Cowles Foundation Paper 67

Reprinted for private circulation from
THE AMERICAN JOURNAL OF SOCIOLOGY
Vol. LVIII, No. 3, November 1952
PRINTED IN U.S.A.

SOCIAL WELFARE FUNCTIONS BASED ON INDIVIDUAL RANKINGS

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ABSTRACT

K. J. Arrow has described five apparently reasonable properties which any voting system or other "social welfare function" should have. He has demonstrated mathematically that none could possibly have all these properties. One of his requirements is questionable, but if it is modified many "voting systems" become acceptable.

A social welfare function is defined as a method for obtaining group preferences, given the preferences of the individual members of the group. An election system, for example, gives a "community choice" of candidates as a function of voters' choices. Kenneth Arrow[2] has shown (1) that if individual preferences are expressed as rankings of various alternatives and (2) that if we require certain "natural" conditions of any "acceptable" social welfare function, then "acceptable" welfare functions do not exist. No welfare function has the properties which, it would seem at first glance, we would require of, for example, a reasonable voting system.

In this paper we argue that the Arrow postulates are not as plausible as they at first appear. The Arrow postulates can be modified somewhat to meet our objections, but then many social welfare functions satisfy the modified postulates. The fact that a large class of social welfare functions exist means that value judgments in addition to those embodied in the modified postulates are needed to choose a unique function. We will then consider which of the many functions not rejected by the modified postulates seem most reasonable.

Let us consider an example of a welfare function which, like all welfare functions, contradicts the "natural" conditions which Arrow requires. This example will be useful later, when we consider the plausibility of the Arrow conditions.

Each "voter" ranks the "candidates," the lowest ranking being assigned to his first preference. Consider the welfare function which prescribes: for each candidate total the ranks given him by the various voters; one candidate is socially preferred to another if his sum of ranks is less than that of the other. Thus, if we have three candidates, *a*, *b*, and *c*, who are ranked by two voters, *A* and *B*, as in Table 1, then *b* is preferred to *a* and *a* is preferred to *c*. But if *c* had not been available, the rankings would have been as in Table 2; *a* and *b* would be socially equivalent. This phenomenon of a changing social choice (*b* was preferred to *a* when *c* was available but was equivalent to *a* when *c* was not available) contradicts a condition which Arrow thinks is natural to impose on "acceptable" social welfare functions.

TABLE 1

	a	*b*	*c*
A	1	2	3
B	3	1	2
Σ	4	3	5

TABLE 2

	a	*b*
A	1	2
B	2	1
Σ	3	3

Let us present another example known as the "paradox of voting," which is described also by Arrow. A natural way of arriving at a social preference might be to say that one candidate is preferred to

[1] Mr. Goodman's contribution was financed in part by the Office of Naval Research. Mr. Markowitz's contribution was initiated while he was a Research Fellow of the Cowles Commission for Research in Economics. This paper will be reprinted as Cowles Commission New Series No. 67.

[2] K. J. Arrow, *Social Choice and Individual Values* (New York: John Wiley & Sons, 1951).

another candidate if a majority of voters in the community prefer the first candidate to the second, i.e., would choose the first over the second if there were only two. Thus, if we have three candidates, *a*, *b*, and *c*, who are ranked by three voters, *A*, *B*, *C*, as in Table 3, then a majority prefer *a* to *b*

TABLE 3

	a	*b*	*c*
A.....	1	2	3
B.....	3	1	2
C.. .	2	3	1

and *b* to *c*. If the community is to be regarded as behaving "rationally," we are forced to say that *a* is preferred to *c*. But in fact a majority prefer *c* to *a*. Hence the social welfare function based on majority rule does not exhibit the "rationality" one might wish to require of such modes of social choice.

If we had used the social welfare function first described (summation of ranks), we would have found that *a*, *b*, and *c* were socially equivalent but that when *b* was not available *c* was preferred to *a*. Hence, again we see the phenomenon of a changing social choice, which contradicts a condition which Arrow thinks is natural.

The Arrow conditions, to be satisfied by any "acceptable" social welfare function, may be paraphrased as follows:

Condition 1.—The social welfare function is a method for obtaining a simple social ordering (which is transitive) defined for a sufficiently wide range of individual orderings (rankings).

