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*A SOCIAL EQUILIBRIUM EXISTENCE THEOREM**

BY GERARD DEBREU

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

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In a wide class of social systems each agent has a range of actions among which he selects one. His choice is not, however, entirely free and the actions of all the other agents determine the subset to which his selection is restricted. Once the action of every agent is given, the outcome of the social activity is known. The preferences of each agent yield *his* complete ordering of the outcomes and each one of them tries by choosing his action in his restricting subset to bring about the best outcome according to his own preferences. The existence theorem presented here gives general conditions under which there is for such a social system an equilibrium, i.e., a situation where the action of every agent belongs to his restricting subset and no agent has incentive to choose another action.

This theorem has been used by Arrow and Debreu[2] to prove the existence of an equilibrium for a classical competitive economic system, it contains the existence of an equilibrium point for an N-person game (see Nash[8] and Section 4) and, naturally, as a still more particular case the existence of a solution for a zero-sum two-person game (see von Neumann and Morgenstern, Ref. 11, Section 17.6).

In Section 1 the topological concepts to be used are defined. In Section 2 an abstract definition of equilibrium is presented with a proof of the theorem. In Section 3 saddle points are presented as particular cases of equilibrium points and in connection with the closely related MinMax operator. Section 4 concludes with a short historical survey of results about saddle points, fixed points for multi-valued transformations and equilibrium points.

Only subsets of finite Euclidean spaces will be considered here.

1. Topological Concepts.—Two sets in R^n are said to be *homeomorphic* when it is possible to set up between them a one-to-one bicontinuous (h and h^{-1} continuous) correspondence h (called a homeomorphism).

A *convex cell* C in R^n is determined by r points $z^k (k = 1, \ldots, r)$; it is the set

$$C = \{z \mid z = \sum_{k=1}^{r} \zeta_k z^k, \zeta_k \geqq 0 \text{ for } k = 1, \ldots, r, \sum_{k=1}^{r} \zeta_k = 1\}.$$

Such a set is closed.

The product of two convex cells $A \subset R^l$ and $B \subset R^m$ is a convex cell $C \subset R^{l+m}$. Let A be generated by the p points x^i $(i = 1, \ldots, p)$ and B by the q points y^j $(j = 1, \ldots, q)$. Denote by C the convex cell in R^{l+m} generated by the pq points (x^i, y^j). Obviously $A \times B \supset C$, and one shows easily that $A \times B \subset C$.

A *geometric polyhedron* is the union of a finite number of convex cells in R^n. It is clearly closed.

The product of two geometric polyhedra P, Q is a geometric polyhedron. Let $P = \bigcup_{i=1}^{p} A_i \left(Q = \bigcup_{j=1}^{q} B_j \right)$ where the A_i (the B_j) are convex cells in R^l (in R^m). The relation $P \times Q = (\bigcup_i A_i) \times (\bigcup_j B_j) = \bigcup_{ij} (A_i \times B_j)$ proves the result.

A *polyhedron* is a set in R^n homeomorphic to a geometric polyhedron (called geometric antecedent of the first one).

The product of two polyhedra is a polyhedron (since it is homeomorphic to the product of the two geometric antecedents).

Let $I = \{t | 0 \leqq t \leqq 1\}$ denote the closed interval $[0, 1]$ on the real line. A nonempty set Z of R^n is said to be *contractible*, or more precisely, deformable into a point $z^0 \epsilon Z$, if there exists a continuous function $H(t, z)$ (called a deformation) taking $I \times Z$ into Z such that for all $z \epsilon Z$, $H(0, z) = z$ and $H(1, z) = z^0$.

The product of two sets $X \subset R^l$, $Y \subset R^m$ deformable into the two points $x^0 \epsilon X$, $y^0 \epsilon Y$, respectively, is clearly deformable into the point (x^0, y^0).

Finally the real function $t = \dfrac{e^{\Theta} - 1}{e^{\Theta} + 1}$ of the real variable Θ is monotonically increasing from -1 to $+1$ when Θ increases from $-\infty$ to $+\infty$. It establishes a one-to-one correspondence between the closed interval $[-1, +1]$ and the set $\bar{R}$ of all real numbers to which are added two elements $-\infty$ and $+\infty$. Open sets in $\bar{R}$ are defined as images of the usual open sets in $[-1, +1]$, an order is defined in $\bar{R}$ as an image of the usual order in $[-1, +1]$. $\bar{R}$ endowed with this topology and this order is called the *completed real line* (which can naturally be defined directly[5]).

