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AN AXIOMATIC APPROACH TO MEASURABLE UTILITY

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1. INTRODUCTION

THE CONCEPT of a measurable utility, that is, of a real-valued function, appropriately linear with respect to probability distributions, measuring an individual's preference ratings, is by no means a new one, tracing its origin as far back as Bernoulli and his "moral expectation." However, a completely rigorous formulation and treatment of the existence of such a utility, on the basis of a well-defined set of conditions or postulates, was completely lacking until the arrival of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* [5]. In order to find a set of axioms that would be more acceptable (to some economists), and to have a simpler derivation mathematically from these axioms, Marschak [2] attacked the subject again. However, in his paper Marschak considered only the case of a finite number of sure prospects. Rubin [4] extended the Marschak system to the case of an infinite number of sure prospects. Herstein [1] and Milnor [3] gave quite different axiom sets for this problem and succeeded in simplifying and shortening the mathematical details considerably. However, topological considerations of the prospect space entered into the axioms. In this paper we remove considerations of the topology of the prospect space itself, weaken the previous axioms, and allow an infinite number of sure prospects; in doing so the treatment actually becomes simpler and more transparent. On the basis of these axioms the existence of a measurable utility is established.

2. NOTATION

In the body of this paper the following notation system is employed: (a) capital script Latin letters will always denote sets; (b) lower case italic letters will always denote elements of sets; (c) lower case Greek letters will always denote real numbers whose values are between 0 and 1, with end values also possible; and (d) $S = \{x \mid P\}$ will denote the set of x having the property P .

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3. THE AXIOMS

A set S is said to be a *mixture set* if for any $a, b \in S$ and for any μ we can associate another element, which we write as $\mu a + (1 - \mu)b$, which is again in S , and where

$$(1) \quad 1a + (1 - 1)b = a,$$

$$(2) \quad \mu a + (1 - \mu)b = (1 - \mu)b + \mu a,$$

$$(3) \quad \lambda[\mu a + (1 - \mu)b] + (1 - \lambda)b = (\lambda\mu)a + (1 - \lambda\mu)b,$$

for all $a, b \in S$ and all λ, μ .

A convex set in a real vector space, where we mean by $\mu a + (1 - \mu)b$ the usual multiplication by scalars and the addition of elements of this vector space, is easily seen to be a mixture set.

The concept of a preference ordering arises naturally in certain phases of economics. We formalize the concept and make the following

DEFINITION: A binary relation, \succsim , defined on a set S is a complete ordering if

$$(i) \quad \text{for any } a, b \in S, \text{ either } a \succsim b \text{ or } b \succsim a \text{ must hold,}$$

$$(ii) \quad \text{if } a, b, c \in S \text{ and } a \succsim b, b \succsim c, \text{ then } a \succsim c.$$

The simultaneous satisfaction of $a \succsim b$ and $b \succsim a$ need not imply that a is identical with b . This prompts us to make the following

DEFINITION: If $a, b, c \in S$, then $a \sim b$ (read: a is indifferent or equivalent to b) if and only if both $a \succsim b$ and $b \succsim a$.

The following three properties then hold:

$$(A) \quad a \sim a,$$

$$(B) \quad a \sim b \text{ implies } b \sim a,$$

$$(C) \quad a \sim b, b \sim c \text{ imply } a \sim c$$

for any $a, b, c \in S$.

Let $I(a) = \{x \in S \mid x \sim a\}$. Properties A, B, C of \sim imply that the distinct $I(a)$'s yield a decomposition of S into mutually disjoint subsets. We call $I(a)$ the *indifference set* of a .

If $a \succ b$ but $a \notin I(b)$, we say that $a > b$.

A real valued function u defined on a completely ordered set S is called *order-preserving* if, for any $a, b \in S$, $u(a) > u(b)$ if and only if $a > b$.

A real-valued function v defined on a mixture set \mathfrak{M} is said to be *linear* if, for all $a, b \in \mathfrak{M}$, and any α ,

$$v[\alpha a + (1 - \alpha)b] = \alpha v(a) + (1 - \alpha)v(b).$$

The economist often considers a set S that is, at the same time, both a mixture set and a completely ordered set, the mixture operation being, in his case, a probability mixture, the ordering being the preference ordering, and S the set of prospects. The economist is interested in finding a real-valued, order-preserving, linear function on S . This function is the so-called *measurable utility*. If there are no restrictions on S other than that it be a completely-ordered mixture set, such a measurable utility need not exist. The problem, then, is to impose "natural" restrictions on the interrelation of the ordering and the mixing which give rise to a measurable utility.