Condition 2.—If alternative (*a*) rises or remains still in the ordering of every individual and no other change takes place in those orderings, then alternative (*a*) rises, or at least does not fall, in the social ordering.

A social welfare function gives a social choice (or set of choices) for every set of available alternatives. We may consider the "choice function" associated with a given social welfare function, and we may consider how this choice function changes as voters' preferences change or as changes take place in the considered (though not necessarily available) candidates. Arrow requires:

Condition 3 (*Independence of irrelevant alternatives*).—If each voter ranks each *available* candidate exactly the same in one situation as he does in another, then, no matter what is true about the rankings of the other (nonavailable) candidates who have been considered, the choice among the available candidates is the same in both situations.

Condition 4.—The social welfare function must not be "imposed"; i.e., it must not be given independently of individual preferences.

Condition 5.—The social welfare function must not be dictatorial; i.e., it must not be identical with the preferences of one individual, irrespective of all other individuals' preferences.

Arrow has shown that no social welfare function satisfies the foregoing, apparently reasonable, conditions. The example of a welfare function which we first considered (summation of ranks) contradicts Condition 3, and the method of majority rule contradicts the transitivity property of Condition 1.

We will first present our objections to the Arrow conditions intuitively and then state our position more formally. Suppose you intended to serve refreshments to two friends. You could serve them either coffee or tea but not both; *A* preferred coffee, *B* preferred tea. It seems clear that a symmetric ("democratic") welfare function would rank coffee and tea equally. Suppose you had other information concerning the preferences of *A* and *B*. While *A* prefers coffee to tea, he prefers tea to cocoa and cocoa to milk. *B*, on the other hand, not only prefers tea to coffee but prefers cocoa to coffee, milk to coffee, tomato juice to coffee; he would rather drink water than coffee; and he preferred tea to cocoa, milk, tomato juice, and water. Given this added information, it seems plausible to serve tea rather than coffee; for it does not make "much difference" to A, and it makes "quite a bit of difference" to B.

In terms of the example of Tables 1 and 2, if we had the information in Table 1, then this information should have been used even when only a and b are available. If initially we had only the information in Table 2 (i.e., the preferences of A and B for a and b) and then had been given the information in Table 1 (i.e., the preferences of A and B for a, b, and c), we might wish to use the fact that it seems to make "more difference" to B than to A. Thus the "irrelevant alternative" is not necessarily irrelevant.[3]

Still looking at the problem intuitively, it may be objected that although B prefers c more than a but less than b, while A prefers b more than c but less than a, still B may feel less "difference" between a and b than does A. This argument might be put forth no matter how many objects were found preferred by B to b but not to a. We can avoid some of the consequences of this argument if we assume that each individual has only a finite number of indifference levels or "levels of discretion." That is, for some N, once we find N states or candidates none of which are indifferent in the individual's preferences, then every other state or candidate is indifferent to one of these. A change from one level to the next represents the minimum difference which is discernible to an individual. This assumption is not unreasonable; we cannot expect individuals to have more than 10^3 or 10^6 or 10^9 levels of discretion.[4]

Let us state our position more formally: Suppose that there are M voters, $1, \ldots, i, \ldots, M$. Each has a finite number of "levels of discretion," $1, 2, \ldots, L_i$. Level one is best; L_i, worst. The number, L_i, of levels may differ from person to person.

Suppose we are considering two candidates (1 and 2) and suppose further—for a moment—that we know the level at which each voter i ranks 1 and 2; i.e., we know 1_{i1}, 1_{i2}. Given the matrix 1_{ij}, a social welfare function will rank the candidates 1 and 2. If candidate 2 fell in the opinion of voter 1 (i.e., 1_{12} increases), everything else remaining the same, we would require that candidate 2 should not rise in the social ordering. This does not contradict the Arrow requirements. We would also admit the following resolutions.

Resolution 1.—A social welfare function shall not be rejected as unreasonable on the sole grounds that candidate j^0 falls in the social ordering, when, for some $i = i^0$, $1_{i^0j^0}$ increases—the other 1_{ij}'s remaining the same.

