{T} \), continuity of \( \varphi(U) \), and the fact that \( \varphi(U) \) is bounded in \( T \) by \( \varphi(U^m) \).

\[11\] The essentials of this proof were furnished by Edmond C. Malinvaud.
We shall prove:

\[(4.7) \quad \text{If } (1) \ U^+ \in T \text{ and } (2) \ \varphi(U^+) \geq \varphi(U) \text{ for all } U \in T \]

\[\text{then } U^+ \in T.\]

Assertion (4.6) is then established by setting \(U^T = U^+.\) Assertion (4.7) clearly holds for \(n = 1\) since \(U^+ \in T\) in that case. If (4.7) holds for \(n - 1,\) its validity for \(n\) follows from Propositions I–IV below.

**Proposition I:** If there is a coordinate \(i\) such that \(U^+\) is in the closure of \(T_{i\!\!i}^+\) (the intersection of \(T\) and the hyperplane given by \(u_i = u_i^+\)), then \(U^+ \in T.\)

**Proof:** Let \(T_{i++}\) be the closure of \(T_{i++}^+,\) \(U^+ \in T_{i++}^+\) and \(\varphi(U^+) \geq \varphi(U)\) for all \(U \in T_{i++}^+.\) Hence by our induction hypothesis \(U^+ \in T_{i++}^+ \subset T.\)

**Proposition II:** If, for all coordinates \(i,\) \(U^+ \in T_{i++}\); then, for each coordinate \(i,\) either \(u_i \geq u_i^+\) for all \(U \in T\) or \(u_i \leq u_i^+\) for all \(U \in T.\)

**Proof:** Suppose there are \(U', U''\) in \(T\) such that \(u_i' < u_i^+ < u_i''\) for some \(i.\) Then there exists a sequence \(\{V^n\}\) in \(T_{i++}^+\) converging to \(U^+\) contradicting the hypothesis that \(U^+ \in T_{i++}^+.\)

To construct \(\{V^n\}\) take a sequence \(\{U^n\}\) in \(T\) converging to \(U^+.\)

Define \(V^n\) as follows: If \(u_i^n = u_i^+\) take \(V^n = U^n.\) If \(u_i^n > u_i^+\) take \(V^n\) as the intersection of \(T_{i++}^+\) with \(U^n U^n+\) (the convex closure of \(U'\) and \(U'\)). Such a point exists because \(T\) is convex.

If \(u_i^n < u_i^+\) take \(V^n\) as the intersection of \(T_{i++}^+\) with \(U^n U^n+\). \(\{V^n\}\) is in \(T_{i++}^+\) and converges to \(U^+.\)

**Proposition III:** If there is a coordinate \(k\) such that \(u_k \geq u_k^+\) for all \(U \in T,\) then \(U^+ \in T.\)

**Proof:** Let \(V\) (and \(V^+\)) be the projections of \(U\) (and \(U^+\)) on the \(n - 1\) dimensional space perpendicular to the \(k\)-axis and let \(T^{(k)}\) be the projection of \(T\). Define \(\varphi^{(k)}(V) = \sum \lambda u_i g(u_i).\) We then have \(V^+ \in T^{(k)}\) and \(\varphi^{(k)}(V^+) \geq \varphi^{(k)}(V)\) for all \(V \in T^{(k)}\). \(T^{(k)}\) fulfills (4.4) and (4.5), hence, by the induction hypothesis \(V^+ \in T^{(k)}\). Thus there exists a \(U^* \in T\) such that \(u_i^* = u_i^+\) for \(i \neq k\) and \(u_k^* \geq u_k^+\). Clearly the last equality must hold or we would have \(\varphi(U^*) \geq \varphi(U^+).\) Thus \(U^* = U^+ \in T.\)

**Proposition IV:** If \(u_i^+ \geq u_i\) for all \(i\) and all \(U \in T,\) then \(U^+ \in T.\)

**Proof:** By taking projections as in the proof of Proposition III we can show that there are elements \(U^{(k)} \in T\) with coordinates \(u_i^{(k)} = u_i^+\) for \(i \neq k\) and for \(k = 1, 2, \ldots, n.\)

If \(u_k^{(k)} = u_k^+\) for some \(k,\) then that \(U^{(k)}\) is equal to \(U^+.\) Otherwise consider the convex hull \(H\) spanned by \(U^+\) and all of the \(U^{(k)}\). The inte-
rior of $H$ is contained in $T$ for given any $V$ in the interior, we can find an element $U^+ \in T$ in a sufficiently small neighborhood of $U^+$ so that the convex hull spanned by $U^+$ and all the $U^{(k)}$ contains $V$.

Consider any coordinate $j$. To each $V$ in the interior of $H$ there corresponds a $W \in T$ such that $w_i = v_i$, $i \neq j$, and $w_j = u^+_j$. To show this let $\alpha$ include all $i \neq j$. By (4.5) there is a $U^+ \in T_{\alpha W}$ such that $u^+_j$ is a maximum for $U \in T_{\alpha W}$. Since $T$ includes the interior of $H$ this maximum cannot be less than $u^+_j$, hence $U^+ = W$.

Thus for any sequence $\{V^n\}$ in the interior of $H$ and approaching $U^+$ there is a corresponding sequence $\{W^n\}$ in $T_{\mu^+}$ (the intersection of $T$ and the hyperplane given by $u_i = u^+_i$) which also approaches $U^+$. $U^+$ is thus in the closure of $T_{\mu^+}$ and $U^+ \in T$ follows from Proposition I.

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REFERENCES