Our aim has been many-fold: to use a system of axioms that was simple and seemingly not too restrictive, that seemed to approximate "economic reality," that was transparent, and that led to the existence of a measurable utility with a minimum of mathematical difficulty or sophistication. Each of the axioms we use is true for S whenever a measurable utility exists on S ; so the axioms are at least necessary conditions. In this paper we prove that they are also sufficient to lead to the existence of the desired order-preserving, linear function.

Let S be a mixture set. We assume

AXIOM 1: S is completely ordered by \succsim .

AXIOM 2: For any $a, b, c \in S$, the sets $\{\alpha \mid \alpha a + (1 - \alpha)b \succsim c\}$ and $\{\alpha \mid c \succsim \alpha a + (1 - \alpha)b\}$ are closed.

AXIOM 3:³ If $a, a' \in S, a \sim a'$, then for any $b \in S, \frac{1}{2}a + \frac{1}{2}b \sim \frac{1}{2}a' + \frac{1}{2}b$.

In economic terms, Axiom 2 approximately states that an individual's preference ordering is continuous with regard to probability distributions; namely, that if $\lim_{i \rightarrow \infty} \mu_i = \mu$ and each $\mu_i a + (1 - \mu_i)b \succsim c$ then $\mu a + (1 - \mu)b \succsim c$ [and similarly for $c \succsim \mu_i a + (1 - \mu_i)b$]. Axiom 3 states that if an individual is indifferent as to a choice between a and a' , then he is also indifferent to a choice between A and A' , where A represents a 50-50 chance of getting a or b and A' a 50-50 chance of getting a' or b , for any prospect b .

4. THE DERIVATION OF THE MEASURABLE UTILITY

THEOREM 1 (*continuity theorem*): If $a, b, c \in S$ and $a \succsim b \succsim c$, then there exists a μ such that $b \sim \mu a + (1 - \mu)c$.

PROOF: Let $T = \{\mu \mid \mu a + (1 - \mu)c \succsim b\}$. By Axiom 2, T is a closed subset of the closed unit interval $[0, 1]$. Since $a \succsim b, 1 \in T$; so T is not

³ The possible use of this form of axiom, rather than the more restrictive form

$$\alpha a + (1 - \alpha)b \sim \alpha a' + (1 - \alpha)b,$$

was suggested by Debreu.

empty. Using Axiom 2 we find that $W = \{\lambda \mid b \succcurlyeq \lambda a + (1 - \lambda)c\}$ is closed in $[0, 1]$; it is not empty since $0 \in W$. S is a completely-ordered mixture set, so $T \cup W = [0, 1]$; the unit interval is connected, so it cannot be decomposed into a union of closed, disjoint subsets; thus $T \cap W$ is not empty. Let $\mu_0 \in T \cap W$; by the definitions of T and W , $b \sim \mu_0 a + (1 - \mu_0)c$, proving Theorem 1. Clearly, if $a > b > c$, then $1 > \mu_0 > 0$.

THEOREM 2:⁴ *If $a, a' \in S$ and $a \sim a'$, then for any $b \in S$ and any $\mu \mu a + (1 - \mu)b \sim \mu a' + (1 - \mu)b$.*

PROOF: We shall prove this result by first establishing several intermediate results. These will be of importance also for many subsequent proofs in this paper.

(a) *If $\mu_i a + (1 - \mu_i)b \sim c$ and $\lim_{i \rightarrow \infty} \mu_i = \mu$, then $\mu a + (1 - \mu)b \sim c$.*

For $\mu_i a + (1 - \mu_i)b \succcurlyeq c$ implies, by Axiom 2, that $\mu a + (1 - \mu)b \succcurlyeq c$. Similarly, using Axiom 2 we obtain $c \succcurlyeq \mu a + (1 - \mu)b$.

(b) *If $a > b$, then $a > \frac{1}{2}a + \frac{1}{2}b > b$.*

Suppose that $\frac{1}{2}a + \frac{1}{2}b \succcurlyeq a > b$. Then, by Theorem 1, there exists a μ such that $a \sim \mu(\frac{1}{2}a + \frac{1}{2}b) + (1 - \mu)b = \frac{1}{2}\mu a + (1 - \frac{1}{2}\mu)b$. Let $T = \{\mu \mid a \sim (\mu/2)a + [1 - (\mu/2)]b\}$. By (a), T is a closed subset of $[0, 1]$, and so has a least element μ_0 ($\mu_0 > 0$ since $a > b$). Now since $a \sim \frac{1}{2}\mu_0 a + (1 - \frac{1}{2}\mu_0)b$, Axiom 3 yields the result that $\frac{1}{2}a + \frac{1}{2}b \sim \frac{1}{4}\mu_0 a + (1 - \frac{1}{4}\mu_0)b \succcurlyeq a > b$; so, for some λ , $a \sim \lambda[\frac{1}{4}\mu_0 a + (1 - \frac{1}{4}\mu_0)b] + (1 - \lambda)b$ by Theorem 1. But $\lambda\mu_0/4 < \mu_0/2$, contradicting the choice of μ_0 . Hence $a > \frac{1}{2}a + \frac{1}{2}b$. Similarly, $\frac{1}{2}a + \frac{1}{2}b > b$.