Typically we will not know the exact levels 1_{ij} or even the number of levels L_i. All we will know are the rankings a_{ij} of n candidates by m voters. This information may be expressed by a matrix $A = (a_{ij})$. (We assume that each of the m voters has ranked all the n candidates.) Our previous discussion justifies.

Resolution 2.—A social welfare function shall not be rejected on the sole grounds that it changes the ordering of j_1 and j_2 as the state of information changes.

The condition that the welfare function should be independent of irrelevant alternatives may be preserved somewhat; it seems reasonable to require.

Resolution 3.—For a given state of information, the welfare function should order the "candidates" independently of their availability.

[3] L. J. Savage ("The Theory of Statistical Decision," *Journal of the American Statistical Association*, XLVI, No. 253 [1951], 64) has pointed out in a different context that "when the new act is admitted the group may well change its choice to arrive at a compromise with some members who prefer the new possibility, without actually adopting the new possibility itself."

[4] We realize that continuity assumptions are often made; but these, we feel, are made for reasons of mathematical convenience rather than out of the conviction that the individual has a noncountable—or even denumerable—number of "discretion levels."

If we modify the Arrow conditions to satisfy our three resolutions, then we have a set of conditions satisfied by many welfare functions. The fact that a large class of social welfare functions exists means that value judgments in addition to those embodied in the resolutions are needed to choose a unique function. We will now consider which functions we think are "most plausible" in our context. We will follow Arrow in the use of the axiomatic approach; that is, we will lay down conditions to be required of any welfare function and will then seek those welfare functions which satisfy these conditions. Our conditions will be somewhat different from those of Arrow.

For any given state of information (given by a matrix $A = [a_{ij}]$ of orderings) the social welfare function gives a simple ordering of alternatives, independently of their availability; i.e., the social welfare function orders the vectors

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{pmatrix},$$

a set in an m-dimensional Euclidean space. The social welfare function is defined for all states of information ("Universal Applicability"), i.e., the ordering is defined for all matrices (a_{ij}) of positive integers. This ordering satisfies the following conditions:

Condition 1 (*Pareto optimality*).—If nobody prefers j_2 to j_1 and somebody prefers j_1 to j_2, then j_1 is socially preferred to j_2, i.e., if $a_{ij_2} \geq a_{ij_1}$ for all i and $a_{ij_2} > a_{ij_1}$ for some i, then $\{a_{ij_2}\} > \{a_{ij_1}\}$.

In some cases the following condition is desirable:

Condition 2 (*Symmetry*).—The social ordering is unchanged if the rows of A are interchanged.

The first two conditions, we feel, are self-explanatory. The next condition says, roughly, that the significance of a change from one discretion level to the next is the same, no matter what level one starts from.

Condition 3.—Suppose voter i has exhibited L_i levels of discretion. The social ordering among candidates 1 and 2 remains unchanged if we replace a_{i1} and a_{i2} by $a_{i1} + c$ and $a_{i2} + c$, respectively. The constant c must be an integer such that

$$1 \leq a_{ij} + c \leq \max_i . [L_i]$$

for all j.

If we require only Condition 1, any monotonic ordering function defines a social welfare function, and these are the only acceptable ones.

If we insist on Conditions 1-3, then one and only one social welfare function is acceptable. This prescribes that

$$\{a_{ij_1}\} > \{a_{ij_2}\}$$

if and only if

$$\sum_i a_{ij_1} > \sum_i a_{ij_2} .$$

This will be shown in the next section.[5]

[5] Arrow has stated: "The close relationship between inter-personal comparability of utility differences (marginal utilities) and the use of the summation-of-utilities criterion for social welfare decisions has, of course, long been recognized vaguely; however, Goodman-Markowitz have greatly clarified the meaning of the proposition, and, in particular, have shown that the two conditions are necessary and sufficient for each other" ("The Meaning of Social Welfare: A Comment on Some Recent Proposals," Technical Report No. 2, Department of Economics, Stanford University, 1951).