2. *Equilibrium Points.*—Let there be ν agents characterized by a subscript $\iota = 1, \ldots, \nu$.

The ιth agent chooses an action a_ι in a set $\mathfrak{A}_\iota$. The ν-tuple of actions $(a_1, \ldots, a_\nu)$, denoted by a, is an element of $\mathfrak{A} = \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_\nu$. The payoff to the ιth agent is a function $f_\iota(a)$ from $\mathfrak{A}$ to the completed real line.

Denote further by $\bar{a}_\iota$ the $(\nu - 1)$-tuple $(a_1, \ldots, a_{\iota-1}, a_{\iota+1}, \ldots, a_\nu)$ and by $\bar{\mathfrak{A}}_\iota$ the product $\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_{\iota-1} \times \mathfrak{A}_{\iota+1} \times \ldots \times \mathfrak{A}_\nu$. Given $\bar{a}_\iota$ (the actions of all the others), the choice of the ιth agent is restricted to a *non-empty, compact* set $A_\iota(\bar{a}_\iota) \subset \mathfrak{A}_\iota$; the ιth agent chooses a_ι in $A_\iota(\bar{a}_\iota)$ so as to maximize $f_\iota(\bar{a}_\iota, a_\iota)$, assumed to be continuous in a_ι on $A_\iota(\bar{a}_\iota)$.

This background makes the following formal definition intuitive:

Definition a^* is an equilibrium point if for all $\iota = 1, \ldots, \nu$ $a_\iota^* \epsilon A_\iota(\bar{a}_\iota^*)$ and $f_\iota(a^*) = \underset{a_\iota \epsilon A_\iota(\bar{a}_\iota^*)}{\mathrm{Max}} f_\iota(\bar{a}_\iota^*, a_\iota)$.

The *graph* of the function $A_\iota(\bar{a}_\iota)$ is defined as the subset of $\bar{\mathfrak{A}}_\iota \times \mathfrak{A}_\iota$, $G_\iota = \{(\bar{a}_\iota, a_\iota) | a_\iota \epsilon A_\iota(\bar{a}_\iota)\}$. For any $\bar{a}_\iota$, $A_\iota(\bar{a}_\iota)$ is always understood to be non-void.

THEOREM. *For all $\iota = 1, \ldots, \nu$, let $\mathfrak{A}_\iota$ be a contractible polyhedron, $A_\iota(\bar{a}_\iota)$ a multi-valued function from $\bar{\mathfrak{A}}_\iota$ to $\mathfrak{A}_\iota$ whose graph G_ι is closed, f_ι a continuous function from G_ι to the completed real line such that $\varphi_\iota(\bar{a}_\iota) = \underset{a_\iota \epsilon A_\iota(\bar{a}_\iota)}{\mathrm{Max}} f_\iota(\bar{a}_\iota, a_\iota)$ is continuous. If for every ι and $\bar{a}_\iota$ the set $M_{\bar{a}_\iota} = \{a_\iota \epsilon A_\iota(\bar{a}_\iota) | f_\iota(\bar{a}_\iota, a_\iota) = \varphi_\iota(\bar{a}_\iota)\}$ is contractible, then there exists an equilibrium point.*

The proof uses as a lemma a particular case of the fixed point theorem of S. Eilenberg and D. Montgomery[6] or of the even more general result of E. G. Begle.[3]

Let Z be a set and ϕ a function associating with each $z \epsilon Z$ a subset $\phi(z)$ of Z. We have defined above the graph of ϕ as the subset of $Z \times Z$, $\{(z, z') |$

$z' \epsilon \phi(z)\}$. ϕ is said to be *semicontinuous* if its graph is closed. A *fixed point* of ϕ is a point z^* such that $z^* \epsilon \phi(z^*)$.

LEMMA. *Let Z be a contractible polyhedron and ϕ: $Z \to Z$ a semicontinuous multi-valued function such that for every $z \epsilon Z$ the set $\phi(z)$ is contractible. Then ϕ has a fixed point.*†

$\mathfrak{A}$, the product of ν contractible polyhedra, is a contractible polyhedron (Section 1). Define on $\mathfrak{A}$ the multi-valued function ϕ as follows:

$$\phi(a) = M_{\bar{a}_1} \times \ldots \times M_{\bar{a}_\nu}.$$

Since $M_{\bar{a}_\iota}$ is contractible for all ι and $\bar{a}_\iota$, $\phi(a)$ is contractible for all $a \epsilon \mathfrak{A}$ (Section 1). To be able to apply the lemma it remains only to show that ϕ is semicontinuous.