(c) *If $a > b$, then, for any $0 < \mu < 1$, $a > \mu a + (1 - \mu)b > b$.*

In the rest of the proof of Theorem 2, ρ will generically stand for a rational number of the form $\rho = \sum_{i=1}^{n(\rho)} \alpha_i/2^i$, where $\alpha_i = 0$ or 1. Successive applications of remark (b) yield the result that, if $\rho_2 > \rho_1$, then $a > \rho_2 a + (1 - \rho_2)b > \rho_1 a + (1 - \rho_1)b > b$. For any $0 < \mu < 1$, pick ρ_2, ρ_1 such that $0 < \rho_1 < \mu < \rho_2 < 1$. Let $a' = \rho_2 a + (1 - \rho_2)b$, $b' = \rho_1 a + (1 - \rho_1)b$. Then for any ρ_i lying between ρ_1 and ρ_2 , we have $a' \succcurlyeq \rho_i a + (1 - \rho_i)b \succcurlyeq b'$. Now $\mu = \lim \rho_i$ for suitable $\rho_2 \geq \rho_i \geq \rho_1$ (using the binary expansion of μ to obtain such approximants). Thus Axiom 2 yields the result that $a > a' \succcurlyeq \mu a + (1 - \mu)b \succcurlyeq b' > b$, proving remark (c).

(d) *If $a \sim a'$, then $\mu a + (1 - \mu)a' \sim a$.*

⁴The reader may prefer to simplify the proofs by taking this theorem as an axiom, in place of Axiom 3. However, we felt that the gain, both in generality and intuitive appeal, of our Axiom 3, was worth the resulting complication in the mathematics.

Successive use of Axiom 3 yields the result that $\rho_i a + (1 - \rho_i)a' \sim a$. Pick ρ_i so that $\lim_{i \rightarrow \infty} \rho_i = \mu$. Then, by remark (a), $\mu a + (1 - \mu)a' \sim a$.

We are now in a position to prove Theorem 2. Suppose $a \sim a'$. By remark (d), if $b \sim a$, then $a \sim \mu a + (1 - \mu)b \sim a' \sim \mu a' + (1 - \mu)b$; so to prove Theorem 2, suppose $a > b$. From Axiom 3, for any ρ , $\rho a + (1 - \rho)b \sim \rho a' + (1 - \rho)b$. Given μ , let $T = \{\lambda \mid \lambda a + (1 - \lambda)b \succeq \mu a' + (1 - \mu)b\}$. We pick a sequence ρ_i approaching μ such that $\rho_i \geq \mu$ for each i . Then $\rho_i a + (1 - \rho_i)b \sim \rho_i a' + (1 - \rho_i)b \succeq \mu a' + (1 - \mu)b$ follows easily from remark (c).⁵ Therefore, $\rho_i \in T$; hence, by Axiom 2, $\mu = \lim \rho_i \in T$. That is, $\mu a + (1 - \mu)b \succeq \mu a' + (1 - \mu)b$.

This argument and construction is symmetric in a and a' , so we also can obtain $\mu a' + (1 - \mu)b \succeq \mu a + (1 - \mu)b$. This leads to the desired result. If, on the other hand, $b > a$, the same technique employed for $a > b$ gives the result.

Because remark (c) of the proof of Theorem 2 will be needed often, we exhibit it as

THEOREM 3: *If $a > b$, then for every $0 < \mu < 1$, $a > \mu a + (1 - \mu)b > b$.*

Suppose $a > b$ and $\lambda > \mu > 0$. Then $\lambda a + (1 - \lambda)b > b$. Since $0 < \mu/\lambda < 1$, by Theorem 3, $\lambda a + (1 - \lambda)b > (\mu/\lambda)[\lambda a + (1 - \lambda)b] + [1 - (\mu/\lambda)]b = \mu a + (1 - \mu)b$. Similarly, if $a > b$ and $\lambda a + (1 - \lambda)b > \mu a + (1 - \mu)b$, then $\lambda > \mu$. Summarizing, we have

THEOREM 4: *If $a > b$ then $\lambda a + (1 - \lambda)b > \mu a + (1 - \mu)b$ if and only if $\lambda > \mu$.*

As a consequence, we obtain

THEOREM 5: *There is only one indifference set or an infinite number of distinct indifference sets in S .*

PROOF: If for some $a, b \in S$, $a > b$, then Theorem 4 yields a distinct indifference set for each λ .