Arrow has also shown that Conditions 1–3 imply that the less discriminating individual is completely discriminated against. He points out that this result has obviously a strong affinity with the discussion by J. von Neumann and O. Morgenstern of games with discrete utility scales; there too it develops that the less discerning player is completely discriminated against (cf. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* [Princeton University Press, 1947], pp. 614–16). It should be mentioned that by removing the symmetry condition and using an appropriate system of weights, w_i, this problem may be avoided.

The authors also had conjectured that if only Conditions 1 and 3 are required, then a simple class of welfare functions is acceptable (some continuity condition on the function is also needed). A welfare function is a member of this acceptable class if and only if there exists a set of weights w_i such that

$$\{a_{ij_1}\} > \{a_{ij_2}\}$$

if and only if

$$\sum_i w_i a_{ij_1} > \sum_i w_i a_{ij_2}.$$

A result of the work of D. Blackwell and M. A. Girshick, presented by Blackwell at the Santa Monica meetings of the Institute of Mathematical Statistics, August, 1951, verifies this conjecture if certain continuity conditions are added.

Clifford Hildreth[6] has shown one way of avoiding the Arrow paradox. In the Hildreth system, probability combinations of alternatives are introduced explicitly. The present writers feel that the Hildreth results do not completely remove the sting from the Arrow paradox. In some practical situations we know only the rankings by individuals. In an election, for example, we usually know only which candidate is most preferred by each voter. We might reasonably ask each voter to rank all candidates. But it does not seem too reasonable to require that voters consider probability combinations of candidates.

If we combine one aspect of the Hildreth approach with ours, an interesting light is cast on Arrow's Possibility Theorem. Arrow's result was originally derived from a requirement that the ordering be independent of the choice of the utility function to represent the individual orderings; i.e., there is some kind of invariance under monotone transformations of the individual indifference map. We find that even requiring invariance under linear transformations will lead to the same paradox, essentially.

[6] C. Hildreth, Cowles Commission Discussion Papers, "Economics," Nos. 2002 and 2028.

PROOFS

Let there be m voters. An acceptable social welfare function is a weak ordering of the points $\{a_i\}$ in a subset of m-dimensional Euclidean space such that

1. *Pareto optimality.*—If $a_i'' \geq a_i'$ for all values of i, except i_0; then $\{a_i''\} > \{a_i'\}$ (a_i' preferred to a_i'') when $a_{i_0}'' > a_{i_0}'$.

2. *Symmetry.*—$\{a_i\} = \{b_i\}$ when there exists a one-to-one mapping of the indices $i \rightleftarrows j$ such that $a_i = b_j$.

3*a*. $\{a_i\} > \{b_i\}$ if and only if $\{a_i\} + \{c_i\} > \{b_i\} + \{c_i\}$.

THEOREM 1: (A) The ordering relation defined by $\{a_i\} = \{b_i\}$ if $\Sigma a_i = \Sigma b_i$, $\{a_i\} > \{b_i\}$ if $\Sigma a_i > \Sigma b_i$ is an acceptable social welfare function. (*B*) It is the only acceptable social welfare function.

Proof.—*A* is trivial since Conditions 1, 2, and 3*a* are easily verified. *B* is more difficult.

Let us first consider the case (*a*) where $\Sigma a_i = \Sigma b_i$. We must show that any social welfare function must necessarily prescribe $\{a_i\} = \{b_i\}$. Let us add

$$\left\{ -a_i + \sum_{s=1}^{i-1} (b_s - a_s) \right\} = \{c_i\}$$

to the two vectors. We then have

$$a_i + c_i = \sum_{s=1}^{i-1} (b_s - a_s) = b_{i-1} - a_{i-1} + \sum_{s=1}^{i-2} (b_s - a_s) = b_{i-1} + c_{i-1},$$

and $a_0 + c_0 = b_m + c_m = 0$. Hence by (3*a*) we need only consider the case where $a_i = b_{i-1}$ and $a_0 = b_m = 0$ when $\Sigma a_i = \Sigma b_i$. Hence by (2) $\{a_i\} = \{b_i\}$.