For this first define in $\bar{\mathfrak{A}}_\iota \times \mathfrak{A}_\iota$ the set

$$M_\iota = \{(\bar{a}_\iota, a_\iota) | a_\iota \epsilon M_{\bar{a}_\iota}\}.$$

The equivalent definition

$$M_\iota = \{(\bar{a}_\iota, a_\iota) \epsilon G_\iota | f_\iota(\bar{a}_\iota, a_\iota) = \varphi_\iota(\bar{a}_\iota)\}$$

shows that M_ι is closed since G_ι is closed and f_ι and φ_ι are continuous.

The graph Γ of ϕ is the subset of $\mathfrak{A} \times \mathfrak{A}$

$$\Gamma = \{(a, a') | a' \epsilon \phi(a)\} = \{(a, a') | a'_\iota \epsilon M_{\bar{a}_\iota} \text{ for all } \iota\} = \{(a, a') | (\bar{a}_\iota, a'_\iota) \epsilon M_\iota \text{ for all } \iota\}.$$

Consider the subset of $\mathfrak{A} \times \mathfrak{A}$

$$\mathfrak{M}_\iota = \{(a, a') | (\bar{a}_\iota, a'_\iota) \epsilon M_\iota\};$$

$\mathfrak{M}_\iota$ is closed since M_ι is. As $\Gamma = \bigcap_\iota \mathfrak{M}_\iota$, Γ is closed.

The conclusion of the lemma is then that there exists $a^* \epsilon \mathfrak{A}$ such that $a^* \epsilon \phi(a^*)$, i.e., for all ι, $a^*_\iota \epsilon M_{\bar{a}^*_\iota}$; this is the definition of an equilibrium point a^*.

The requirement that $\varphi_\iota(\bar{a}_\iota)$ be continuous is a *joint* requirement on the two functions f_ι and $A_\iota(\bar{a}_\iota)$; it is therefore not well adapted to applications. The following *Remark* tries to overcome this.

The function $A_\iota(\bar{a}_\iota)$ is said to be *continuous* at $\bar{a}^0_\iota$ if for any $a^0_\iota \epsilon A_\iota(\bar{a}^0_\iota)$ and any sequence $(\bar{a}^n_\iota)$ converging to $\bar{a}^0_\iota$, there exists a sequence (a^n_ι) converging to a^0_ι such that for all n, $a^n_\iota \epsilon A_\iota(\bar{a}^n_\iota)$.

Remark: If $A_\iota(\bar{a}_\iota)$ has a compact graph G_ι and is continuous at $\bar{a}^0_\iota$, if f_ι is a continuous function from G_ι to the completed real line, then $\varphi_\iota(\bar{a}_\iota)$ is continuous at $\bar{a}^0_\iota$.

We drop subscripts ι everywhere and reason as if f took its values in the real line (the isomorphism $\dfrac{e^f - 1}{e^f + 1}$ between the completed real line and the closed interval $[-1, +1]$ immediately extends the results to the general case).

(α) *Using only the compactness of G and the continuity of f we first prove:* For any sequence $(\bar{a}^n)$ converging to $\bar{a}^0$ and any $\epsilon > 0$, there is an N such that $n > N$ implies $\varphi(\bar{a}^n) < \varphi(\bar{a}^0) + \epsilon$ (in other words, $\varphi(\bar{a})$ is upper semicontinuous at $\bar{a}^0$).

For every n, choose $a^n \epsilon A(\bar{a}^n)$ such that $f(\bar{a}^n, a^n) = \varphi(\bar{a}^n)$. Since G is compact it is possible to extract from the sequence $(\bar{a}^n, a^n)$ a subsequence $(\bar{a}^{n'}, a^{n'})$ converging to $(\bar{a}^0, a^0)$.