If there were only one indifference set, the problem of finding a measurable utility on S would be trivial, namely, to define $\mu(a) = 1$ for every $a \in S$. So we assume, henceforth, that there is more than one indifference set, that is, in view of Theorem 5, that there are an infinite number of them.

In the presence of Theorem 4, Theorem 1 can now be sharpened to

THEOREM 6: *If $a > b > c$, then there is a unique μ such that $b \sim \mu a + (1 - \mu)c$.*

PROOF: Given $a > b$, let

$$S_{ab} = \{x \in S \mid a \succeq x \succeq b\}.$$

⁵ See, for instance, the proof of Theorem 4.

For $x \in S_{ab}$ we define $\mu_{ab}(x)$ by

$$x \sim \mu_{ab}(x)a + [1 - \mu_{ab}(x)]b.$$

In light of Theorem 6 this definition is meaningful. By Theorem 4, $\mu_{ab}(x) > \mu_{ab}(y)$ if and only if $x > y$. As a trivial consequence of Theorem 2, $\mu_{ab}[\alpha x + (1 - \alpha)y] = \alpha\mu_{ab}(x) + (1 - \alpha)\mu_{ab}(y)$ for all $x, y \in S_{ab}$, and all α . Thus μ_{ab} has the requisite properties of being both linear and order-preserving. It is then natural to expect these to serve as the building stones for our final measurable utility. All we need is a mechanism to extend μ_{ab} from S_{ab} to all of S .

Pick $r_1 > r_0$ and consider these as fixed henceforth. In what follows we consider only a, b , which satisfy

$$(I) \quad r_0, r_1 \in S_{ab}.$$

For any $x \in S_{ab}$ we define

$$M_{ab}(x) = \frac{\mu_{ab}(x) - \mu_{ab}(r_0)}{\mu_{ab}(r_1) - \mu_{ab}(r_0)}.$$

Clearly $M_{ab}(r_0) = 0$ and $M_{ab}(r_1) = 1$.

We need

THEOREM 7: *Let f, g be linear, order-preserving functions defined on S_{ab} such that $f(r_0) = g(r_0)$, $f(r_1) = g(r_1)$. Then f is identical with g on S_{ab} .*

PROOF: Let $x \in S_{ab}$. If $r_1 \succeq x \succeq r_0$, then $x \sim \alpha r_1 + (1 - \alpha)r_0$; so $f(x) = \alpha f(r_1) + (1 - \alpha)f(r_0) = \alpha g(r_1) + (1 - \alpha)g(r_0) = g[\alpha r_1 + (1 - \alpha)r_0] = g(x)$. If $x \succeq r_1 > r_0$, then $r_1 \sim \mu x + (1 - \mu)r_0$, $\mu > 0$; therefore $f(r_1) = \mu f(x) + (1 - \mu)f(r_0) = g(r_1) = \mu g(x) + (1 - \mu)g(r_0)$; therefore $f(x) = g(x)$ since $\mu > 0$. The proof holds similarly for $r_1 > r_0 \succeq x$.

We are now ready to define our measurable utility, u . For any $x \in S$, pick $a > b$ so that

$$(I) \quad r_1, r_0 \in S_{ab},$$

$$(II) \quad x \in S_{ab}.$$

We then define u by $u(x) = M_{ab}(x)$.

By Theorem 7, for all pairs a, b and c, d , each satisfying (I) and (II), $M_{ab}(x) = M_{cd}(x)$, since $M_{ab}(r_0) = M_{cd}(r_0) = 0$, $M_{ab}(r_1) = M_{cd}(r_1) = 1$; therefore we need not worry about the particular choice of a, b used in defining u . Given any x, y , pick a, b so that all of $x, y, r_0, r_1 \in S_{ab}$. Then $u(x) > u(y)$ if and only if $x > y$ since μ_{ab} , and so M_{ab} , are order-

preserving in S_{ab} . The same holds for $u[\alpha x + (1 - \alpha)y] = \alpha u(x) + (1 - \alpha)u(y)$. Thus u is the desired measurable utility.

We have proved

THEOREM 8: *If a mixture set S satisfies Axioms 1, 2, and 3, a measurable utility can be defined on S .*

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