Let us now consider the case when $\Sigma a_i > \Sigma b_i$. We must show that any social welfare function must necessarily prescribe $\{a_i\} > \{b_i\}$. Let us consider a vector $\{c_i\}$ such that $\Sigma c_i = \Sigma(a_i - b_i)$, $c_i = 0$ for $a_i < b_i$ and $c_i + b_i \leq a_i$, otherwise. Then

$\{b_i + c_i\} = \{a_i\}$, by the preceding result, and $\{b_i + c_i\} > \{b_i\}$ by (1). Hence, $\{a_i\} > \{b_i\}$.

In the proof of Theorem 1, we added, using (3*a*), constants to the vectors $\{a_i\}$ and $\{b_i\}$ in proving the first case (*a*). It is interesting to note that we need not postulate the existence of any numbers a_i, b_i greater than those observed.

We prove this fact by means of mathematical induction on m, the number of voters. For $m = 2$, it is clear since the constants added are such that the resulting vectors contain only the numbers 0 and $b_1 - a_1$, which may be taken as nonnegative. Let us assume the result is true for vectors in $m - 1$ space, and let us choose two vectors satisfying case (*a*) in m space. Rearrange these vectors so that $a_m - b_m > 0$. Now consider the vectors $\{a_1, a_2, \ldots, a_{m-2}, a_{m-1}\}$ and $\{b_1, b_2, \ldots, b_{m-2}, b_{m-1} - (a_m - b_m)\}$ which satisfy case (*a*) in $m - 1$ space. By the induction hypothesis, we assume that constants may be added to these vectors such that none of the resulting numbers are greater than those appearing in the two vectors and a fortiori in the vectors $\{a_i\}$, $\{b_i\}$. In the original vectors, b_{m-1} and a_m are transformed into $a_m - b_m$, and b_m is transformed into 0. Hence the result is proved.

The preceding argument suffices to show also that the constants added need never be such as to obtain negative numbers.

If the original subset in the m space is restricted, furthermore, by the condition that a_i should be nonnegative integers (rankings), the results still hold, since, in that case, all constants added in the proof of Theorem 1 were also integers.

Suppose condition (3*a*) is changed to read 3*b*. The a_i are determined except for arbitrary scale factors; i.e., $\{a_i\} > \{b_i\}$, if and only if $\{c_i a_i\} > \{c_i b_i\}$.

Then by replacing a_i by $a_i = e^{x_i}$ and using a slight modification of Theorem 1 in the x space, we obtain

THEOREM 2: (*A*) When (3*a*) is replaced by (3*b*), then the ordering relation defined by $\{a_i\} = \{b_i\}$ if $\Pi a_i = \Pi b_i$, $\{a_i\} > \{b_i\}$ if $\Pi a_i > \Pi b_i$ is an acceptable social welfare function. (*B*) It is the only acceptable social welfare function (when [3*a*] is replaced by [3*b*]).

From Theorems 1 and 2 we have

THEOREM 3: When (3*a*) is replaced by the condition that the a_i are determined except for arbitrary linear transformations; i.e., $\{a_i\} > \{b_i\}$, if and only if $\{c_i + d_i a_i\} > \{c_i + d_i b_i\}$, then no social welfare function is possible.

This result gives us some insight into the "Possibility Theorem" of Arrow.

Suppose now that a social welfare function is not required to satisfy conditions (2) and (3*a*). Then we have

THEOREM 4: (*A*) If (2) and (3*a*) are not required, the ordering relation defined by $\{a_i\} = \{b_i\}$ if $f(\{a_i\}) = f(\{b_i\})$, $\{a_i\} > \{b_i\}$ if $f(\{a_i\}) > f(\{b_i\})$, is an acceptable social welfare function, when f is monotonic; i.e., for $c_{i_1} > 0$ and $c_i \geq 0, f(\{b_i + c_i\}) > f(\{b_i\})$. (*B*) These are the only acceptable social welfare functions (if [2] and [3*a*] are not required).

Proof.—*A* is trivial. Now for *B*. Suppose we are given a social welfare function (i.e., an ordering of the points, in a subset of m space). By ordering we may define a real value function $g(\{a_i\})$ which has the property that $g(\{a_i\}) = g(\{b_i\})$ if and only if $\{a_i\} = \{b_i\}$ and $g(\{a_i\}) > g(\{b_i\})$ if and only if $\{a_i\} > \{b_i\}$. Since the social welfare function satisfies (1), g is monotonic.

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