By the continuity of f, $f(\bar{a}^{n'}, a^{n'})$ [which is $= \varphi(\bar{a}^{n'})$] tends to $f(\bar{a}^0, a^0)$ [which is $\leqq \varphi(\bar{a}^0)$]. Therefore there exists N' such that $n' > N'$ implies $\varphi(\bar{a}^{n'}) < \varphi(\bar{a}^0) + \epsilon$. Since from any sequence $(\bar{a}^n)$ converging to $\bar{a}^0$ it is possible to extract a subsequence $(\bar{a}^{n'})$ having the desired property, any sequence $(\bar{a}^n)$ converging to $\bar{a}^0$ has the property.

(β) *Using in addition the continuity of $A(\bar{a})$ at $\bar{a}^0$ we prove:* For any sequence $(\bar{a}^n)$ converging to $\bar{a}^0$ and any $\epsilon > 0$, there is an N such that $n > N$ implies $\varphi(\bar{a}^n) > \varphi(\bar{a}^0) - \epsilon$ (in other words, $\varphi(\bar{a})$ is lower semicontinuous at $\bar{a}^0$).

Choose $a^0 \epsilon A(\bar{a}^0)$ such that $f(\bar{a}^0, a^0) = \varphi(a^0)$. By continuity of $A(\bar{a})$ at $\bar{a}^0$, there is a sequence (a^n) converging to a^0 such that for all n, $a^n \epsilon A(\bar{a}^n)$. By the continuity of f, $f(\bar{a}^n, a^n)$ [which is $\leqq \varphi(\bar{a}^n)$] tends to $f(\bar{a}^0, a^0)$ [which is $= \varphi(\bar{a}^0)$]. Therefore there exists N such that $n > N$ implies $\varphi(\bar{a}^n) > \varphi(\bar{a}^0) - \epsilon$.

(α) and (β) together naturally prove that $\varphi(\bar{a})$ is continuous at $\bar{a}^0$.

3. *Saddle Points and MinMax Operator.*—In this section $x \epsilon X \subset R^l$, $y \epsilon Y \subset R^m$ and $f(x, y)$ is a function from $X \times Y$ to the completed real line. A *saddle point* of f is a point (x^0, y^0) such that

$$\operatorname*{Min}_{y} f(x^0, y) = f(x^0, y^0) = \operatorname*{Max}_{x} f(x, y^0). \quad (1)$$

It is a very particular case of an equilibrium point for two agents:

$$\begin{array}{llll} a_1 = x, & \mathfrak{A}_1 = X, & A_1(\bar{a}_1) \equiv X, & f_1(a) = f(x, y) \\ a_2 = y, & \mathfrak{A}_2 = Y, & A_2(\bar{a}_2) \equiv Y, & f_2(a) = -f(x, y) \end{array}$$

One obtains therefore by using the remark:

COROLLARY. *Let X, Y be two contractible polyhedra, and $f(x, y)$ a continuous function from $X \times Y$ to the completed real line. If for every $x^0 \epsilon X$, $U_{x^0} = \{y \epsilon Y | f(x^0, y) = \operatorname*{Min}_{y \epsilon Y} f(x^0, y)\}$ is contractible and for every $y^0 \epsilon Y$, $V_{y^0} = \{x \epsilon X | f(x, y^0) = \operatorname*{Max}_{x \epsilon X} f(x, y^0)\}$ is contractible, then f has a saddle point.*

This corollary contains as more and more particular cases the saddle point theorems of Kakutani,[7] von Neumann (Ref. 9, p. 307, and 10), and von Neumann and Morgenstern (Ref. 11, Section 17.6).

The special interest of saddle points comes from their intimate relation with the MinMax operator.

From now on X, Y are assumed to be compact and $f(x, y)$ to be continuous.

We know from the Remark that $\operatorname{Min}_y f(x, y)$ [resp., $\operatorname{Max}_x f(x, y)$] *is a continuous function of x* [resp., *y*]. The following results, already given in Ref. 11, Section 13, are proved here for completeness.

(*a*) $\operatorname{Max}_x \operatorname{Min}_y f(x, y) \leqq \operatorname{Min}_y \operatorname{Max}_x f(x, y)$.

Let $A = \{x' \mid \operatorname{Min}_y f(x', y) = \operatorname{Max}_x \operatorname{Min}_y f(x, y)\}$, $B = \{y' \mid \operatorname{Max}_x f(x, y') = \operatorname{Min}_y \operatorname{Max}_x f(x, y)\}$.

If $x' \epsilon A$ and $y' \epsilon B$,

$$\operatorname{Max}_x \operatorname{Min}_y f(x, y) = \operatorname{Min}_y f(x', y) \leqq f(x', y') \leqq \operatorname{Max}_x f(x, y') = \operatorname{Min}_y \operatorname{Max}_x f(x, y). \quad (2)$$

The result follows from a comparison of the first and last terms.

(*b*) *The existence of a saddle point* (x^0, y^0) *implies the equality*

$$\operatorname{Max}_x \operatorname{Min}_y f = \operatorname{Min}_y \operatorname{Max}_x f[= f(x^0, y^0)].$$

From the definition 1 it follows that

$$\operatorname{Max}_x \operatorname{Min}_y f(x, y) \geqq \operatorname{Min}_y f(x^0, y) = f(x^0, y^0) = \operatorname{Max}_x f(x, y^0) \geqq \operatorname{Min}_y \operatorname{Max}_x f(x, y) \quad (3)$$

which together with (*a*) gives the result. It also gives $\operatorname{Max}_x f(x, y^0) = \operatorname{Min}_y \operatorname{Max}_x f(x, y)$ i.e., $y^0 \epsilon B$, and similarly $x^0 \epsilon A$.

(*c*) *The equality* $\operatorname{Max}_x \operatorname{Min}_y f = \operatorname{Min}_y \operatorname{Max}_x f$ *implies the existence of a saddle point.*

Assume that the equality holds and take $x^0 \epsilon A$, $y^0 \epsilon B$, Eq. (2) gives $\operatorname{Min}_y f(x^0 y) = f(x^0, y^0) = \operatorname{Max}_x f(x, y^0)$, which is the definition Eq. (1) of a saddle point (x^0, y^0). We have, incidentally, proved

(*d*) *the set of saddle points is either empty or equal to* $A \times B$.

4. *Historical Note.*—A function $f(z)$ from a set Z to the completed real line $\bar{R}$ is said to be *quasi-convex* (resp., *quasi-concave*) if for any $\alpha \epsilon \bar{R}$, the set of $z \epsilon Z$ such that $f(z) \leqq \alpha$ (resp., $f(z) \geqq \alpha$) is convex.

Let $S_n = \{z \epsilon R^n \mid z_k \geqq 0 \text{ for } k = 1, \ldots, n \text{ and } \sum_{k=1}^{n} z_k = 1\}$.

In his first study on the theory of games, J. von Neumann[9] proved:

(I). *Let $f(x, y)$ be a continuous real-valued function for $x \in S_l$ and $y \in S_m$. If for every $x^0 \in S_l$ the function $f(x^0, y)$ is quasi-convex, and if for every $y^0 \in S_m$ the function $f(x, y^0)$ is quasi-concave, then f has a saddle point.*

In another paper on economics[10] he later proved a closely related lemma which S. Kakutani[7] restated in the more convenient form of the following (equivalent) fixed point theorem:

(II). *Let Z be a compact convex set in R^n and ϕ: $Z \rightarrow Z$ a semicontinuous multi-valued function such that for every $z \in Z$ the set $\phi(z)$ is non-empty and convex. Then ϕ has a fixed point.*

The convexity assumptions were, however, irrelevant and S. Eilenberg and D. Montgomery[6] gave a fixed point theorem where convexity was replaced by acyclicity. Their result was further generalized by E. G. Begle.[3]

These last two theorems deserve particular attention as valuable contributions to topology whose origin can be traced directly to economics.

The notion of an equilibrium point was first formalized by J. F. Nash[8] in the following game context. There are ν players; the ιth player chooses a strategy s_ι in S_{n_ι}; his payoff is $f_\iota(s)$, a polylinear function of $s_1, \ldots, s_\nu$. An equilibrium point is a ν-tuple s^* such that for all ι, $f_\iota(s^*) = \underset{s_\iota \in S_n}{\text{Max}} f_\iota(\bar{s}_\iota^*, s_\iota)$. Nash proved the existence of such an equilibrium point.

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† The statement of E. G. Begle (Ref. 3, p. 546) is indeed much more general and the existence theorem can accordingly be generalized. Instead of a contractible polyhedron one might take for example an Absolute Retract (as defined in [Ref. 4, p. 222]) using the fact that the product of two A.R. is an A.R. [Ref. 1, p. 197]. For finite dimensions "Absolute Retract" is equivalent to "contractible and locally contractible (Ref. 4, pp. 235–236) compact metric space" [Ref. 4, p. 240].

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